Lecture 2 Geometric Satake
— statement and A_1 case

quiver gauge theory

$Q = (Q_0, Q_1)$ $\qquad \underset{o(h)}{\circ} \xrightarrow{h} \underset{i(h)}{\circ}$

V, W : Q_0-graded vector sp. $/\mathbb{C}$

$$G = \prod_{i \in Q_0} GL(V_i)$$

$$G \curvearrowright N = \bigoplus_{h} \mathrm{Hom}(V_{o(h)}, V_{i(h)}) \oplus \bigoplus_{i} \mathrm{Hom}(W_i, V_i)$$

<u>Assume</u> $W \neq 0$ $\qquad$ Notation $\begin{cases} \lambda = \sum \dim W_i \cdot \Lambda_i \\ \mu = \lambda - \sum \dim V_i \alpha_i \end{cases}$

FI parameter $\quad \zeta \in \underbrace{\mathrm{Lie}(T_c)_{\mathbb{R}}}_{= \pi_1(G)^\wedge} \otimes_{\mathbb{R}} \mathrm{Im}\,\mathbb{H}$

Assume $\zeta = (\zeta_i)$, $\zeta_i \in \mathbb{R}_{>0}\ \forall i$

(cpx part $= 0$)

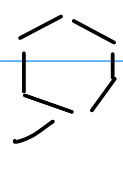
Kähler param.

mass parameter

$T_F = \tilde{G}/G$ $\qquad T_F = \prod_i T(W_i) \times (\mathbb{C}^\times)^{b_1(Q)}$

If $\left(\begin{array}{l} Q : \text{finite} \\ \text{or affine} \end{array} \right.$

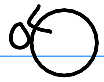
Only affine type A

 $\to b_1(Q) = 1$

Higgs branch $\mathcal{M}_H \xleftarrow{\pi} \mathcal{M}_H^{\zeta}$ (quiver variety)

$\uparrow$
($\zeta = 0$ version)

- Q : Jordan quiver $\quad \mathcal{M}_H$: ADHM description of $U(w)$-instantons on $\mathbb{R}^4$



- Q : affine ADE $\quad$ Kronheimer-N ADHM descr. of instantons on ALE sp. $\widetilde{\mathbb{R}^4/\Gamma_Q}$

Coulomb branch $\quad$ Hanany-Witten, Kapustin, Cherkis, ...

$\mathcal{M}_C^m \longrightarrow \mathcal{M}_C$
$\quad\quad\quad\quad m=0$

Q : affine type $\quad \mathcal{M}_C^m$ is the Uhlenbeck partial compactification of $G_{Q_{fin}}$-instantons on multi-Taub-NUT sp.

$\widetilde{\mathbb{C}^2/(\mathbb{Z}/\ell\mathbb{Z})}$ with hyperKähler metric $\neq$ Euclidean

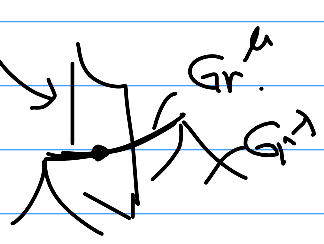
Rigorously proved (N-Takayama) for affine type A.

Q : finite type $\quad$ singular G_Q-monopoles on $\mathbb{R}^3$

If μ is dominant, moduli of singular monopole (good or ugly)

$$\cong$$

slice to $Gr^{\mu}_{G_Q}$ in $\overline{Gr^{\lambda}_{G_Q}}$



$$G_Q(\mathcal{O}) \curvearrowright Gr_{G_Q} = \bigsqcup_{\lambda} \underbrace{Gr^{\lambda}_{G_Q}}_{=\, G_Q(\mathcal{O})\cdot z^{\lambda}}$$

$\mathcal{O} = \mathbb{C}[[z]]$

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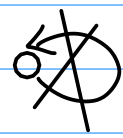
topology of Higgs / Coulomb   Assume 

<u>Fact</u>. [N '94 ~ 98]

$\mathfrak{g}_Q :=$ Kac-Moody Lie alg. corresp. to Q

e.g. Q: affine type $\Rightarrow$ $\mathfrak{g}_Q$ = affine Lie alg.

$$\Longrightarrow \bigoplus_{\mu} \underbrace{H_{mid}}_{\text{middle degree}}(\mathcal{M}^{\zeta}_H(\lambda,\mu)) = \bigoplus_{\mu} H_{top}(\mathfrak{L}^{\zeta}_H(\lambda,\mu))$$

has a str. of a representation of $\mathfrak{g}_Q$

$\mathfrak{L}^{\zeta}_H(\lambda,\mu)$ = inverse image of 0 under π — Lagrangian subvariety

$$\begin{array}{ccc} \mathcal{M}^{\zeta}_H & \xrightarrow{\pi} & \mathcal{M}_H \\ \cup & & \cup \\ \mathfrak{L}^{\zeta}_H & \longrightarrow & 0 \end{array}$$

$V(\lambda)$: integrable highest wt rep.

A better formulation for mathematicians

$\mathbb{C}_{\mathcal{M}_H^\zeta(\lambda,\mu)}[\dim]$: constant sheaf ~~of~~ on $\mathcal{M}_H^\zeta$ shifted

$$\pi_* \mathbb{C}_{\mathcal{M}_H^\zeta(\lambda,\mu)}[\ ] = \bigoplus IC(\text{stratum}) \otimes H_{top}(\text{fiber})$$

semi-small

Q: finite

$$\left(\mathcal{M}_H(\lambda,\mu) = \bigsqcup_\nu \mathcal{M}_H^{reg}(\lambda,\nu) \right)$$

0 : smallest stratum.

$\mathcal{L}_H^\zeta$: fiber over 0.

How about $\mathcal{M}_C$?

If Q: finite type

$\mathcal{M}_C(\lambda,\mu)$: slice in the affine Grassmannian

$\xRightarrow[\text{geometric Satake}]{}$ Get representation of $\mathfrak{g}_Q$.

<u>Conj</u> (Braverman-Finkelberg-N)

This is true for arbitrary type Q.

Mirkovic-Vilonen

FI parameter $\zeta \in$ ~~Lie~~ T_C

$\mathrm{Hom}(\mathbb{C}^\times, T_C)$

$$\mathbb{C}^* \xrightarrow{\zeta} T_C \curvearrowright \mathcal{M}_C(\lambda,\mu)$$

<u>Conjecture</u>(1) $\mathcal{M}_C(\lambda,\mu)^{\zeta=0}$ is either ϕ
or a single point

and it is a single $\overset{\text{fixed pt set}}{\text{pt}}$ $\Leftrightarrow$ $V(\lambda)_\mu \neq 0$
$\Leftrightarrow$ $\mathcal{M}_H^\zeta(\lambda,\mu) \neq \phi$

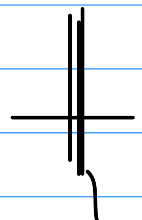
$$\mathcal{M}_C(\lambda,\mu)^{\zeta\leqq 0} \overset{\text{def.}}{=} \{x \in \mathcal{M}_C(\lambda,\mu) \mid \lim_{t\to\infty} \zeta(t)\cdot x \text{ exists}\}$$

repelling set

<u>Prop.</u> $\mathcal{M}_C(\lambda,\mu)^{\zeta\leqq 0} \cap$ regular locus of $\mathcal{M}_C(\lambda,\mu)$: lagrangian

$\uparrow$
symplectic mfd

<u>e.g.</u>



$\mathbb{C}^2$ $dx\wedge dy$ $t(x,y) = (tx, t^{-1}y)$

repelling set

<u>Conjecture</u>(2)
This is true for any intersection
with symplectic leaves.

A better formulation : use IC
and Braden's
hyperbolic restriction
function

<u>Main Conj</u> $H^{BM}_{top}(\mathcal{M}_C(\lambda,\mu)^{\zeta\leq 0}) \cong V(\lambda)_\mu$

$$\begin{array}{c} V(\lambda)_\mu \\ \wr\| \\ H_{top}(\mathcal{L}^{\zeta}_H(\lambda,\mu)) \end{array}$$

$\left(\begin{array}{c} H^{BM}_2(X) = \mathbb{C}[X] \\ X=\mathbb{R}^2 \end{array} \right)$ True for geometric Satake

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<u>Examples</u> (A_1) $\begin{array}{c} \circ\, v \\ | \\ \square\, w \end{array}$

Higgs

$$\begin{array}{ccccc} \mathcal{M}^{\zeta}_H(\lambda,\mu) = & T^*Gr(v,w) & \xrightarrow{\pi} & \mathcal{M}_H(\lambda,\mu) \\ \cup & \cup & & \\ \mathcal{L}^{\zeta}_H(\lambda,\mu) = & Gr(v,w) & & \end{array}$$

$$\bigoplus_\mu \underbrace{H_{top}(\mathcal{L}^{\zeta}_H(\lambda,\mu))}_{\text{1-dim. for each } 0\leq v\leq w} \cong (w+1)\text{-dim'l irr. rep. of } \mathfrak{sl}_2$$

$\lambda, w, = w$

Consider mass parameter $m : \mathbb{C}^\times \longrightarrow T_F$ generic $\ni t$ $\begin{bmatrix} t^{m_1} & & \\ & t^{m_2} & \\ & & \ddots \; t^{m_r} \end{bmatrix}$

equivariant variable

$\mathcal{M}^{\zeta}_H(\lambda,\mu)^{m\leq 0}$

$\uparrow$

repelling set

$\left(\begin{array}{c} \subset_{\!\times} \\ \mathcal{L}^{\zeta}_H(\lambda,\mu) \end{array} \right)$ Lagrangian

$$\bigoplus_\mu H_{top}(\mathcal{M}^{\zeta}_H(\lambda,\mu)^{m\leq 0}) \cong (\mathbb{C}^2)^{\otimes m}$$

vector

<u>Coulomb</u> $\mathcal{M}_C(\lambda,\mu)^{\zeta\leq 0}$ $\rightsquigarrow$ h.w. $= w$ irr rep.

$\uparrow$

m: generic $\mathcal{M}_C^m(\lambda,\mu)^{\zeta\leq 0} \rightsquigarrow (\mathbb{C}^2)^{\otimes m}$

$\bigoplus_\mu H_{top}(\mathcal{M}_C^m(\lambda,\mu)^{\zeta\leq 0}) \cong$

<u>Conj</u> $\mathcal{M}_C^m(\lambda,\mu) \longrightarrow \mathcal{M}_C(\lambda,\mu)$ m: generic (including $(\mathbb{C}^\times)^{b_1(Q)}$)

$\uparrow$

smooth (symplectic resolution)

$\overset{?}{\Longleftrightarrow}$ $\mathcal{M}_H(\lambda,\mu)^{m=0}$ is a point.

(If Q: ADE, both $\Leftrightarrow$ λ is a sum of minuscule weights)