Lecture 3 Geometric Satake (cont'd)

Setting $Q = (Q_0, Q_1)$: quiver
V, W : Q_0-graded vector spaces
$\lambda = \sum \dim W_i \cdot \Lambda_i$, $\mu = \lambda - \sum \dim V_i \alpha_i$

$\rightsquigarrow$ $\mathcal{M}_H(\lambda, \mu)$: Higgs branch
$\mathcal{M}_C(\lambda, \mu)$: Coulomb branch

ζ : FI parameter $\in i\mathbb{R}_{>0}^{Q_0}$

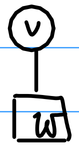
$\rightsquigarrow$ $\pi : \mathcal{M}_H^\zeta(\lambda, \mu) \longrightarrow \mathcal{M}_H(\lambda, \mu)$ (often) symplectic resolution

$\mathcal{M}_H^\zeta(\lambda, \mu) \supset \mathcal{L}_H^\zeta(\lambda, \mu) = \pi^{-1}(0)$ cpt Lagrangian

$\rightsquigarrow$ $\underset{\text{repelling set}}{\mathcal{M}_C(\lambda, \mu)^{\zeta \leqq 0}} \subset \mathcal{M}_C(\lambda, \mu)$ noncpt Lagrangian

$$H_{top}(\mathcal{L}_H^\zeta(\lambda, \mu)) \cong V(\lambda)_\mu \cong H_{top}(\mathcal{M}_C(\lambda, \mu)^{\zeta \leqq 0})$$

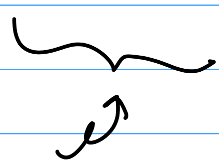
[N. 1994~98] geometric Satake for Kac-Moody

Ex. A_1 

$\lambda = \Lambda$
$\mu = \Lambda - v\alpha$

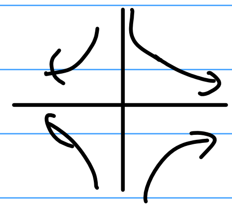
$\mathcal{M}_C(\lambda, \lambda)$, $\mathcal{M}_C(\lambda, \lambda - \alpha)$, $\mathcal{M}_C(\lambda, \lambda - 2\alpha)$ ....

$\underline{W = 1}$ λ $\quad$ $\mathcal{M}_C(\lambda,\lambda) = \mathrm{pt}$ $\quad$ $\mathcal{M}_C(\lambda,\lambda-\alpha) \cong \mathbb{C}^2$



has no fixed pts.

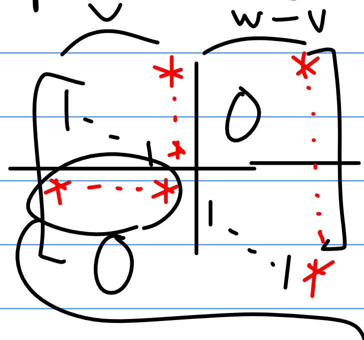
$$\mathcal{M}_C(\lambda,\lambda)^{\zeta\le 0} = \{pt\}, \quad \underbrace{\mathcal{M}_C(\lambda,\lambda-\alpha)^{\zeta\le 0}}_{\cong\, \mathbb{C}}, \quad \underbrace{\cdots}_{\emptyset}$$



$$H_{top}(\mathcal{M}_C(\lambda,\mu)^{\zeta\le 0}) = \begin{cases} \mathbb{C} & \text{if } \mu=\lambda \text{ or } \lambda-\alpha \\ 0 & \text{otherwise} \end{cases}$$

$\rightsquigarrow$ $\mathbb{C}^2$: vectr rep. of $\mathfrak{sl}_2$

general w

$v<w$ $\quad \mathcal{M}_C(\lambda,\mu) = \{$  $\} \cap N$

repelling condition $\Longleftrightarrow$ only survive

$\cong$

$\mathbb{C}^v$

$v=w$ $\quad \mathcal{M}_C(\lambda,\mu)^{\zeta\le 0} \cong \mathbb{C}^v$

$v>w$ $\quad \qquad\qquad = \emptyset$

$$H_{top}(\mathcal{M}_C(\lambda,\mu)^{\zeta\le 0}) = \begin{cases} \mathbb{C} & v=0,\dots,w \\ 0 & \text{otherwise} \end{cases}$$

$\mathbb{C}^{w+1}$ = irr. rep. of $\mathfrak{sl}_2$ with h.w. $= w$

m : mass parameter

$$\mathcal{M}_C^m(\lambda,\mu) \longrightarrow \mathcal{M}_C(\lambda,\mu)$$

$\bigoplus_\mu H_{top}(\mathcal{M}^m_C(\lambda,\mu)^{\zeta\equiv 0}) \cong (\mathbb{C}^2)^{\otimes w}$

$\downarrow$ analog of π $\qquad \uparrow$

$\bigoplus_\mu H_{top}(\mathcal{M}_C(\lambda,\mu)^{\zeta\equiv 0}) \cong \mathbb{C}^{w+1}$ $\quad w=1$ direct summand.

$\mathbb{C}^{w+1} = \mathrm{Sym}^w(\mathbb{C}^2)$

$\pi^{-1}([\quad]) \subset T^*\mathrm{flag}$

$\downarrow \pi$ $\quad$ Springer representation $\circlearrowleft S_w$

N

General Q.

strategy: ① define generators e_i, f_i by reduction to A_1-case via $\zeta' \leftarrow$ non generic FI param.

② Check relations $[e_i, f_j] = 0$ (if $i \neq j$)

(integrability is easy to check)

Choose $i \in Q_0$ $\quad \zeta' = (\zeta'_j)$ s.t. $\zeta'_i = 0$

$\zeta'_j \in \mathbb{R}_{>0}$ $(j \neq i)$

$\uparrow$

FI param. $\qquad$ contained in the hyperplane $\alpha_i = 0$

$\swarrow \qquad \searrow$

GIT param. for $\mathcal{M}_H$ $\qquad$ cochar. in $T_C \curvearrowright \mathcal{M}_C$

$$
\begin{tikzcd}
\mathcal{M}_H^{\zeta}(\lambda,\mu) \arrow[r, "\pi^{\zeta\zeta'}"] \arrow[dr, "\pi"'] & \mathcal{M}_H^{\zeta'}(\lambda,\mu) \arrow[d] \arrow[dl, phantom, "\circlearrowright"] \\
& \mathcal{M}_H(\lambda,\mu)
\end{tikzcd}
$$

← in many cases partial resolution

↑ no FI param.

<u>Key Fact</u>. $\underbrace{\pi^{\zeta\zeta'}}_{\text{locally}}$ looks like $\pi : \mathcal{M}_H^{\zeta} \to \mathcal{M}_H$ for type A_1 with appropriate dimension vectors

e.g. fibers of $\pi^{\zeta\zeta'}$ are Grassmannian

e_i, f_i

$\rightsquigarrow$ use constr. of $\mathfrak{sl}_2$ for A_1 case

<u>mathematicians</u>

$$\pi^{\zeta\zeta'}_*\left(\mathbb{C}_{\mathcal{M}_H^{\zeta}(\lambda,\mu)}[\dim]\right) = \bigoplus IC(\text{strata}) \otimes H_{top}(\text{fiber})$$

of $\mathcal{M}_H^{\zeta'}$ $\qquad$ $= \circlearrowright e_i, f_i$

← rep. of $\mathfrak{g}_Q$

$$\mathrm{Res}^{\mathfrak{g}_Q}_{\mathfrak{sl}_2} V(\lambda) = \bigoplus_w M_w \otimes V_{\mathfrak{sl}_2}(w)$$

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single

$$\mathcal{M}_C(\lambda,\mu)^{\zeta=0} \left(= \mathcal{M}_C(\lambda,\mu)^T\right) = \phi \text{ or pt}$$

$$\cap$$

$$\mathcal{M}_C(\lambda,\mu)^{\zeta'=0}$$

fixed pt set w.r.t. ζ' $\quad \left\{ \begin{array}{l} \zeta'_i = 0 \\ \zeta'_j > 0 \end{array} \right.$

Key conjecture (true for Q: finite or affine type A)

$$\mathcal{M}_C(\lambda,\mu)^{\zeta'=0} \cong \mathcal{M}_C(\lambda',\mu') \text{ for type } A_1$$

for $\left(\begin{array}{l}\text{appropriate } \lambda' \\ \mu' = \langle \mu, \alpha_i^\vee \rangle\end{array}\right.$

Moreover $\mathcal{M}_C(\lambda,\mu)^{\zeta\leqq 0}$ $\cong \underbrace{\mathcal{M}_C(\lambda,\mu)^{\zeta'\leqq 0}}_{\text{"multiplicity space"}} \times_{\mathcal{M}_C(\lambda,\mu)^{\zeta'=0}} \underbrace{\mathcal{M}_C(\lambda',\mu')^{\zeta\leqq 0}}_{e_i,\, f_i}$

original repelling

$\mathcal{M}_C(\lambda,\mu)^{\zeta'=0} = \mathcal{M}_C(\lambda',\mu')$ for A_1

more precisely

Use Braden's hyperbolic restriction functor

$\Phi(IC(\mathcal{M}_C(\lambda,\mu)))$ = semisimple perverse sheaf on fixed point set.

$$\Phi^{\zeta} = \Phi^{\zeta}_{A_1} \circ \Phi^{\zeta'}$$

$$\Phi^{\zeta'}(IC(\mathcal{M}_C(\lambda,\mu)))$$

$$\cong \bigoplus_{\kappa} M_\kappa \otimes IC(\mathcal{M}_C(\kappa,\mu'))$$

$\mathcal{M}_C(\lambda',\mu') = \bigsqcup \mathcal{M}_C^{reg}(\kappa,\mu')$ strata affine type $\underline{A}$ quiver gauge theory

$\mathcal{M}_H$ quiver variety aff. $\subset$ bow varieties (cobalanced cond)

$\mathcal{M}_C$ Coulomb branch $\subset$ (balanced cond.)

Hanany-Witten transition

$$\to \left(\bar{V}_1 \to \bar{V}_2 \rightleftarrows \bar{V}_3\right) \cdots \xrightarrow[HW]{} \cdots \left(\bar{V}_1 \leftrightarrows \bar{V}_2' \to \bar{V}_3\right) \cdots$$

$$\bar{V}_1 \to \mathbb{C} \to \bar{V}_2 \qquad \bar{V}_2' \to \mathbb{C} \to \bar{V}_3$$

$$\dim \bar{V}_2' + \dim \bar{V}_2 = \dim V_1 + \dim \bar{V}_3 + 1$$

Costalk of equivariant IC

work in progress with Dinakar Muthiah

$T_{\mathbb{C}} \curvearrowright \mathcal{M}_{\mathbb{C}}$ $\qquad \mathbb{T} := T_{\mathbb{C}} \times \mathbb{C}^{\times}_{\hbar} \curvearrowright \mathcal{M}_{\mathbb{C}}$

(modified) homological degree

Let $i_\mu : \underset{\{z^\mu\}}{\underset{\shortparallel}{\mathcal{M}_{\mathbb{C}}(\lambda,\mu)^{\zeta=0}}} \longrightarrow \mathcal{M}_{\mathbb{C}}(\lambda,\mu)$ inclusion.

We consider the $\mathbb{T}$-equivariant costalk:

$$H^*_{\mathbb{T}}(i_\mu^! IC(\mathcal{M}_{\mathbb{C}}(\lambda,\mu)))$$

$$\text{\rotatebox{90}{$\cong$}}$$

$$\left(\text{If } \mu : \text{dominant}, \quad IH^*_{\mathbb{C},\mathbb{T}}(\mathcal{M}_{\mathbb{C}}(\lambda,\mu))\right)$$

This is different from what we have discussed so far (even $\mathbb{T}$-equivariant version)

$$H^*_{\mathbb{T}}(\Phi(IC(\mathcal{M}_{\mathbb{C}}(\lambda,\mu)))) \cong H_{top}(\mathcal{M}_{\mathbb{C}}(\lambda,\mu)^{\zeta\leq 0}) \otimes \underbrace{H^*_{\mathbb{T}}(pt)}_{\mathbb{C}[\hbar, a_1, \cdots, a_\ell]}$$

($\mathbb{T}$-equiv. hyperbolic stalk)

There is a natural homomorphism (which is injective)

$\mathbb{T}$-equiv. costalk $\longrightarrow$ $\mathbb{T}$-equiv. hyperbolic stalk

$$\text{\rotatebox{90}{$\cong$}}$$

geometric Satake $\quad V(\lambda)_\mu \otimes H^*_{\mathbb{T}}(pt)$

Th [Muthiah–N]

Image of $\mathbb{T}$-equiv. costalk $\cong (\mathcal{V}(\lambda)\otimes \mathbb{M}(\mu))^{B_Q}$ ← Borel subgrp.

where $\mathbb{M}(\mu)$ is the universal asymptotic Verma module.

$(H^*_{\mathbb{T}}(pt), U_\hbar(\mathfrak{g}_Q))$-bimodule

$XY-YX=\hbar[X,Y]$

s.t. $\quad v_\mu\cdot t = (t+\hbar(\mu+\rho)(t))\, v_\mu \qquad (t\in\mathfrak{h})$

Application: A new proof of AGT and its higher level analog.

$$\begin{cases}\lambda=\Lambda_0 & 0^{\text{th}}\text{ fundamental wt}\\ \mu=\Lambda_0-k\delta\end{cases}$$

$\mathcal{M}_C(\lambda,\mu)$ = Uhlenbeck partial compactification of moduli of $G_{Q,\text{cpt}}$-instantons on $\mathbb{R}^4$

Th AGT (in the framework of BFN)

$$\bigoplus_{k\geq 0}\mathbb{T}\text{-equiv. costalk}\longrightarrow\bigoplus_{k\geq 0}\mathbb{T}\text{-equiv. hyperbolic stalk}$$

the universal Verma module of the W-algebra for $G_Q^\vee$

Fock module of Heisenberg (free fields)

The above theorem

$\Rightarrow$ RHS = universal Verma for the coset VOA $(\lambda=\Lambda_0)$ $\cong$ [Arakawa–Creutzig–Linshaw] W-alg.

The corresponding result in finite dim'l case was proved by Ginzburg–Riche 2015.

closely related result : Arkhipov–Bezrukavnikov–Ginzburg 2004

Set $\hbar = 0$.

$$\mathrm{RHS} = \left(V(\lambda) \otimes \mathbb{C}[(\mathfrak{g}/\mathfrak{u})^*] \otimes \mathbb{C}_{-\mu}\right)^B$$

$$= \left(\overline{V}(\lambda) \otimes \underbrace{\mathrm{Ind}_B^G\left(\mathbb{C}[(\mathfrak{g}/\mathfrak{u})^*] \otimes \mathbb{C}_{-\mu}\right)}_{\shortparallel}\right)^G$$

$$H^0(\underbrace{G \times_B (\mathfrak{g}/\mathfrak{u})^*}_{\text{deformation of } T^*(G/B)}, \mathcal{O}(\mu))$$

This result can be regarded as a variant of Coulomb branch construction (ring objects)

$T^*(G/B) \leftsquigarrow$ intersection cohomology of $\mathcal{M}_C(\lambda, \mu)$

Hence [Muthiah-N] can be considered as a Coulomb branch construction

where the gauge group = loop group.