From faces of Weyl polytopes, to weights and characters of highest weight modules

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Introduction

- $\mathfrak{g}$ = complex semisimple/Kac–Moody Lie algebra (just work with $\mathfrak{sl}_n$), $U(\mathfrak{g})$ = universal enveloping algebra.

  Triangular decomposition: $\mathfrak{g} = \mathfrak{n}^- \oplus \mathfrak{h} \oplus \mathfrak{n}^+$, simple roots π, Weyl group W.

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  Weyl group W.

- Fix a (highest) weight $\lambda \in \mathfrak{h}^*$.
  - $M(\lambda)$ = Verma module;
  - $L(\lambda)$ = simple quotient;
  - V = highest weight module: $M(\lambda) \twoheadrightarrow V \twoheadrightarrow L(\lambda)$.

- We are interested in the structure of highest weight modules,
  e.g. simple *non-integrable* modules.
  (Integrable modules, Verma modules well-studied.)

The start of the journey

In 2009, Chari–Greenstein used certain combinatorial subsets of root system Δ to:

- Study modules over classical and quantum loop algebras,
- Construct Koszul algebras of all finite global dimensions from graded $\mathfrak{g}$-modules (via endomorphism algebras as in blocks of $\mathcal{O}$),

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- Construct Koszul algebras of all finite global dimensions from graded $\mathfrak{g}$-modules (via endomorphism algebras as in blocks of $\mathcal{O}$),
- Obtain a graded character formula (at $q = 1$) of a Kirillov–Reshetikhin module over $U_q(\widehat{\mathfrak{g}})$,
- Connect parabolic subalgebras of $\mathfrak{g}$ to ad-nilpotent ideals.

These combinatorial subsets $Y \subseteq \Delta$ are given by:

$$y_1 + y_2 = \alpha_1 + \alpha_2 \quad (y_1, y_2 \in Y,\ \alpha_1, \alpha_2 \in \Delta) \quad \Longrightarrow \quad \alpha_1, \alpha_2 \in Y.$$

The start of the journey (cont.)

Then in 2009, Chari–Dolbin–Ridenour classified all such subsets. Related to roots on the faces of root polytopes $\operatorname{conv} \Delta$.

- Chari–K.–Ridenour (2012) extended to faces of *Weyl polytopes* $\operatorname{conv}(\operatorname{wt} L(\lambda)), \quad \lambda \in P^+$, and constructed larger families of Koszul (endomorphism) algebras.
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- The study of weights of all simple modules $L(\lambda)$ (even for $\lambda \notin P^+$) and of **all** highest weight modules – over semisimple and also Kac–Moody $\mathfrak{g}$;
- A hitherto unstudied (even over $\mathfrak{sl}_4$) class of "universal" highest weight modules $\mathbb{M}$;
- BGG resolutions and Weyl–Kac character formulas for $\mathbb{M}$.

1a. First-order theory: weights of simple modules

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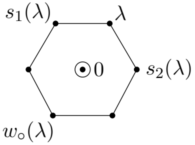

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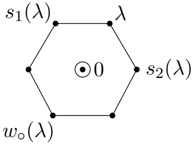

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If instead $L(\lambda)$ is infinite-dimensional:

- *Quantitatively:* character known through Kazhdan–Lusztig theory, e.g.:

$$\operatorname{ch} L(ww_\circ \bullet 0) = \sum_{x \leqslant w} (-1)^{\ell(w)-\ell(x)} P_{x,w}(1) \operatorname{ch} M(xw_\circ \bullet 0).$$

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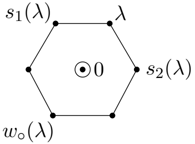

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- *Qualitatively:* which weights occur in $L(\lambda)$? What is their convex hull?
  (Was not written down until recently.)

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What if $\mathfrak{g}$ is of infinite type?

- If $\mathfrak{g}$ is affine, or symmetrizable, $\operatorname{ch} L(\lambda)$ is not known for *all* critical λ.
- If $\mathfrak{g}$ is non-symmetrizable, formulas for $\operatorname{ch} L(\lambda)$ are not known even for integrable simple modules.

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- These formulas involve the weights of a "first-order" (= parabolic) Verma module $M(\lambda, J)$.
- We then extend this (uniform) formula to *all* highest weight $\mathfrak{g}$-modules – now involves the weights of "higher order Verma modules" $\mathbb{M}(\lambda, \mathcal{H})$.

Notation for Kac–Moody $\mathfrak{g}$

For every Kac–Moody Lie algebra $\mathfrak{g}$ (e.g. $\mathfrak{sl}_n$):

- Generalized Cartan matrix A, indexed by Dynkin diagram nodes I;
- Realization $(\mathfrak{h}, \pi, \pi^\vee)$ of simple (co)roots satisfying: $\langle \alpha_j, \alpha_i^\vee \rangle = a_{ij}$.

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Parabolic analogues: For a subset $J \subseteq I$, we have analogues:

- Parabolic Weyl group W_J generated by $\{s_j : j \in J\}$.
- Define $\pi_J := \{\alpha_j : j \in J\}$.
- Roots $\Delta_J = \Delta_J^+ \sqcup \Delta_J^-$.
- The Levi subalgebra $\mathfrak{l}_J$ is generated by $\{e_j, f_j : j \in J\}$ and $\mathfrak{h}$.

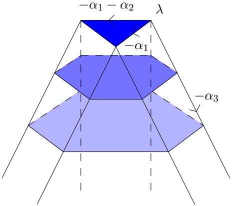
Integrable Slice Decomposition of the weights

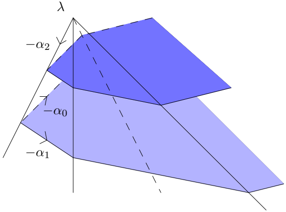
Theorem (K. (2016), Dhillon–K. (2022))

Given $J \subseteq I$ *and* $\nu \in \mathfrak{h}^*$ (P_J^+), *let* $L_J(\nu)$ *denote the simple (integrable)* $\mathfrak{l}_J$*-module with highest weight* ν. *Then*

$$\operatorname{wt} L(\lambda) = \bigsqcup_{\mu \in \mathbb{Z}_{\geqslant 0}(\pi \setminus \pi_{J_\lambda})} \operatorname{wt} L_{J_\lambda}(\lambda - \mu),$$

where $J_\lambda = J_{L(\lambda)}$ *is the* integrability $\{i \in I : \langle \lambda, \alpha_i^\vee \rangle \in \mathbb{Z}_{\geqslant 0}\}$.





$$\mathfrak{g} = \begin{array}{c} \\ \check\alpha_1 \\ \check\alpha_2 \\ \check\alpha_3 \end{array} \begin{array}{c} \begin{array}{ccc} \alpha_1 & \alpha_2 & \alpha_3 \end{array} \\ \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix} \end{array}, \quad J = \{1, 2\}$$

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Example of Integrable Slice Decomposition in rank 2

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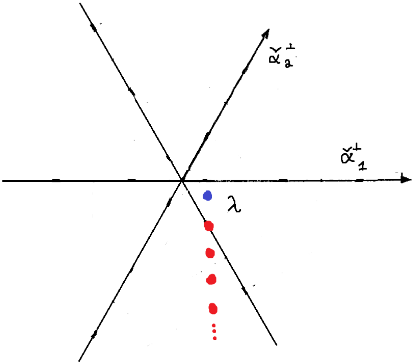

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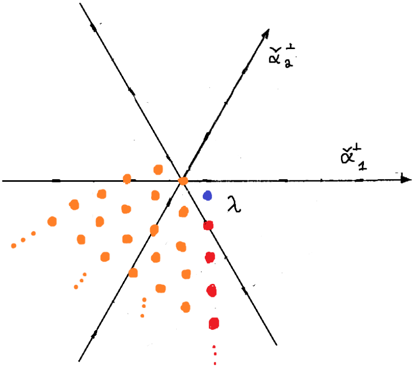

A question of Bump on the weights

Recall, Verma modules and finite-dimensional simple modules have "no holes":

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$$\operatorname{wt} M(\lambda) = (\lambda - \mathbb{Z}\pi) \cap \operatorname{conv}(\operatorname{wt} M(\lambda)).$$

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These formulas follow from weight-formulas for a class of universal "first-order" highest weight modules $M(\lambda, J)$:

Parabolic Verma modules

Key tool in proving the above weight-formulas: *parabolic Verma modules*.

Say $\lambda \in \mathfrak{h}^*$ and $J \subseteq J_\lambda$ (so $\langle \lambda, \alpha_j^\vee \rangle \in \mathbb{Z}_{\geqslant 0}$). Define

$$M(\lambda, J) := \frac{U\mathfrak{g}}{U\mathfrak{g} \cdot (\ker \lambda, \mathfrak{n}^+, \{f_j^{\langle \lambda, \alpha_j^\vee \rangle + 1}\})} = \frac{M(\lambda)}{\sum_{j \in J} U\mathfrak{g} \cdot f_j^{\langle \lambda, \alpha_j^\vee \rangle + 1} M(\lambda)_\lambda}.$$

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  Character: Kostant partition function

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- In first order, if e.g. $J = I$ (so $\lambda \in P^+$): $M(\lambda, I) = L^{\max}(\lambda)$, maximal integrable module (simple if $\mathfrak{g}$ is symmetrizable).
  Character: Weyl–Kac character formula

$$\operatorname{ch} L^{\max}(\lambda) = \sum_{w \in W} \frac{(-1)^{\ell(w)} e^{w \bullet \lambda}}{\prod_{\alpha \in \Delta^+} (1 - e^{-\alpha})^{\dim \mathfrak{g}_\alpha}}, \qquad \forall \lambda \in P^+.$$

Weight-formula 3 for $L(\lambda)$: Minkowski difference

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Because (a) They turn out to hold for all parabolic Verma modules $M(\lambda, J)$,
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$\operatorname{wt} L(\lambda) = \operatorname{wt} M(\lambda, J_\lambda)$, *for all* $\lambda \in \mathfrak{h}^*$ *(and all Kac–Moody* $\mathfrak{g}$*).*

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Now set $J = J_\lambda \rightsquigarrow$ gives third weight-formula for $\mathrm{wt}\, L(\lambda)$.

Theorem (G.V.K. Teja, 2020): "Minimal description" of all $\mathrm{wt}\, M(\lambda, J)$ (and hence for all simple $L(\lambda)$), using *parabolic* partial sum property.

1b. First-order invariant & convex hull, of all modules

A first-order invariant of a highest weight module

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2. $\operatorname{conv}(\operatorname{wt} V)$, *the convex hull of the weights of* V.

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1. I_V, *the* integrability *of* V, *i.e.* $I_V = \{i \in I : f_i$ *acts locally nilpotently (= non-freely) on* $V_\lambda\}$.
2. $\mathrm{conv}(\mathrm{wt}\, V)$, *the convex hull of the weights of* V.
3. *The stabilizer of the character of* V *in* W.

A first-order invariant of a highest weight module

The "discrete" Minkowski difference formula for $\mathrm{wt}\, L(\lambda)$, $\mathrm{wt}\, M(\lambda, J)$
$\rightsquigarrow$ akin to V-decomposition of convex polyhedra (from Motzkin's 1936 thesis)
$\rightsquigarrow$ extends to convex hull formula – for **all** highest weight modules!

This is an application of a "first-order invariant" of highest weight module V:

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Moreover, for special classes of highest weight modules, including simple modules $L(\lambda)$, *these data determine the weights.*

Convex hull of weights

Recall – integrability of V is: $I_V := \{i \in I : f_i$ acts locally nilpotently on $V_\lambda\}$.

Theorem (Dhillon–K., 2022)

For all highest weight modules V over Kac–Moody $\mathfrak{g}$, $\mathrm{conv}(\mathrm{wt}\, V)$ is the Minkowski sum of

- *the hull $\mathrm{conv}\, W_{I_V}(\lambda)$, and*
- *the cone $-\mathbb{R}_{\geqslant 0} W_{I_V}(\pi_{I \setminus I_V})$.*

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Corollary: $\operatorname{conv}(\operatorname{wt} V)$ is always a W_{I_V}-invariant polyhedron.
(Novel even in finite type.)

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Let $\mathfrak{g}$ be a Kac–Moody Lie algebra, $\lambda \in \mathfrak{h}^*$, and V a highest weight $\mathfrak{g}$-module.

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Also: complete determination of the face lattice (for all $\mathfrak{g}, \lambda, V$).

From (exposed) faces to weak faces to 212-closed subsets

Recall the 2009 property studied by Chari with coauthors:

$$y_1 + y_2 = \alpha_1 + \alpha_2 \quad (y_1, y_2 \in Y,\ \alpha_1, \alpha_2 \in \Delta) \quad \implies \quad \alpha_1, \alpha_2 \in Y.$$

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Given a subset $Y \subseteq \mathbf{X} := \operatorname{conv}(\operatorname{wt} V)$, *each statement implies the next:*

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4. Y *satisfies the 2009-property above (via* $\lambda = \frac{1}{2}$*).*
   (We call such Y *as **212-closed subsets** of* $\mathbf{X}$*.)*

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Chari et al: 212-closed subsets of $\mathbf{X} = \Delta$; weak faces of $\mathbf{X} = \operatorname{wt} L(\lambda), \lambda \in P^+$.

Preceding slide: $(1) \iff (2)$ for $\mathbf{X} = \operatorname{conv}(\operatorname{wt} V)$.

From exposed to weak faces to 212-closed subsets (cont.)

How restrictive are (3), (4)? (In general – no "nice" answer.)
What about for "special" subsets $\mathbf{X}$ in representation theory?

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What about for "special" subsets $\mathbf{X}$ in representation theory?

For these subsets – these are **equivalent** to (the weights on) exposed faces!

Theorem (G.V.K. Teja (*Transform. Groups*, in press))

These notions (1)–(4) are equivalent *for* $\mathbf{X} = \mathrm{wt}\, V$ *and* $\mathbf{X} = \mathrm{conv}(\mathrm{wt}\, V)$, *for all highest weight modules over Kac–Moody* $\mathfrak{g}$.

2a. Higher-order theory: holes, higher-order Vermas

Holes in the set of weights

Recall, Verma modules and integrable simple modules have "no holes":

- For all $\lambda \in \mathfrak{h}^*$, $\quad \operatorname{wt} M(\lambda) = (\lambda - \mathbb{Z}\pi) \cap \operatorname{conv}(\operatorname{wt} M(\lambda))$.
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- This in fact extends to all parabolic Verma modules:
  $\operatorname{wt} M(\lambda, J) = (\lambda - \mathbb{Z}\pi) \cap \operatorname{conv}(\operatorname{wt} M(\lambda, J))$,
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Does this hold for *all* highest weight modules V?

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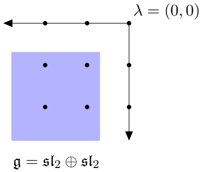
No: Consider $\mathfrak{g} = \mathfrak{sl}_2 \oplus \mathfrak{sl}_2$, and $V' = M(0)/M(s_1 s_2 \bullet 0)$.



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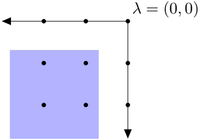

$\mathfrak{g} = \mathfrak{sl}_2 \oplus \mathfrak{sl}_2$

Deleted (blue) portion: example of a *second order hole*.

__**Question**__ (Lepowsky): Is this the only way holes arise?
__*Answer*__ (Dhillon–K., 2022): Yes.

Holes in the set of weights (cont.) + Weight formula 1

In general:

Definition

The *holes* $\mathcal{H}_V$ in a module $M(\lambda) \twoheadrightarrow V$ are all $H \subseteq J_\lambda \subseteq I$ such that

(a) the Dynkin subdiagram on H has no edges, and

(b) $\prod_{h \in H} f_h^{\langle \lambda, \alpha_h^\vee \rangle + 1} \cdot V_\lambda = 0$. (Note: $H \subseteq J_\lambda$.)

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Using (higher-order) holes yields a positive weight-formula <u>for all V</u>:

Theorem (K.–Teja, 2022)

Given a Kac–Moody $\mathfrak{g}$, a weight $\lambda \in \mathfrak{h}^*$, and a nonzero module $M(\lambda) \twoheadrightarrow V$,

$$\operatorname{wt} V = \bigcup_{J \subseteq J_\lambda \,:\, J \cap H \neq \emptyset \ \forall H \in \mathcal{H}_V} \operatorname{wt} M(\lambda, J).$$

Higher-order Verma modules

Definition: Given any weight $\lambda \in \mathfrak{h}^*$, and any subset $\mathcal{H} \subseteq \mathrm{Indep}(J_\lambda)$, define the (universal) **higher-order Verma module**

$$\mathbb{M}(\lambda, \mathcal{H}) := \frac{M(\lambda)}{\sum_{H \in \mathcal{H}} \left(U\mathfrak{g} \cdot \prod_{h \in H} f_h^{\langle \lambda, \alpha_h^\vee \rangle + 1} \right) \cdot M(\lambda)_\lambda}.$$

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- The extremal ones are $M(0)$ itself, and zero:

$$\mathbb{M}(0, \emptyset) = M(0), \qquad \mathbb{M}(0, \{\emptyset\}) = 0.$$

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$$\begin{aligned} \mathbb{M}(0, \{\{i\}\}) &= M(0)/U(\mathfrak{g}) \cdot f_i \cdot M(0)_0 = M(0, \{i\}), \quad && \forall i \in I; \\ \mathbb{M}(0, \{\{i\}, \{j\}\}) &= M(0, \{i, j\}), && \forall i \neq j \in I; \\ \mathbb{M}(0, \{\{1\}, \{2\}, \{3\}\}) &= M(0, I) = L(0). \end{aligned}$$

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- There are two *second-order* Verma modules:

$$\mathbb{M}(0, \{\{1, 3\}\}) = M(0)/U(\mathfrak{g}) \cdot f_1 f_3 \cdot M(0)_0;$$

$$V' = \mathbb{M}(0, \{\{1, 3\}, \{2\}\}).$$

All highest weight modules: Weight formula 2

Recall, simples and first-order Vermas have the same weights:

Theorem (K. (2016), Dhillon–K. (2022))

$\mathrm{wt}\, L(\lambda) = \mathrm{wt}\, M(\lambda, J_\lambda)$, *for all* $\lambda \in \mathfrak{h}^*$ *(and all Kac–Moody* $\mathfrak{g}$*).*

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Such an equality of weights holds in full generality:

Theorem (K.–Teja, 2022)

Fix any Kac–Moody $\mathfrak{g}$*, weight* λ*, and nonzero module* $M(\lambda) \twoheadrightarrow V$*. Then*

$$\mathrm{wt}\, V = \mathrm{wt}\, \mathbb{M}(\lambda, \mathcal{H}_V).$$

Thus, need to better understand higher-order Vermas.

2b. Higher-order Vermas: characters, BGG resolutions

From weights to characters, to resolutions

The higher order Verma modules are crucial in understanding $\mathrm{wt}\, V$ for all modules $M(\lambda) \twoheadrightarrow V$.
We understand their weights (hence, weights of all V).
Can we understand their *characters*?

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Can we understand their *characters*?

- (0th order – usual Vermas) Character = Kostant partition function.
- (1st order – parabolic Vermas) Weyl–Kac character formula
  = Euler characteristic of a BGG-type resolution.

Question: What happens in higher order, i.e. for $\mathbb{M}(\lambda, \mathcal{H})$?

We can answer this for two classes of modules (we explain one below).

BGG resolution: 1. Pairwise orthogonal minimal holes

Example: $\mathfrak{g} = \mathfrak{sl}_n$ and

$$V'' := \frac{M(0)}{U\mathfrak{g}(f_1 f_3, f_5) \cdot M(0)_0} = \frac{U\mathfrak{n}^-}{U\mathfrak{n}^-(f_1 f_3, f_5)}.$$

(Thus $V'' = \mathbb{M}(0, \{\{1,3\},\{5\}\})$.)

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(Thus $V'' = \mathbb{M}(0, \{\{1,3\},\{5\}\})$.) Now check:

$$0 \to M(s_1 s_3 s_5 \bullet 0) \xrightarrow{d_2} M(s_1 s_3 \bullet 0) \oplus M(s_5 \bullet 0) \xrightarrow{d_1} M(0) \xrightarrow{d_0} V'' \to 0,$$

where d_0 is the natural projection, and

- $d_1(X_1 m_{s_1 s_3 \bullet 0} + X_2 m_{s_5 \bullet 0}) := X_1 \cdot f_1 f_3 m_0 + X_2 \cdot f_5 m_0.$
- $d_2(X m_{s_1 s_3 s_5 \bullet 0}) := (-X f_5 \cdot m_{s_1 s_3 \bullet 0}, X f_1 f_3 \cdot m_{s_5 \bullet 0}).$

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This is easily verified, but also – special case of the *Koszul resolution* over $R := \mathbb{C}[f_1 f_3, f_5]$,
subsequently tensored with the free R-module $U(\mathfrak{n}^-) \otimes_R -$.

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where d_0 is the natural projection, and

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- $d_2(X m_{s_1 s_3 s_5 \bullet 0}) := (-X f_5 \cdot m_{s_1 s_3 \bullet 0}, X f_1 f_3 \cdot m_{s_5 \bullet 0})$.

This is easily verified, but also – special case of the *Koszul resolution* over $R := \mathbb{C}[f_1 f_3, f_5]$,
subsequently tensored with the free R-module $U(\mathfrak{n}^-) \otimes_R -$.

- It is also the BGG resolution over

$$W_{\mathcal{H}} = \langle s_{H_1} := s_1 s_3,\ s_{H_2} := s_5 \rangle \simeq (\mathbb{Z}/2\mathbb{Z})^2,$$

with length $\ell_{\mathcal{H}}(s_{H_1}) = \ell_{\mathcal{H}}(s_{H_2}) := 1$.

BGG resolution: 1. Pairwise orthogonal minimal holes

The above example – and proof – is completely general:

Theorem (K.–Teja, 2022)

Fix Kac-Moody $\mathfrak{g}$ *and a weight* λ*. Suppose* $\mathcal{H} \subseteq \mathrm{Indep}(J_\lambda)$ *is such that* $\mathcal{H}^{\min}$ *consists of pairwise orthogonal subsets* $H_1, \ldots, H_k \subseteq J_\lambda$*. Define* $s_H := \prod_{h \in H} s_h$*. Then* $\mathbb{M}(\lambda, \mathcal{H})$ *has a BGG resolution:*

$$0 \longrightarrow M_k \xrightarrow{d_k} M_{k-1} \xrightarrow{d_{k-1}} \cdots \xrightarrow{d_2} M_1 \xrightarrow{d_1} M_0 \xrightarrow{d_0} \mathbb{M}(\lambda, \mathcal{H}) \to 0,$$

with M_p *the direct sum of Vermas* $M(s_{H_{i_1}} \cdots s_{H_{i_p}} \bullet \lambda)$ *over all* p*-tuples of indices* $1 \leqslant i_1 < \cdots < i_p \leqslant k$*.*

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with M_p the direct sum of Vermas $M(s_{H_{i_1}} \cdots s_{H_{i_p}} \bullet \lambda)$ over all p-tuples of indices $1 \leqslant i_1 < \cdots < i_p \leqslant k$. In particular, with $W_{\mathcal{H}} \simeq (\mathbb{Z}/2\mathbb{Z})^k$,

$$\mathrm{ch}\, \mathbb{M}(\lambda, \mathcal{H}) = \sum_{w \in W_{\mathcal{H}}} \frac{(-1)^{\ell_{\mathcal{H}}(w)} e^{w \bullet \lambda}}{\prod_{\alpha \in \Delta^+} (1 - e^{-\alpha})^{\dim \mathfrak{g}_\alpha}}.$$

Resembles the Weyl–Kac character formula:

$$\mathrm{ch}\, M(\lambda, J) = \sum_{w \in W_J} \frac{(-1)^{\ell(w)} e^{w \bullet \lambda}}{\prod_{\alpha \in \Delta^+} (1 - e^{-\alpha})^{\dim \mathfrak{g}_\alpha}}.$$

Disjoint but non-orthogonal holes?

Speculation: Suppose $\mathcal{H}^{\min} = \{H_1, H_2\}$, with the H_i disjoint independent sets but not pairwise orthogonal. Then s_{H_1}, s_{H_2} generate a dihedral subgroup $W_{\mathcal{H}}$ of W. Does this provide a BGG resolution?

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"Simplest case": Consider $\mathfrak{g} = \mathfrak{sl}_4(\mathbb{C})$, $\lambda = 0$, and the module mentioned above:

$$V' := \frac{M(0)}{U\mathfrak{g}(f_1 f_3, f_2) \cdot M(0)_0} = \frac{U\mathfrak{n}^-}{U\mathfrak{n}^-(f_1 f_3, f_2)} = \mathbb{M}(0, \{\{1,3\}, \{2\}\}).$$

Then $s_{H_1} := s_1 s_3$ and $s_{H_2} := s_2$ generate a dihedral subgroup $W_{\mathcal{H}} \leqslant W$ of size 8, say with longest element $\mathbf{w}_\circ$.

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Question 1: Does V' have the following resolution? *(Only unknown case!)*

$$\begin{aligned} 0 \to M(\mathbf{w}_\circ \bullet 0) &\to M(\mathbf{w}_\circ s_{H_1} \bullet 0) \oplus M(\mathbf{w}_\circ s_{H_2} \bullet 0) \\ &\to M(s_{H_1} s_{H_2} \bullet 0) \oplus M(s_{H_2} s_{H_1} \bullet 0) \\ &\to M(s_{H_1} \bullet 0) \oplus M(s_{H_2} \bullet 0) \to M(0) \to V' \to 0 \end{aligned}$$

(Writing down the explicit maps is tedious, but not hard.) More generally, *find a resolution for $\mathbb{M}(\lambda, \mathcal{H})$ using Vermas, when $\mathcal{H}^{\min}$ consists of disjoint subsets.*

Future directions

1. BGG resolutions – or simpler, characters – for higher-order Verma modules $\mathbb{M}(\lambda, \mathcal{H})$?

   (The other setting in which we can provide a resolution involves summing over a Weyl *semigroup* orbit, not Weyl group – see paper for details.)

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2. Define the *higher order BGG category* $\mathcal{O}^{\mathcal{H}}$ –
   full subcategory of $\mathcal{O}$ with objects on which the lowering operators $\mathbf{f}_H := \prod_{h \in H} f_h$ $(H \in \mathcal{H})$ act locally nilpotently.

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5. Analogues over quantum groups $U_q(\mathfrak{g}), U_q(\widehat{\mathfrak{g}})$?

Happy birthday, Vyjayanthi!

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