Schubert calculus on polytopes and semi-toric degenerations of Schubert varieties

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Combinatorics, Geometry, and Representation Theory

November 27, 2025

Schubert calculus for G/B

- $G := GL_{n+1}(\mathbb{C})$,
- $B \subseteq G$: the subgroup of upper triangular matrices,
- $B^- \subseteq G$: the subgroup of lower triangular matrices.

The quotient variety G/B is called the **full flag variety**. Let

$$G/B = \bigsqcup_{w \in W} B\overline{w}B/B = \bigsqcup_{w \in W} B^-\overline{w}B/B$$

be the orbit decompositions of G/B, where $W = S_{n+1}$ is the Weyl group and $\overline{w} \in G$ denotes the permutation matrix.

Definition

For $w \in W$, the **Schubert variety** X_w and the **opposite Schubert variety** X^w are defined by

$$X_w := \overline{B\overline{w}B/B} \subseteq G/B, \qquad X^w := \overline{B^-\overline{w}B/B} \subseteq G/B.$$

- $X^w \simeq X_{w_0 w}$ for some $w_0 \in W$ called the longest element,
- $[X^w] = [X_{w_0 w}]$ in $H^*(G/B; \mathbb{Z})$, called a **Schubert class**.

Schubert calculus for G/B

Properties

- $H^*(G/B;\mathbb{Z}) = \sum_{w\in W} \mathbb{Z}[X_w] = \sum_{w\in W} \mathbb{Z}[X^w]$,
- $[X^u]\cdot[X^v] = \sum_{w\in W} c^w_{u,v}[X^w]$ for some $c^w_{u,v} \in \mathbb{Z}_{\geq 0}$.

Aim (of Schubert calculus)

to compute $c^w_{u,v}$ explicitly.

Approach

to realize $[X^w]$ as a concrete combinatorial object such as Schubert polynomials.

Slogan in this talk

Semi-toric degenerations of X^w can be expected as combinatorial models of $[X^w]$.

Reduced pipe dreams

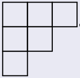
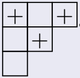
Consider the Schubert polynomial $\mathfrak{S}_w \in \mathbb{Z}[x_1, \ldots, x_{n+1}]$ for $w \in W = S_{n+1}$, introduced by Lascoux–Schützenberger (1982). Its explicit formula is given by **reduced pipe dreams**.

- $Y_n \coloneqq \{(i,j) \in \mathbb{Z}^2 \mid 1 \leq i \leq n,\ 1 \leq j \leq n-i+1\}$,
- $\mathcal{PD}_n$: the power set of Y_n.

Definition (Knutson–Miller 2005)

An element of $\mathcal{PD}_n$ is called a **pipe dream**.

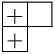
Example

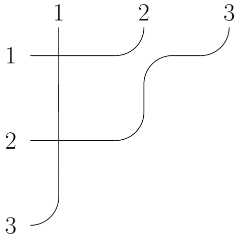
If $n = 3$, then $Y_n = Y_3 =$ . In addition, we represent a pipe dream $D = \{(1,1),(1,3),(2,2)\} \in \mathcal{PD}_3$ as $D =$ .

Reduced pipe dreams

We describe a pipe dream $D \in \mathcal{PD}_n$ as a diagram of $(n+1)$ pipes (see the example below).

- A pipe dream $D \in \mathcal{PD}_n$ is **reduced** if arbitrary two pipes do not cross more than once.
- $RP(w)$: the set of reduced pipe dreams $D \in \mathcal{PD}_n$ whose diagram gives a permutation $w \in S_{n+1}$.

If $n = 2$ and $w = (1\ 2\ 3) \in S_3$, then $D =$  $\in RP(w)$ corresponds to the following diagram of 3 pipes.



Combinatorial properties of reduced pipe dreams

- $s_i := (i\ i+1) \in S_{n+1} \quad (i \in I := \{1, 2, \ldots, n\})$,
- a sequence $\boldsymbol{i} := (i_1, i_2, \ldots, i_m) \in I^m$ is a **reduced word** for $w \in S_{n+1}$ if $w = s_{i_1} s_{i_2} \cdots s_{i_m}$ and if m is the minimum among such expressions of w,
- $R(w)$: the set of reduced words for $w \in W = S_{n+1}$.

The set $RP(w)$ of reduced pipe dreams has the following two kinds of remarkable combinatorial properties.

- **(reduced subwords)** There exists a natural bijection between $RP(w)$ and the set of reduced words for w appearing as subwords of

$$(n, n-1, n, n-2, n-1, n, \ldots, 1, 2, \ldots, n) \in R(w_0).$$

- **(mitosis recursion)** $RP(w)$ is obtained from $RP(w_0) = \{Y_n\}$ by a sequence of transposed mitosis operators:

$$RP(w) = \mathrm{mitosis}_{j_\ell}^\top \cdots \mathrm{mitosis}_{j_1}^\top (Y_n)$$

for $(j_1, \ldots, j_\ell) \in R(w_0 w^{-1})$ (Knutson–Miller 2005, Miller 2003).

Pipe dream formula for Schubert polynomials

Theorem (Billey–Jockusch–Stanley 1993 and Fomin–Stanley 1994)

For $w \in W = S_{n+1}$, the following equality holds:

$$\mathfrak{S}_w = \sum_{D \in RP(w)} \boldsymbol{x}^D,$$

where $\boldsymbol{x}^D \coloneqq \prod_{(i,j) \in D} x_i$.

Knutson–Miller (2005) gave a geometric proof of this formula using a semi-toric degeneration of the (matrix) opposite Schubert variety $\overline{X}^{w_0^{-1} w w_0}$. As a quotient of this degeneration, Kogan–Miller (2005) constructed a semi-toric degeneration of $X^{w_0^{-1} w w_0}$.

Gelfand–Tsetlin polytopes

For $\lambda = (\lambda_1, \ldots, \lambda_n) \in P_{++} := \mathbb{Z}^n_{>0}$, the **Gelfand–Tsetlin polytope** $GT(\lambda)$ is defined to be the set of $(a_1^{(1)}, a_1^{(2)}, a_2^{(1)}, a_1^{(3)}, a_2^{(2)}, a_3^{(1)}, \ldots, a_1^{(n)}, \ldots, a_n^{(1)}) \in \mathbb{R}^N$ satisfying the following inequalities:

$$\begin{array}{cccccccccc}
a_1^{(0)} & & a_2^{(0)} & & \cdots & & & a_n^{(0)} & & a_{n+1}^{(0)} \\
& a_1^{(1)} & & a_2^{(1)} & & \cdots & & & a_n^{(1)} & \\
& & a_1^{(2)} & & \cdots & & & a_{n-1}^{(2)} & & \\
& & & \ddots & & \cdots & & & & \\
& & & & a_1^{(n-1)} & & a_2^{(n-1)} & & & \\
& & & & & a_1^{(n)}, & & & &
\end{array}$$

where $N := \frac{n(n+1)}{2}$, $a_k^{(0)} := \sum_{k \leq \ell \leq n} \lambda_\ell$ for $1 \leq k \leq n+1$, and we mean by $\begin{array}{ccc} a & & c \\ & b & \end{array}$ that $a \geq b \geq c$.

Gelfand–Tsetlin polytopes

- $GT(\lambda)$ is an N-dimensional integral convex polytope for all $\lambda \in P_{++}$.
- The set $GT(\lambda) \cap \mathbb{Z}^N$ of lattice points gives a natural parametrization of a specific basis of an irreducible G-module, called the Gelfand–Tsetlin basis.
- (**toric degeneration**) Gonciulea–Lakshmibai (1996) constructed a flat degeneration of G/B to the toric variety $Z(GT(\lambda))$ corresponding to $GT(\lambda)$.
- (**Minkowski decomposition property**)

$$GT(\lambda + \mu) = GT(\lambda) + GT(\mu)$$

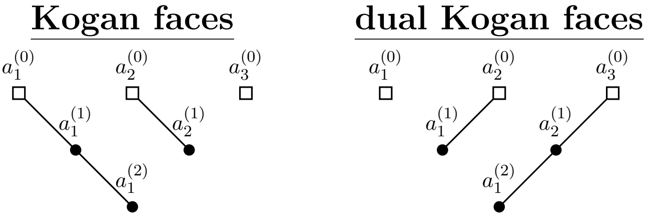
  for all λ, μ, which implies that $GT(\lambda)$, $\lambda \in P_{++}$, have the same normal fan. In particular, the combinatorial structure of faces of $GT(\lambda)$ is independent of the choice of λ.

Kogan faces and dual Kogan faces

Definition (Kogan 2000)

A face of $GT(\lambda)$ given by equations of the type $a_k^{(l)} = a_k^{(l+1)}$ (resp., $a_k^{(l)} = a_{k+1}^{(l-1)}$) is called a **Kogan face** (resp., a **dual Kogan face**).

We associate to each $D \in \mathcal{PD}_n$ a Kogan face $F_D(GT(\lambda))$ and a dual Kogan face $F_D^\vee(GT(\lambda))$.



Kogan faces and dual Kogan faces

A Kogan face $F_D(GT(\lambda))$ and a dual Kogan face $F_D^\vee(GT(\lambda))$ are said to be **reduced** if D is reduced. The reduced Kogan faces and reduced dual Kogan faces inherit information on the Schubert class $[X^w] = [X_{w_0 w}]$ in several ways:

- through the pipe dream formula of Schubert polynomials,
- through the theory of Gelfand–Tsetlin integrable systems (Kogan 2000),
- through the theory of polytope rings (Kiritchenko–Smirnov–Timorin 2012).

Relation with semi-toric degenerations

Theorem (Kogan–Miller 2005)

For $\lambda \in P_{++}$, Gonciulea–Lakshmibai's toric degeneration of G/B induces a degeneration of X^w to a reduced union of toric subvarieties of $Z(GT(\lambda))$, called a **semi-toric degeneration** of X^w. In addition, the semi-toric limit of X^w corresponds to

$$GT(\lambda, X^w) := \bigcup_{D \in RP(w_0^{-1} w w_0)} F_D^\vee(GT(\lambda)) \subseteq GT(\lambda).$$

Kogan–Miller's semi-toric degeneration

Example

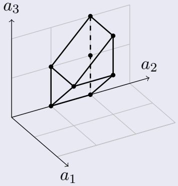
Let $n = 2$. Then $G/B = GL_3(\mathbb{C})/B$ is isomorphic to a hypersurface

$$\{\boldsymbol{p} = ([p_1, p_2, p_3], [p_{1,2}, p_{1,3}, p_{2,3}]) \in \mathbb{P}^2 \times \mathbb{P}^2 \mid p_1 p_{2,3} = p_2 p_{1,3} - p_3 p_{1,2}\}$$

of $\mathbb{P}^2 \times \mathbb{P}^2$. Define a hypersurface $\mathfrak{X}$ of $\mathbb{P}^2 \times \mathbb{P}^2 \times \mathbb{C}$ by

$$\mathfrak{X} = \{(\boldsymbol{p}, t) \in \mathbb{P}^2 \times \mathbb{P}^2 \times \mathbb{C} \mid p_1 p_{2,3} = p_2 p_{1,3} - t p_3 p_{1,2}\},$$

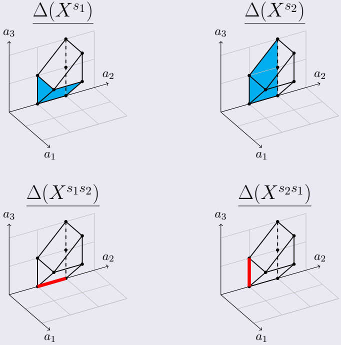
and set $\pi \colon \mathfrak{X} \to \mathbb{C}$, $(\boldsymbol{p}, t) \mapsto t$. Then π gives a toric degeneration of G/B. In particular, the toric variety $\pi^{-1}(0)$ corresponds to the following polytope $\Delta = GT((1, 1))$:



Kogan–Miller's semi-toric degeneration

Example (continued)

The induced semi-toric degenerations of X^w are given as follows. In particular, $\Delta(X^{s_1}) \cap \Delta(X^{s_2}) = \Delta(X^{s_1 s_2}) \cup \Delta(X^{s_2 s_1})$, where $\Delta(X^w) \coloneqq GT((1,1), X^w)$. This relation corresponds to $[X^{s_1}] \cdot [X^{s_2}] = [X^{s_1 s_2}] + [X^{s_2 s_1}]$.



Borel–Weil theory

Aim

to give a generalization of Kogan–Miller's semi-toric degenerations using the theory of crystal bases and cluster algebras.

For $\lambda \in P_{++}$, the irreducible highest weight G-module $V(\lambda)$ with highest weight λ can be realized as $V(\lambda) = H^0(G/B, \mathcal{L}_\lambda)^*$ for some ample line bundle $\mathcal{L}_\lambda$ by the Borel–Weil theorem.
Set

$$R_\lambda \coloneqq H^0(G/B, \mathcal{L}_\lambda) = V(\lambda)^*$$

and

$$R^{(\lambda)} \coloneqq \bigoplus_{k \in \mathbb{Z}_{\geq 0}} R_{k\lambda}.$$

Then we have

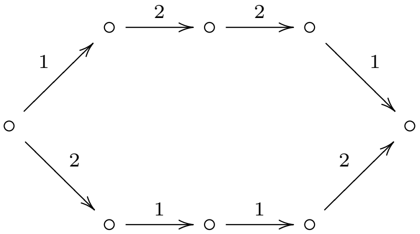
$$G/B = \mathrm{Proj}(R^{(\lambda)}).$$

Crystal bases

Let $\mathbf{B}^{\mathrm{up}}(\lambda) = \{G_\lambda^{\mathrm{up}}(b) \mid b \in \mathcal{B}(\lambda)\}$ denote Lusztig's dual canonical basis (= Kashiwara's upper global basis) of $H^0(G/B, \mathcal{L}_\lambda)$. The index set $\mathcal{B}(\lambda)$ equipped with the Kashiwara operators $\{\tilde{e}_i \mid i \in I\} \cup \{\tilde{f}_i \mid i \in I\}$ is called a **crystal basis**. Define an I-colored directed graph structure on $\mathcal{B}(\lambda)$ by

$$b \xrightarrow{i} b' \text{ if and only if } b' = \tilde{f}_i b.$$

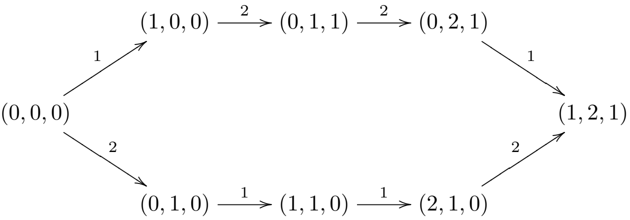
If $G = GL_3(\mathbb{C})$ and $\lambda = (1, 1) \in P_{++}$, then $\mathcal{B}(\lambda)$ is given by



String parametrizations

For $b \in \mathcal{B}(\lambda)$ and $\boldsymbol{i} = (i_1, \ldots, i_N) \in R(w_0)$, define $\Phi_{\boldsymbol{i}}(b) = (a_1, \ldots, a_N) \in \mathbb{Z}_{\geq 0}^N$, called the **string parametrization**, by

$$
\begin{aligned}
a_1 &\coloneqq \max\{a \in \mathbb{Z}_{\geq 0} \mid \tilde{e}_{i_1}^{a} b \neq 0\}, \\
a_2 &\coloneqq \max\{a \in \mathbb{Z}_{\geq 0} \mid \tilde{e}_{i_2}^{a} \tilde{e}_{i_1}^{a_1} b \neq 0\}, \\
&\ \vdots \\
a_N &\coloneqq \max\{a \in \mathbb{Z}_{\geq 0} \mid \tilde{e}_{i_N}^{a} \tilde{e}_{i_{N-1}}^{a_{N-1}} \cdots \tilde{e}_{i_1}^{a_1} b \neq 0\}.
\end{aligned}
$$



String polytopes

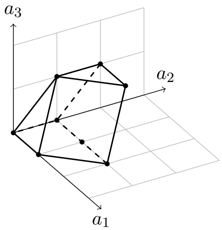
We set

$$\mathcal{S}_{\boldsymbol{i}}(\lambda) := \bigcup_{k \in \mathbb{Z}_{>0}} \{(k, \Phi_{\boldsymbol{i}}(b)) \mid b \in \mathcal{B}(k\lambda)\} \subseteq \mathbb{Z}_{>0} \times \mathbb{Z}^N,$$

$$\mathcal{C}_{\boldsymbol{i}}(\lambda)\text{: the smallest real closed cone containing } \mathcal{S}_{\boldsymbol{i}}(\lambda),$$

$$\Delta_{\boldsymbol{i}}(\lambda) := \{\boldsymbol{a} \in \mathbb{R}^N \mid (1, \boldsymbol{a}) \in \mathcal{C}_{\boldsymbol{i}}(\lambda)\}.$$

The set $\Delta_{\boldsymbol{i}}(\lambda)$ is called **Berenstein–Littelmann–Zelevinsky's string polytope**. If $G = GL_3(\mathbb{C})$, $\lambda = (1,1) \in P_{++}$, and $\boldsymbol{i} = (1,2,1)$, then the string polytope $\Delta_{\boldsymbol{i}}(\lambda)$ is given by



Caldero's toric degenerations

Fix $\lambda \in P_{++}$, and recall $R^{(\lambda)} := \bigoplus_{k \in \mathbb{Z}_{\geq 0}} R_{k\lambda}$.

Theorem (Caldero 2002)

For $b_1 \in \mathcal{B}(k\lambda)$ and $b_2 \in \mathcal{B}(k'\lambda)$, there exists $b \in \mathcal{B}((k+k')\lambda)$ such that $\Phi_{\boldsymbol{i}}(b) = \Phi_{\boldsymbol{i}}(b_1) + \Phi_{\boldsymbol{i}}(b_2)$ and such that

$$G^{\mathrm{up}}_{k\lambda}(b_1) \cdot G^{\mathrm{up}}_{k'\lambda}(b_2) \in \mathbb{C}^{\times} G^{\mathrm{up}}_{(k+k')\lambda}(b) + \sum_{\substack{b' \in \mathcal{B}((k+k')\lambda); \\ \Phi_{\boldsymbol{i}}(b') < \Phi_{\boldsymbol{i}}(b)}} \mathbb{C} G^{\mathrm{up}}_{(k+k')\lambda}(b'),$$

where we consider the lexicographic order on $\{\Phi_{\boldsymbol{i}}(b') \mid b' \in \mathcal{B}((k+k')\lambda)\}$.

Considering $\mathrm{gr}(R^{(\lambda)})$, we can degenerate the relation above as follows:

$$\overline{G^{\mathrm{up}}_{k\lambda}(b_1)} \cdot \overline{G^{\mathrm{up}}_{k'\lambda}(b_2)} \in \mathbb{C}^{\times} \overline{G^{\mathrm{up}}_{(k+k')\lambda}(b)},$$

which gives the toric variety $Z(\Delta_{\boldsymbol{i}}(\lambda))$ corresponding to $\Delta_{\boldsymbol{i}}(\lambda)$. Through the theory of Rees algebras, we have a toric degeneration of $G/B = \mathrm{Proj}(R^{(\lambda)})$ to $Z(\Delta_{\boldsymbol{i}}(\lambda)) = \mathrm{Proj}(\mathrm{gr}(R^{(\lambda)}))$.

Realization as Newton–Okounkov polytopes

The filtration on $R^{(\lambda)}$ is induced from the parametrization

$$\mathbf{B}^{\mathrm{up}}(k\lambda) \ni G^{\mathrm{up}}_{k\lambda}(b) \mapsto (k, \Phi_{\boldsymbol{i}}(b)) \in \mathbb{Z}_{\geq 0} \times \mathbb{Z}^N.$$

Theorem (Kaveh 2015)

This parametrization is extended to a valuation $\nu_{\boldsymbol{i}} \colon R^{(\lambda)} \setminus \{0\} \to \mathbb{Z}_{\geq 0} \times \mathbb{Z}^N$ with respect to some total order $\prec$ on $\mathbb{Z}_{\geq 0} \times \mathbb{Z}^N$, and

$$\Delta_{\boldsymbol{i}}(\lambda) = \overline{\bigcup_{k \in \mathbb{Z}_{>0}} \frac{1}{k} \nu_{\boldsymbol{i}}(R_{k\lambda} \setminus \{0\})}.$$

- The right hand side is the **Newton–Okounkov body (polytope)** of $G/B = \mathrm{Proj}(R^{(\lambda)})$ associated with the valuation $\nu_{\boldsymbol{i}}$ (Lazarsfeld–Mustata 2009, Kaveh–Khovanskii 2012).
- If we consider a different kind of valuation ν, then we obtain a different Newton–Okounkov body of $G/B = \mathrm{Proj}(R^{(\lambda)})$, which induces a toric degeneration under some conditions (Anderson 2013).

Induced semi-toric degenerations

A map $\nu\colon R^{(\lambda)} \setminus \{0\} \to \mathbb{Z}_{\geq 0} \times \mathbb{Z}^N$ is a **valuation** on $R^{(\lambda)}$ if for $\sigma, \tau \in R^{(\lambda)} \setminus \{0\}$ and $c \in \mathbb{C}^\times$,

(i) $\nu(\sigma + \tau) \succeq \min\{\nu(\sigma), \nu(\tau)\}$ unless $\sigma + \tau = 0$,

(ii) $\nu(c \cdot \sigma) = \nu(\sigma)$,

(iii) $\nu(\sigma \cdot \tau) = \nu(\sigma) + \nu(\tau)$.

Replacing the condition (iii) by

(iv) $\nu(\sigma \cdot \tau) \succeq \nu(\sigma) + \nu(\tau)$,

we reach the definition of **quasi-valuations**.

Question (see Chirivì–Fang–Littelmann 2023)

Can we develop the Newton–Okounkov theory using quasi-valuations instead of valuations?
$\rightsquigarrow$ Yes, but the resulting Newton–Okounkov set is not necessarily convex.

Induced semi-toric degenerations

Let $R_{w,\lambda} \coloneqq H^0(X_w, \mathcal{L}_\lambda)$, and set $R_w^{(\lambda)} \coloneqq \bigoplus_{k \in \mathbb{Z}_{\geq 0}} R_{w,k\lambda}$. Then we have $X_w = \mathrm{Proj}(R_w^{(\lambda)})$ and the restriction map $R^{(\lambda)} \twoheadrightarrow R_w^{(\lambda)}$. Through $R^{(\lambda)} \twoheadrightarrow R_w^{(\lambda)}$, Kaveh's valuation $\nu_{\boldsymbol{i}}$ on $R^{(\lambda)}$ induces a quasi-valuation $\nu_{\boldsymbol{i}, X_w} : R_w^{(\lambda)} \setminus \{0\} \to \mathbb{Z}_{\geq 0} \times \mathbb{Z}^N$.

Definition

The **Newton–Okounkov set** $\Delta(X_w, \mathcal{L}_\lambda, \nu_{\boldsymbol{i}, X_w})$ of $X_w = \mathrm{Proj}(R_w^{(\lambda)})$ associated with $\nu_{\boldsymbol{i}, X_w}$ is defined by

$$\Delta(X_w, \mathcal{L}_\lambda, \nu_{\boldsymbol{i}, X_w}) = \overline{\bigcup_{k \in \mathbb{Z}_{>0}} \frac{1}{k} \nu_{\boldsymbol{i}, X_w}(R_{w,k\lambda} \setminus \{0\})}.$$

Induced semi-toric degenerations

Write $\Delta_{\boldsymbol{i}}(\lambda, X_w) := \Delta(X_w, \mathcal{L}_\lambda, \nu_{\boldsymbol{i}, X_w})$

Theorem (see Morier-Genoud 2008)

Let $\boldsymbol{i} \in R(w_0)$, $\lambda \in P_{++}$, and $w \in W$.

(1) The Newton–Okounkov set $\Delta_{\boldsymbol{i}}(\lambda, X_w)$ is a union of faces of $\Delta_{\boldsymbol{i}}(\lambda)$.

(2) $\Delta_{\boldsymbol{i}}(\lambda, X_w) \cap \mathbb{Z}^N = \Phi_{\boldsymbol{i}}(\mathcal{B}_w(\lambda))$ for the Demazure crystal $\mathcal{B}_w(\lambda) \subseteq \mathcal{B}(\lambda)$.

(3) Caldero's toric degeneration of G/B induces a semi-toric degeneration of X_w. In addition, the semi-toric limit of X_w corresponds to $\Delta_{\boldsymbol{i}}(\lambda, X_w)$.

(4) Similar statements hold for X^w.

Induced semi-toric degenerations

For $\boldsymbol{i} = (i_1, \ldots, i_N) \in R(w_0)$ and $w \in W$, we set

$$R(\boldsymbol{i}, w) \coloneqq \{(k_1, \ldots, k_\ell) \mid 1 \leq k_1 < \cdots < k_\ell \leq N,\ (i_{k_1}, \ldots, i_{k_\ell}) \in R(w)\}.$$

Example

Let $G = GL_4(\mathbb{C})$, $\boldsymbol{i} \coloneqq (2, 1, 3, 2, 3, 1) \in R(w_0)$, and $w \coloneqq s_1 s_3 = s_3 s_1$.
Then we have $R(\boldsymbol{i}, w) = \{(2, 3), (2, 5), (3, 6), (5, 6)\}$.

Theorem (F. 2022)

For $\boldsymbol{i} \in R(w_0)$, $w \in W$, and $\lambda \in P_{++}$, the maximal faces of Morier-Genoud's semi-toric limit $\Delta_{\boldsymbol{i}}(\lambda, X^w) \subseteq \Delta_{\boldsymbol{i}}(\lambda)$ are naturally parametrized by $R(\boldsymbol{i}, w)$.

Case of Gelfand–Tsetlin polytopes

Let

$$\boldsymbol{i}_A \coloneqq (1, \underbrace{2, 1}_{2}, \underbrace{3, 2, 1}_{3}, \ldots, \underbrace{n, n-1, \ldots, 1}_{n}) \in R(w_0).$$

Theorem (Littelmann 1998)

For $\lambda \in P_{++}$, the string polytope $\Delta_{\boldsymbol{i}_A}(\lambda)$ is unimodularly equivalent to the Gelfand–Tsetlin polytope $GT(\lambda)$.

Theorem (F. 2022)

For $w \in W$ and $\lambda \in P_{++}$, Morier-Genoud's semi-toric limit $\Delta_{\boldsymbol{i}_A}(\lambda, X^w)$ of X^w in the string polytope $\Delta_{\boldsymbol{i}_A}(\lambda)$ corresponds to Kogan–Miller's semi-toric limit $GT(\lambda, X^w)$ in $GT(\lambda)$ under the unimodular affine transformation $\Delta_{\boldsymbol{i}_A}(\lambda) \simeq GT(\lambda)$.

Case of Gelfand–Tsetlin polytopes

Claim (F. 2022)

Under the unimodular affine transformation $\Delta_{\boldsymbol{i}_A}(\lambda) \simeq GT(\lambda)$, the (transposed) mitosis operator $\mathrm{mitosis}_j^\top$ for reduced Kogan faces $F_D(GT(\lambda))$ naturally corresponds to iterated actions $\bigcup_{k \in \mathbb{Z}_{\geq 0}} (\tilde{f}_j^*)^k$ of a certain twisted Kashiwara operator $\tilde{f}_j^*$, where $\tilde{f}_j^*$ is an operator on the set of lattice points of the string cone

$$\mathcal{C}_{\boldsymbol{i}_A} = \bigcup_{\lambda \in P_{++}} \Delta_{\boldsymbol{i}_A}(\lambda).$$

Theorem (F. 2022)

Let $w \in W$ and $\lambda \in P_{++}$. Under the unimodular affine transformation $\Delta_{\boldsymbol{i}_A}(\lambda) \simeq GT(\lambda)$, Morier-Genoud's semi-toric limit $\Delta_{\boldsymbol{i}_A}(\lambda, X_w) \subseteq \Delta_{\boldsymbol{i}_A}(\lambda)$ of X_w corresponds to $\bigcup_{D \in RP(w^{-1}w_0)} F_D(GT(\lambda))$.

Generalization to type C case

Let $G = Sp_{2n}(\mathbb{C})$ of type C_n. Then the Weyl group W_{C_n} is regarded as the group

$$\{\sigma \in S_{\{-n,\ldots,-1,1,\ldots,n\}} \mid \sigma(-i) = -\sigma(i),\ 1 \leq i \leq n\}$$

of signed permutations. Consider a reduced word

$$\boldsymbol{i}_C \coloneqq (1, \underbrace{2, 1, 2}_{3}, \ldots, \underbrace{n, n-1, \ldots, 1, \ldots, n-1, n}_{2n-1})$$

for $w_0 \in W_{C_n}$.

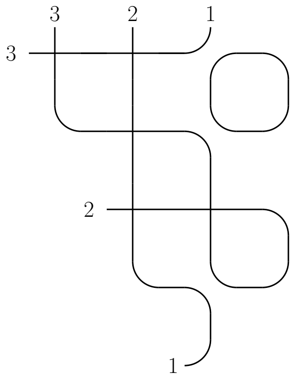
Theorem (F.–Nishiyama 2024)

Let $\lambda \in P_{++}$, and $w \in W_{C_n}$. Then the maximal faces of Morier-Genoud's semi-toric limit $\Delta_{\boldsymbol{i}_C}(\lambda, X_w)$ of X_w bijectively correspond to **reduced skew pipe dreams** for w, which can be constructed using (transposed) mitosis operators for skew pipe dreams introduced by Kiritchenko (2016).

Generalization to type C case

Let $n = 3$. Then the following diagram is an example of a reduced skew pipe dream for

$$w = \begin{pmatrix} -3 & -2 & -1 & 1 & 2 & 3 \\ -2 & 1 & 3 & -3 & -1 & 2 \end{pmatrix} \in W_{C_3}.$$



Generalization to type C case

- Billey–Haimann (1995) and Fomin–Kirillov (1996) introduced several kinds of Schubert polynomials for types B, C, D.
- Type C pipe dreams were already considered in this context by Kirillov–Naruse (2017).
- Our reduced skew pipe dreams have different combinatorics from such type C pipe dreams. The former inherits information on a semi-toric degeneration of X_w while the latter corresponds to a semi-toric degeneration of X^w.

Toric degenerations arising from cluster structures

Let $\lambda \in P_{++}$. Using a cluster variety structure of (a unipotent cell in) G/B, we obtain a family of toric degenerations of G/B.

Theorem (Gross–Hacking–Keel–Kontsevich 2018, F.–Oya 2025)

(1) There exists a family of toric degenerations of G/B to toric varieties $Z(\Delta_{\mathbf{s}}(\lambda))$ associated with g-vector polytopes $\Delta_{\mathbf{s}}(\lambda)$, which are parametrized by seeds $\mathbf{s}$ for the cluster structure, and the lattice points in $\Delta_{\mathbf{s}}(\lambda)$ parametrize some extended g-vectors in cluster theory.

(2) The g-vector polytopes $\Delta_{\mathbf{s}}(\lambda)$ are realized as Newton–Okounkov polytopes of G/B.

Toric degenerations arising from cluster structures

- The Newton–Okounkov body $\Delta_{\mathbf{s}}(\lambda)$ and the associated toric degeneration of G/B can be naturally extended to flag varieties in general Lie type.
- In the case $G = GL_{n+1}(\mathbb{C})$, the polytope $\Delta_{\mathbf{s}}(\lambda)$ coincides with Gross–Hacking–Keel–Kontsevich's superpotential polytope.

Theorem (F.–Oya 2025)

For each $\boldsymbol{i} \in R(w_0)$, there exists a seed $\mathbf{s}_{\boldsymbol{i}}$ such that $\Delta_{\mathbf{s}_{\boldsymbol{i}}}(\lambda)$ is unimodularly equivalent to the string polytope $\Delta_{\boldsymbol{i}}(\lambda)$. In addition, the toric degeneration to $Z(\Delta_{\mathbf{s}_{\boldsymbol{i}}}(\lambda))$ coincides with Caldero's toric degeneration to $Z(\Delta_{\boldsymbol{i}}(\lambda))$ up to $\Delta_{\mathbf{s}_{\boldsymbol{i}}}(\lambda) \simeq \Delta_{\boldsymbol{i}}(\lambda)$.

Semi-toric degenerations arising from cluster structures

Theorem (F. preprint 2021)

Let $\lambda \in P_{++}$, and $\mathbf{s}$ a seed for the cluster structure. Then the toric degeneration of G/B to $Z(\Delta_{\mathbf{s}}(\lambda))$ induces semi-toric degenerations of X_w and X^w, which generalize Morier-Genoud's semi-toric degenerations.

$$
\begin{aligned}
&\text{(Kogan–Miller's semi-toric degeneration)} \\
&\in \{\text{Morier-Genoud's semi-toric degenerations}\} \\
&\subseteq \{\text{semi-toric degenerations arising from cluster structures}\}.
\end{aligned}
$$

Problem

Describe the limits of X_w and X^w explicitly for semi-toric degenerations arising from cluster structures.