The sphere eversion project

PATRICK MASSOT

Université Paris-Saclay at Orsay
with Oliver Nash and Floris van Doorn

April 30, 2025

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- Proofs mix up bureaucracy and actual content.

Note: competing against non-proofs is very difficult.

Sphere immersions

$\mathbb{S}^2 = \{x \in \mathbb{R}^3 \mid \|x\|^2 = 1\}$
Tangent plane at $x \in \mathbb{S}^2$: $T_x\mathbb{S}^2 = \{v \in \mathbb{R}^3 \mid v \perp x\}$.

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Definition

An immersion of $\mathbb{S}^2$ into $\mathbb{R}^3$ is a smooth map j defined near $\mathbb{S}^2$ such that, for every $x \in \mathbb{S}^2$, $Dj(x)|_{T_x\mathbb{S}^2}$ is injective.

Smale's sphere eversion theorem

Theorem

There is a smooth map j defined near $[0,1] \times \mathbb{S}^2$ to $\mathbb{R}^3$ such that

- *each $j_t = j(t, \cdot)$ is an immersion of $\mathbb{S}^2$*
- *j_0 is the inclusion.*
- *j_1 is the antipodal map $x \mapsto -x$.*

How to turn any map into an immersion in two easy steps

Given $f_0 : \mathbb{S}^2 \to \mathbb{R}^3$, $K \subseteq \mathbb{S}^2$ compact, $U \supseteq K$ open, and (e_1, e_2) local basis of $T\mathbb{S}^2$ on U.

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- build f_2 such that

$$\forall x \text{ near } K, \begin{cases} Df_2(x)e_1 \text{ very close to } Df_1(x)e_1 \\ Df_2(x)e_2 \in \mathbb{R}^3 \setminus \operatorname{Span}(Df_1(x)e_1) \end{cases}$$

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Observe f_2 is an immersion near K. Make sure all modification have support in U. Use cover of $\mathbb{S}^2$ by such $K \subseteq U$.

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We improve Df one direction at a time. Improvement has shape $Df(x)e_i \in \Omega_i(x)$, with $\Omega_i(x)$ open.

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Would need C^0-small modifications in a more global setup

Every step need to almost preserve derivative in direction already improved.

Convex integration by Theillière

$K \subseteq U \ \subseteq \mathbb{R}^n$. $f_0 : U \to \mathbb{R}^d$. $i \in \{1, ...d\}$.

$\gamma : U \times \mathbb{S}^1 \to \mathbb{R}^d$ familly of loops in $\mathbb{R}^d$ indexed by U.

Each $\gamma_x := \gamma(x, \cdot)$ takes value in $\Omega_i(x)$ with average $\bar{\gamma}_x = \partial_i f_0(x)$.

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$$f_1(x) := f_0(x) + \frac{1}{N} \int_0^{Nx_i} (\gamma_x(s) - \bar{\gamma}_x)\, ds$$

- $f_1(x) = f_0(x) + O(\frac{1}{N})$
- $\forall j \neq i, \quad \partial_j f_1(x) = \partial_j f_0(x) + O(\frac{1}{N})$
- $\partial_i f_1(x) = \gamma(Nx_i) + O(\frac{1}{N})$

Is there hope to find such γ?

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No hope unless $\partial_i f_0(x) \in \text{ConvexHull}(\Omega_i(x))$

In sphere immersion case:

- $\Omega_1(x) = \mathbb{R}^3 \smallsetminus \{0\}$
- $\Omega_2(x) = \mathbb{R}^3 \smallsetminus \mathrm{Span}(Df_1(x)e_1)$

Both are connected open sets whose convex hull is everything.

Generalized sets

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This is bureaucracy. We can't afford that in a proof assistant.

We need more subsets

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Given any set X, there is an injective map

$$P : \mathcal{P}(X) \hookrightarrow \mathcal{F}(X)$$

from subsets of X into "generalized subsets of X".

Examples:

$\mathcal{N}_{+\infty} \in \mathcal{F}(\mathbb{N})$ the generalized subset of very large natural numbers

$\mathcal{N}_{x_0} \in \mathcal{F}(\mathbb{R})$ the generalized subset of points that are close to x_0

Generalized quantifiers

X a set, Q a predicate on X, $F \in \mathcal{F}\, X$ get statements

- $\forall^f x \in F,\ Q(x)$
- $\exists^f x \in F,\ Q(x)$

Get back usual notation when $F = P(A)$ for some $A \in \mathcal{P}(X)$.

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Fact:
$[(\forall^f x \in F, Q(x)) \wedge (\forall^f x \in F, Q'(x))] \Rightarrow \forall^f x \in F, Q(x) \wedge Q'(x)$
To be proven *only once*.

Order, intersections and unions

$\mathcal{F}(X)$ is ordered and $P : \mathcal{P}(X) \to \mathcal{F}(X)$ is order preserving

$\mathcal{F}(X)$ is a complete lattice: there is a notion of intersection and union for any family of generalized sets.

Say X is equipped with a topology. Fix $A \subseteq X$

- A open $\Leftrightarrow \forall x_0 \in A, \mathcal{N}_{x_0} \leq P(A)$
- A closed $\Leftrightarrow \forall x_0 \in X, \mathcal{N}_{x_0} \cap P(A) \neq \varnothing \Rightarrow x_0 \in A$
- X is Hausdorff $\Leftrightarrow \forall x\, x',\ x \neq x' \Rightarrow \mathcal{N}_x \cap \mathcal{N}_{x'} = \varnothing$

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Warning: P does *not* commute with infinite intersection:

$$\mathcal{N}_x = \bigcap_{U \text{ nhd of } X} P(U)$$

Direct image and inverse image

Given any map $f : X \to Y$ get order-preserving maps

$$\begin{array}{ccc} \mathcal{P}(X) & \xrightarrow{f} & \mathcal{P}(Y) \\ \downarrow P & & \downarrow P \\ \mathcal{F}(X) & \xrightarrow{f_*} & \mathcal{F}(Y) \end{array} \qquad \text{and} \qquad \begin{array}{ccc} \mathcal{P}(X) & \xleftarrow{f^{-1}} & \mathcal{P}(Y) \\ \downarrow P & & \downarrow P \\ \mathcal{F}(X) & \xleftarrow{f^*} & \mathcal{F}(Y) \end{array}$$

$$(g \circ f)_* = g_* \circ f_* \quad \text{and} \quad (g \circ f)^* = f^* \circ g^*$$

Convergence

Given $f : X \to Y$ map between topological spaces and $x \in X$

$f_* \mathcal{N}_x \leq \mathcal{N}_{f(x)}$ means

"direct image under f of the generalized subset of points that are close to x is contained into the generalized subset of points that are close to $f(x)$" aka "f is continuous at x".

All common notions of limits correspond to filters.

Exercise: prove uniqueness of limits in a Hausdorff space.

Induction principles: natural numbers

P: property on natural numbers

If

- $P(0)$
- $\forall n, P(n) \Rightarrow P(n+1)$

Then $\forall n, P(n)$.

Induction principles: compact spaces

P: properties on subsets of a *compact* topological space X

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- $P(\varnothing)$
- $\forall A, \forall B, \; [P(B) \text{ and } A \subset B] \Rightarrow P(A)$
- $\forall A, \forall B, \; [P(A) \text{ and } P(B)] \Rightarrow P(A \cup B)$
- $\forall x, \; \exists A$ nhd of $x, P(A)$

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This characterizes compact spaces.

Lebesgue's number lemma

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Claim: $\exists \varepsilon > 0,\ \forall x,\ \exists i,\ B_\varepsilon(x) \subseteq U_i$.

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- Local existence: Fix x. U covers so we get i such that $x \in U_i$.
  U_i open so we get $\delta > 0$, $B_\delta(x) \subseteq U_i$.
  $\varepsilon := \delta/2$ works to prove $P(B_{\delta/2}(x))$.

Induction principles: functions

E, F: finite-dimensional real vector spaces
P: property on $E \times F$

If

- $\forall x \in E,\ \exists U$ nhd of $x, \exists f \in \mathcal{C}^\infty(U, F),\ \forall x' \in U,\ P(x', f(x'))$
- $\forall x \in E,\ \{y \mid P(x, y)\}$ is convex.

Then $\exists f \in \mathcal{C}^\infty(E, F),\ \forall x \in E,\ P(x, f(x))$.

Example

Given E finite-dimensional real vector space and $\varepsilon : E \to \mathbb{R}$ positive and continuous, there exists $\varepsilon' : E \to \mathbb{R}$ positive and *smooth* such that $\forall x, \varepsilon'(x) \leq \varepsilon(x)$.

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$P(x,t) : 0 < t \wedge t \leq \varepsilon(x)$

Local existence: fix x. $U := \bar{B}_1(x)$ is compact (non-empty) neighborhood of x. Continuity of ε gives $\delta > 0$, $\varepsilon|_U > \delta$. Set $\varepsilon' = \delta$ on U.

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Variation: Assume ε is smooth near C closed, ensure $\varepsilon' = \varepsilon$ on C.

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More importantly, we've seen how to clean up reasonning.