Essential minimum of height functions on the projective line

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1 Height of points in algebraic varieties

The height of an algebraic point is a measure of its arithmetic complexity. Roughly speaking, it can be understood as the amount of information needed to determine it. The height is a fundamental tool in number theory and appears in many finiteness results. A very good introduction to the theory of heights is the monograph [3].

We will construct a height function called the Weil height. We start defining the height of a rational number. Let $\alpha \in \mathbb{Q}^{\times}$, and write $\alpha = p/q$, with $p, q \in \mathbb{Z}$, $\gcd(p, q) = 1$. Then we define

$$h_W(\alpha) = \max\{\log |p|, \log |q|\}.$$

Recall that the logarithm of a number is roughly speaking its number of digits when expressed on the base of the logarithm. Thus this definition is in concordance with the idea that the height should be related with the amount of information needed to determine the number.

We can easily imagine several variants like

$$h_1(\alpha) = \log(|p| + |q|), \qquad \text{or } h_2(\alpha) = h_{\mathrm{FS}}(\alpha) = \frac{1}{2}\log(|p|^2 + |q|^2),$$

that will give different definitions of heights.

What happens if $\alpha \in \overline{\mathbb{Q}}$ is an algebraic number? We know that there exists a unique polynomial, called the minimal polynomial $P_\alpha(x) = a_d x^d + \cdots + a_0 \in \mathbb{Z}[x]$, characterized by the conditions

1. $P_\alpha(\alpha) = 0$,
2. P_α is irreducible over $\mathbb{Q}$.
3. P_α is primitive: $\gcd(a_d, \dots, a_0) = 1$.
4. $a_d > 0$.

The number d is called the degree of α and the complex roots of P_α,

$$Z(P_\alpha) = \{\beta \in \mathbb{C} \mid P_\alpha(\beta) = 0\}$$

are the conjugates of α. From an algebraic point of view, all the conjugates of α are equivalent. In fact if $\beta \in Z(P_\alpha)$ then there exists an embedding $\iota\colon \overline{\mathbb{Q}} \to \mathbb{C}$ with $\iota(\alpha) = \beta$. The set $Z(P_\alpha)$ is also called the Galois orbit of α in $\mathbb{C}$ and denoted $O(\alpha)$. Note that, if $\alpha \in \overline{\mathbb{Q}}$, and $\iota\colon \overline{\mathbb{Q}} \to \mathbb{C}$ is an embedding, then $\iota(\alpha)$ is a complex number that represents α but is not canonical, it depends on the choice of the embedding. By contrast the set $O(\alpha)$ is canonical.

So it makes sense that the height of all the conjugates of α has the same value and that the height of α only depends on the minimal polynomial P_α. Thus we define

$$h_W(\alpha) = \frac{1}{d}\Big(\log(a_d) + \sum_{\beta\in O(\alpha)} \log^+|\beta|\Big),$$

where $\log^+|\beta| = \max\{\log|\beta|, 0\}$. Thus the height of α has two components, the average size of the conjugates of α and the "denominator" a_d. It is easy to see that if $\alpha \in \mathbb{Q}$ we recover the previous definition. Moreover, we can extend the Weil height to points in $\mathbb{P}^1(\overline{\mathbb{Q}})$ by declaring that $h_W(0) = h_W(\infty) = 0$.

The definition we just have given of the Weil height is a global definition involving the "denominator" a_d. There is also a definition as a sum of local terms. Let $\mathcal{M}_\mathbb{Q}$ be the set of places of $\mathbb{Q}$. The set of non-equivalent non-trivial absolute values on $\mathbb{Q}$ suitably normalized. The elements of $\mathcal{M}_\mathbb{Q}$ are represented by the prime numbers p and the special symbol ∞.

If $\infty \in \mathcal{M}_\mathbb{Q}$, then $|\cdot|_\infty$ is the usual absolute value, while if $p \in \mathcal{M}_\mathbb{Q}$ is a prime number, then $|\cdot|_p$ is the p-adic absolute value. The absolute value $|\cdot|_\infty$ extends to $\mathbb{C}_\infty = \mathbb{C}$, while the p-adic absolute value extends to $\mathbb{C}_p$, the completion of $\overline{\mathbb{Q}}$ with respect to the p-adic absolute value. These absolute values satisfy the product formula, for all $\alpha \in \mathbb{Q}$,

$$\prod_{v\in\mathcal{M}_\mathbb{Q}} |\alpha| = 1, \quad \text{equivalently} \sum_{v\in\mathcal{M}_\mathbb{Q}} \log|\alpha|_v = 0.$$

In the same way that we have defined the Galois orbit $O(\alpha) \subset \mathbb{C}$ as the set of complex zeros of P_α, we can define the Galois orbits $O(\alpha) \subset \mathbb{C}_v$ for every $v \in \mathcal{M}_\mathbb{Q}$. The definition of the Weil height as a sum of local contributions is

$$h_W(\alpha) = \frac{1}{d}\sum_{v\in\mathcal{M}_\mathbb{Q}}\ \sum_{\beta\in O(\alpha)\subset\mathbb{C}_v} \log^+|\beta|_v.$$

One can check that the above sum only involves a finite number of nonzero terms and that it agrees with the previous definition of height.

Following the same pattern we can now define the Weil height of points in projective space. Let $p = (x_0 : x_1 : \cdots : x_n) \in \mathbb{P}^n(\overline{\mathbb{Q}})$ be an algebraic point of the projective space. Fix an embedding $\iota_v\colon \overline{\mathbb{Q}}_v \to \mathbb{C}_v$ for each place $v \in \mathcal{M}_\mathbb{Q}$. Choose a finite Galois extension K of $\mathbb{Q}$ such that $p \in \mathbb{P}^n(K)$. Write

$$d = [K : \mathbb{Q}] = \#\operatorname{Gal}(K/\mathbb{Q}).$$

Then the Weil height of the point p is given by

$$h_W(p) = \frac{1}{d}\sum_{v\in\mathcal{M}_\mathbb{Q}}\ \sum_{\sigma\in\operatorname{Gal}(K/\mathbb{Q})} \log(\max_{i=0,\dots,n}\{|\iota_v(\sigma(x_i))|_v\}).$$

One can check that this definition does not depend on the choice of the representative of the point p thanks to the product formula, nor on the field K or the embeddings ι_v, and that it agrees with the previously defined height for $n = 1$.

Finally we can define heights for arbitrary projective varieties. Let X be a projective variety defined over $\mathbb{Q}$. Since it is projective, there is an embedding $\varphi\colon X \to \mathbb{P}^N$ for some N and we can define a height function as

$$h_\varphi(x) = h_W(\varphi(x)).$$

Of course this height depends on the projective embedding φ.

Example 1.1. Consider $X = \mathbb{P}^1$. We have the embedding $\varphi_1 = \mathrm{Id}\colon \mathbb{P}^1 \to \mathbb{P}^1$. With this embedding the height of the point $(\alpha : 1)$ is

$$h_{\varphi_1}((\alpha:1)) = h_W(\alpha) = \frac{1}{d}\Big(\log(a_d) + \sum_{\beta\in O(\alpha)} \log^+(|\beta|)\Big).$$

We can consider now the embedding $\varphi_2 \colon \mathbb{P}^1 \to \mathbb{P}^2$ given by $\varphi_2((x:y)) = (x:(y-x):y)$. In this case

$$h_{\varphi_2}((\alpha:1)) = \frac{1}{d}\Big(\log(a_d) + \sum_{\beta\in O(\alpha)} \log(\max\{|\beta|, |1-\beta|, 1\})\Big).$$

One of the main properties of a height function is the Northcott property that is behind many finiteness results.

Theorem 1.2 (Northcott property). *Let X be a projective variety defined over $\mathbb{Q}$ and φ a projective embedding. Then, for every pair of real numbers A, B,*

$$\#\left\{p \in X(\overline{\mathbb{Q}}) \,\middle|\, \begin{array}{c} h_\varphi(p) \leq A \\ [k(p):\mathbb{Q}] \leq B \end{array}\right\} < \infty.$$

In general there are many possible definitions of heights. Depending on the problem at hand it may be better to use one or another. For instance in his proof of Mordell's conjecture, Faltings [11] uses a height that has a modular interpretation. In another example, the Gross–Zagier formula [14] computes also a modular height and is used in the proof of the BSD conjecture in rank one. Or in the study of dynamical systems one would like to use a height function that is invariant under the dynamical system [8]. Arakelov geometry gives us a very flexible framework to define heights.

2 Successive minima and Bogomolov property

Let X be a projective variety of dimension n defined over $\mathbb{Q}$ and h a height function. Typically height functions are bounded below, so the first interesting invariant of a height function is the minimal value that it can have. This is called the absolute minimum:

$$\operatorname{abs}(h) = \inf\{h(x) \mid x \in X(\overline{\mathbb{Q}})\}.$$

In many situations it may happen that there is a proper subvariety that contains points of small height but that the generic points have always a bigger height. To understand this phenomenon one introduces the successive minima.

$$e_i(h) = \sup_{\substack{Y\subset X \\ \operatorname{cod}(Y)=i}} \inf\{h(x) \mid x \in (X\setminus Y)(\overline{\mathbb{Q}})\}.$$

Then $\operatorname{abs}(h) = e_{n+1}(h)$. At the other extreme is the essential minimum,

$$\operatorname{ess}(h) = e_1(h).$$

The essential minimum is the minimal value of the height on "generic" points.

Given a height function, an interesting and difficult problem is to find the successive minima of the height. For instance on toric varieties, it is possible to determine all the successive minima of toric heights [7]. For curves there are only two minima. The absolute minimum and the essential one.

To see the interest of the essential minimum we recall Bogomolov's conjecture. Mordell's conjecture, proved by Faltings, states that, given a curve C defined over a number field K, if the genus $g(C) \geq 2$, then the number of rational points $\#C(K)$ is finite. This is a classical theme in arithmetic geometry. Curves of genus greater than or equal to 2 have few *special* points. In this case special means having coordinates in K. In a similar vein, there is the Manin–Mumford conjecture, proved by Raynaud [21]. After fixing a base point, a curve C of genus greater than or equal to 2 can be embedded in its Jacobian JC. The set of algebraic points $JC(\overline{\mathbb{Q}})$ is an abelian group and we denote by $JC(\overline{\mathbb{Q}})_{\mathrm{tor}}$ the subgroup of torsion elements. Then the Manin–Mumford conjecture states that

$$\#(C(\overline{\mathbb{Q}}) \cap JC(\overline{\mathbb{Q}})_{\mathrm{tor}}) < \infty.$$

Here the special points are the torsion points in the Jacobian and, again, the conjecture says that for genus greater than or equal to 2 there is only a finite set of special points in the curve.

Bogomolov formulated a strengthening of this conjecture. In the abelian variety JC there is a canonical height function (the Nerón–Tate height) $\hat{h}$ with the properties

1. $\hat{h}(x) \geq 0$ for all $x \in JC(\mathbb{Q})$,
2. $\hat{h}(x) = 0$ if and only if $x \in JC(\mathbb{Q})_{\text{tor}}$.

The Bogomolov conjecture states that there is an $\varepsilon > 0$ such that

$$\#\{x \in C(\overline{\mathbb{Q}}) | \hat{h}(x) \leq \varepsilon\} < \infty.$$

That is, not only there is a finite number of torsion points, but that there is a finite number of points with small canonical height. This conjecture can be restated by saying that

$$\text{ess}(\hat{h}, C) > 0.$$

That is, the Bogomolov conjecture predicts that the essential minimum of the canonical height function is strictly bigger than zero.

The Bogomolov conjecture was proved by Ullmo [27] in the case of curves and by Zhang [30] in higher dimensions by using the Szpiro–Ullmo–Zhang [26] equidistribution result. We will discuss the meaning of equidistribution later in Section 5.

3 Height functions on the projective line

We want to study the problem of determining the essential minimum in the simplest possible non trivial case. We start with a continuous function $g\colon \mathbb{C} \to \mathbb{R}$ that is invariant under complex conjugation, that is, $g(\bar{z}) = g(z)$ and that satisfies the asymptotic estimate

$$g(z) = \log|z| + o(\log|z|), \quad |z| \to \infty.$$

We define the associated height as

$$h_g((\alpha : 1)) = \frac{1}{d}\Big(\log a_d + \sum_{\beta \in O(\alpha)} g(\beta)\Big),$$

where $P_\alpha(x) = a_d x^d + \cdots + a_0$ is the minimal polynomial of α. We also write

$$h_g(\infty) = h_g((1:0)) = \lim_{|z| \to \infty} g(z) - \log|z|$$

if the limit exists. We have several examples of such heights.

1. $g(z) = \log^+|z|$ we recover the Weil height.
2. $g(z) = (1/2)\log(1 + |z|^2)$ gives the Fubini–Study height.
3. $g(z) = (1/2)(\log^+|z| + \log^+|1-z|)$ determines the so called Zhang–Zagier height. Studied by Zagier [29], Doche [9], [10] and Flammang [13].
4. $g(z) = \log(\max\{|z|, |1-z|, 1\})$ gives the height associated to the embedding $\mathbb{P}^1 \to \mathbb{P}^2$ that sends $(x : y)$ to $(x : (y-x) : y)$. This is a more symmetric version of the Zhang–Zagier height also studied in [29].

5. $g(z) = \log(1 + |z| + |1 - z|) + \log(2)$ gives a height associated with the Grassmannian $\mathrm{Gr}(2,4)$ as we will see in section 9.

6. $g(z) = \log|2z - 1 + 2\sqrt{z^2 - z}| - 2\log 2$ gives a height related with the integral Chebyshev constant of $[0,1]$ and Montgomery's conjecture.

All the previous examples satisfy the asymptotic

$$g(z) = \log|z| + \text{ continuous at } \infty.$$

We next give a more exotic example.

Example 3.1. Let $Y(1)$ be the modular curve and $X(1)$ its compactification at the cusp. We have a map $j\colon X(1) \to \mathbb{P}^1$ that is essentially an isomorphism if we forget the orbifold structure of $X(1)$ at the elliptic points. We can define a new height on $\mathbb{P}^1$ by declaring

$$h(j(E)) = h_F(E),$$

where E is an elliptic curve, $j(E)$ its j-invariant and $h_F(E)$ its Faltings height. This height can also be cast into the setting of this section. Let $\mathfrak{H} = \{\tau = x + iy \in \mathbb{C} | y > 0\}$ be the upper half plane. The group $\mathrm{SL}(2,\mathbb{Z})$ acts on $\mathfrak{H}$ and $Y(1)(\mathbb{C}) = \mathfrak{H}/\mathrm{SL}(2,\mathbb{Z})$. Consider the function

$$F(\tau) = -\log(4\pi \,\mathrm{Im}(\tau)^6 |\Delta(\tau)|),$$

where

$$\Delta(\tau) = q\prod_{n=1}^{\infty}(1 - q^n)^{24}, \quad q = e^{2\pi i \tau},$$

is the modular discriminant, a modular form of weight 12.

The function F is invariant under the action of $\mathrm{SL}(2,\mathbb{Z})$, hence descends to a function of $Y(1)$ and defines a function of $\mathbb{C}$ by the rule

$$g(z) = F(\tau), \quad \text{for } z = j(\tau).$$

The Faltings height is just the height associated with this function $h_g(j(E)) = h_F(E)$. The function g satisfies the estimate

$$g(z) \sim \log|z| - 6\log\log|z\bar{z}|, \quad \text{as } |z| \to \infty.$$

The function $\log\log|z\bar{z}|$ is $o(\log|z|)$ but not continuous at $z = \infty$. The essential minimum of the Faltings height has been studied in [15] and [5].

Statement of the problem: Given a function g as before, can we determine the essential minimum of the height function h_g?

Example 3.2. For the Weil height the problem is simple. By the very definition we have that $h_W(0) = h_W(\infty) = 0$. Moreover $h_W(z) = 0$ as long as z is a root of unity. Thus there are infinitely many points with height 0. Thus $\mathrm{abs}(h_W) = \mathrm{ess}(h_W) = 0$.

Slightly more difficult is the case of the Fubini–Study height. Again $h_{\mathrm{FS}}(z) \geq 0$ and $h_{\mathrm{FS}}(0) = h_{\mathrm{FS}}(\infty) = 0$, so $\mathrm{abs}(h_{\mathrm{FS}}) = 0$. But now, if z is a root of unity we have that $h_{\mathrm{FS}}(z) = (1/2)\log(2)$. For the moment we know

$$0 \leq \mathrm{ess}(h_{\mathrm{FS}}) \leq (1/2)\log(2).$$

We will see at the beginning of next section that, in fact $\mathrm{ess}(h_{\mathrm{FS}}) = (1/2)\log(2)$.

Remark 3.3. For an algebraic integer $\alpha \in \overline{\mathbb{Z}}$, the minimal polynomial is monic. Hence $a_d = 1$ and in this case the term $\log a_d$ is zero. One may ask if the essential minimum can be computed using only algebraic integers. In other words, write

$$\mathrm{ess}_{\overline{\mathbb{Z}}}(h_g) = \liminf\{h_g(\alpha) | \alpha \in \overline{\mathbb{Z}}\}.$$

Is it true in general that $\mathrm{ess}_{\overline{\mathbb{Z}}}(h_g) = \mathrm{ess}(h_g)$.

Remark 3.4. Even if we are not able to compute the essential minimum, it is an important theoretical result to know if the number is theoretically computable. That is, if there is an algorithm that given a desired precision, gives us an approximation of the essential minimum in a finite amount of time (maybe more than the age of the universe).

4 Lower bounds

In order to determine the essential minimum we need both lower and upper bounds that hopefully will be close enough to determine the essential minimum.

We start with the lower bounds that are easier to obtain.

Lemma 4.1. *Let $Q \in \mathbb{Z}[z]$ be a polynomial with integer coefficients and $r \in \mathbb{R}_{\geq 0}$ a positive real number. Let*

$$E = \inf_z (g(z) - r \log |Q(z)|).$$

Then for any $\alpha \notin Z(Q)$, the estimate $h_g(\alpha) \geq E$ holds. Therefore $E \leq \mathrm{ess}(h_g)$.

Proof. Let $\alpha \notin Z(Q)$ and P_α its minimal polynomial. Let d be the degree of P_α and a_d the leading coefficient. Write d_Q for the degree of Q. The polynomials P_α and Q are coprime. It follows that

$$\sum_{\beta \in O(\alpha)} \log |Q(\beta)| + d_Q \log(a_d) = \log \left| a_d^{d_Q} \prod_{b \in Z(P_\alpha)} Q(\beta) \right| = \log |\mathrm{Res}(Q, P_\alpha)| \geq 0,$$

because $\mathrm{Res}(Q, P_\alpha)$ is an integer different from zero. Therefore

$$\begin{aligned} h_g(\alpha) &= \frac{1}{d} \left(\log(a_d) + \sum_{\beta \in O(\alpha)} g(\beta) \right) \\ &\geq \frac{1}{d} \left(\log(a_d) + \sum_{\beta \in O(\alpha)} g(\beta) - r d_Q \log(a_d) - r \sum_{\beta \in O(\alpha)} \log |Q(\beta)| \right) \\ &= \frac{1 - r d_Q}{d} \log(a_d) + \frac{1}{d} \sum_{\beta \in O(\alpha)} g(\beta) - r \log |Q(\beta)|. \end{aligned}$$

We consider now two cases. If $r d_Q \leq 1$, then

$$h_g(\alpha) \geq \frac{1}{d} \sum_{\beta \in O(\alpha)} g(\beta) - r \log |Q(\beta)| \geq E.$$

While if $r d_Q > 1$, then $E = -\infty \leq h_g(\alpha)$, so in any case $h_g(\alpha) \geq E$. $\square$

Example 4.2. We go back to the Fubini–Study height. Recall that we already have the estimates

$$0 \le \mathrm{ess}(h_{\mathrm{FS}}) \le \frac{\log(2)}{2}.$$

We know that $h_{\mathrm{FS}}(0) = 0$. If we want to prove that $\mathrm{ess}(h_{\mathrm{FS}}) > 0$ we need a polynomial that vanishes at 0. The simplest such polynomial is $Q(z) = z$. Consider the function

$$\varphi(z) = g_{\mathrm{FS}}(z) - \frac{1}{2}\log|z| = \frac{1}{2}\log\left(\frac{1+|z|^2}{|z|}\right).$$

It is easy to check that

$$\inf \varphi(z) = \varphi(1) = \frac{\log(2)}{2}.$$

By Lemma 4.1 we conclude that

$$\mathrm{ess}(h_{\mathrm{FS}}) = \frac{\log(2)}{2}.$$

It is no wonder that we can compute exactly the essential minimum for the Fubini–Study height because the function g satisfies

$$|z| = |z'| \Longrightarrow g(z) = g(z').$$

For such functions, the theory of toric metrics applies and the essential minimum is easy to compute.

Remark 4.3. In Lemma 4.1, instead of considering a single polynomial Q, we can consider a finite family of polynomials $Q_i \in \mathbb{Z}[z]$, $i = 1, \dots, n$ and positive real numbers $r_i \ge 0$. Then

$$E = \inf_z \left(g(z) - \sum_{i=1}^{n} r_i \log|Q_i(z)| \right)$$

also satisfies $E \le \mathrm{ess}(h_g)$.

We obtain a different lower bound for each polynomial Q and coefficient r. This family of lower bounds is sharp in the following sense.

Theorem 4.4 (Ballaÿ [1], Qu–Yin [19]). *The estimate*

$$\mathrm{ess}(h_g) = \sup_{\substack{Q\in\mathbb{Z}[z]\\ r\ge 0}} \inf_{z\in\mathbb{C}} g(z) - r\log|Q(z)|$$

holds.

5 Equidistribution of small points

Before discussing upper bounds, we will recall the equidistribution principle that plays an important role in the proof of the Bogomolov conjecture.

Equidistribution Principle. Let X be an irreducible projective variety over $\mathbb{Q}$, $h\colon X(\overline{\mathbb{Q}}) \to \mathbb{R}$ a suitable height. Let $(\alpha_n)_{n\ge 1}$ a sequence of points in $X(\overline{\mathbb{Q}})$ satisfying two conditions.

1. The sequence is generic: for every projective proper subvariety $Y \subset X$, the set $\{n | \alpha_n \in Y(\overline{\mathbb{Q}})\}$ is finite.

2. The limit $\lim_{n\to\infty} h(\alpha_n)$ exists and is as small as possible.

then the shape of the Galois orbits $O(\alpha_n)$ should converge to a fixed shape.

Some comments are in order.

1. It is called equidistribution *principle* because not every height function satisfies it. One of the most general equidistribution theorems can be found in [28] as the culmination of the work of many mathematicians. A complete discussion of when the equidistribution principle holds for toric heights can be found in [6].

2. The minimal possible value of the limit of the heights of a generic sequence is precisely the essential minimum, thus the equidistribution principle is another reason to be interested in the essential minimum.

3. We have to give a precise meaning to the sentence, "the Galois orbits $O(\alpha_n)$ converge to a fixed shape".

In order to make sense to the convergence of "shapes", we associate a counting measure to each finite set. Let F be a finite set, the counting measure δ_F is the Radon measure that, for each continuous function f, is given by

$$\int f \, \mathrm{d}\delta_F = \frac{1}{\#F} \sum_{p \in F} f(p).$$

Thus every Galois orbit $O(\alpha_n)$ has an associated measure and we will say that the Galois orbits converge if the associated measures converge to a given measure.

The measures $\delta_{O(\alpha_n)}$ are measures on the topological space $\mathbb{C}$, that is non compact. Since $\mathbb{C}$ is not compact we will be interested in different definitions of convergence.

Definition 5.1. Let $(\mu_n)_{n\geq 1}$ be a sequence of Radon measures on $\mathbb{C}$ and let μ be a measure on $\mathbb{C}$.

1. We say that $(\mu_n)_{n\geq 1}$ *converges weakly* to μ if for every continuous and bounded function $f\colon \mathbb{C} \to \mathbb{R}$, the convergence
$$\lim_{n\to\infty} \int f \, \mathrm{d}\mu_n = \int f \, \mathrm{d}\mu$$
holds.

2. We say that $(\mu_n)_{n\geq 1}$ *converges logarithmically* to μ if for every continuous function $f\colon \mathbb{C} \to \mathbb{R}$ satisfying $|f(z)| \leq A + B\log^+|z|$ for some constants $A, B \geq 0$, the convergence
$$\lim_{n\to\infty} \int f \, \mathrm{d}\mu_n = \int f \, \mathrm{d}\mu$$
holds.

3. We say that $(\mu_n)_{n\geq 1}$ *converges properly* to μ if for every continuous function $f\colon \mathbb{C} \to \mathbb{R}$, the convergence
$$\lim_{n\to\infty} \int f \, \mathrm{d}\mu_n = \int f \, \mathrm{d}\mu$$
holds.

In the above list each convergence is stronger that the previous one because it holds for a bigger set of test functions. We note also that proper convergence is equivalent to weak convergence and the condition that there is a compact K and a number n_0 such that the support of μ and of μ_n for $n \geq n_0$ is contained in K.

6 Upper bounds

The upper bounds are more difficult to establish than the lower bounds and typically need a deep theorem. The first source of upper bounds is the logarithmic potential theory. For a gentle introduction to the theory of logarithmic potential the reader can follow [24], while for a complete treatment the reader is referred to [20].

We denote by $\mathcal{P}_{\log}(\mathbb{C})$ the vector space of probability measures on $\mathbb{C}$ satisfying

$$\int \log^+ |z| \, \mathrm{d}\mu < +\infty.$$

For instance every probability measure with compact support belongs to $\mathcal{P}_{\log}(\mathbb{C})$. For $\mu \in \mathcal{P}_{\log}(\mathbb{C})$, the potential of μ is the function $U^\mu \colon \mathbb{C} \to \mathbb{R} \cup \{\infty\}$ defined as

$$U^\mu(z) = \int -\log |w - z| \, \mathrm{d}\mu(w).$$

We can recover the measure from the potential as

$$\mu = \frac{1}{4\pi i} \partial \bar{\partial} U^\mu.$$

The energy of a measure is defined as

$$E(\mu) = \int U^\mu \, \mathrm{d}\mu = \iint -\log |z - w| \, \mathrm{d}\mu(w) \, \mathrm{d}\mu(z).$$

If $K \subset \mathbb{C}$ is a compact subset, then

$$E(K) = \inf\{E(\mu) \mid \operatorname{supp}(\mu) \subset K\},$$

and the capacity of K is defined as $\operatorname{Cap}(K) = e^{-E(K)}$.

When $E(K) \neq \infty$, there is a unique measure μ_K such that $\operatorname{supp}(\mu_K) \subset K$ and $E(\mu_K) = E(K)$. Such measure is called the equilibrium measure of K.

For any compact set K and real number $\varepsilon > 0$, we denote by

$$B_\varepsilon(K) = \{x \in \mathbb{C} \mid d(x, K) < \varepsilon\}$$

an ε neighborhood of K.

The first family of upper bounds is a consequence to the following couple of theorems.

Theorem 6.1 (Fekete–Szegő [12]). *Let $K \subset \mathbb{C}$ be a compact subset, invariant under complex conjugation.*

1. *If $\operatorname{Cap}(K) < 1$, then there exists an $\varepsilon > 0$ such that*

$$\#\{x \in \overline{\mathbb{Z}} | O(x) \subset B_\varepsilon(K)\} < \infty.$$

2. *If $\operatorname{Cap}(K) \geq 1$, then for all $\varepsilon > 0$,*

$$\#\{x \in \overline{\mathbb{Z}} | O(x) \subset B_\varepsilon(K)\} = \infty.$$

Theorem 6.2 (Rumely [23]). *Let K be a compact set of capacity 1 invariant under complex conjugation. For every integer $n \geq 1$ choose $\alpha_n \in \overline{\mathbb{Z}}$ such that $O(\alpha_n) \subset B_{1/n}(K)$ and that $O(\alpha_n) \neq O(\alpha_m)$ for $n \neq m$. Then $\delta_{O(\alpha_n)}$ converges to μ_K properly.*

Corollary 6.3. *Let K be a compact set of capacity* 1 *invariant under complex conjugation. Then*

$$\mathrm{ess}(h_g) \leq \int g \, \mathrm{d}\mu_K.$$

Proof. Choose a sequence of algebraic integers α_n as in Rumely's theorem such that $\delta_{O(\alpha_n)}$ converges properly to μ_K. Since the α_n are integers, the corresponding minimal polynomial is monic. Therefore

$$h_g(\alpha_n) = \frac{1}{d} \sum_{\beta \in O(\alpha_n)} g(\beta) = \int g \, \mathrm{d}\delta_{O(\alpha_n)}.$$

Using that $\delta_{O(\alpha_n)}$ converges properly to μ_K and that the sequence α_n is generic, we deduce

$$\mathrm{ess}(h_g) \leq \lim_{n \to \infty} h_g(\alpha_n) = \int g \, \mathrm{d}\mu_K.$$

□

Remark 6.4. In view of Corollary 6.3 we may ask if we are lucky and the family of upper bounds we have constructed is sharp. In other words, is it true that

$$\mathrm{ess}(h_g) = \inf_K \int g \, \mathrm{d}\mu_K,$$

where the infimum runs over all compact subsets of $\mathbb{C}$ of capacity 1 that are invariant under complex conjugation. The answer to this question is no and we need better upper bounds.

The second source of upper bounds is a recent theorem of Smith and Orloski–Sardari, that can be seen as a very flexible version of Fekete–Szegő theorem.

Theorem 6.5 (Smith [25], Orloski–Sardari[18]). *Let μ be a measure on $\mathbb{C}$ with compact support and invariant under complex conjugation. Then the following are equivalent.*

1. *For all $Q \in \mathbb{Z}[z]$, the integral $\int \log |Q(z)| \, \mathrm{d}\mu \geq 0$.*
2. *There exists a sequence of algebraic integers α_n such that $\delta_{O(\alpha_n)}$ converges properly to μ.*

With Menares, Qu and Sombra we have extended this result to measures with non necessarily compact support.

Corollary 6.6 (—, Menares, Qu, Sombra [4]). *Let $\mu \in \mathcal{P}_{\log}(\mathbb{C})$ invariant under complex conjugation. The following are equivalent.*

1. *For all $Q \in \mathbb{Z}[z]$, the integral $\int \log |Q(z)| \, \mathrm{d}\mu \geq 0$.*
2. *There exists a sequence of algebraic integers α_n such that $\delta_{O(\alpha_n)}$ converges logarithmically to μ.*

We immediately obtain the following corollary.

Corollary 6.7. *Let $\mu \in \mathcal{P}_{\log}(\mathbb{C})$ be a measure invariant under complex conjugation and such that*

$$\int \log |Q(z)| \, \mathrm{d}\mu(z) \geq 0$$

for all non-zero polynomials $Q \in \mathbb{Z}[z]$. Then

$$\mathrm{ess}(h_g) \leq \int g \, \mathrm{d}\mu.$$

As we will see, the family of upper bounds provided by Corollary 6.7 is sharp.

7 Linear programing

Now that we have lower and upper bounds for the essential minimum we would like to show that both can be arbitrarily close to the essential minimum. The tool to achieve this is linear programming. A comprehensive reference to linear optimization is the book [2]. We will present an abstract version of linear programming that will be useful for our purposes.

The abstract version of duality in linear optimization has the following ingredients.

1. Let E and F be real vector spaces and $A\colon E \times F \to \mathbb{R}$ a bilinear map that we see as linear maps $A\colon E \to F^\vee$ and $A^t\colon F \to E^\vee$.
2. Let $u \in E^\vee$ and $v \in F^\vee$ be linear maps.
3. Let $\sigma \subset E$ and $\tau \subset F$ be convex cones.

Recall that, given a convex cone σ in a vector space E, the dual cone is defined as

$$\sigma^\vee = \{x \in E^\vee \mid \langle x, u\rangle \geq 0,\ \forall u \in \sigma\}.$$

Linear programming studies the optimization of a linear function subordinated to a set of convex constrains:

$$\mathcal{P} = \sup\{u(x) \mid x \in \sigma,\ v - A(x) \in \tau^\vee\}. \tag{7.1}$$

By abuse of notation we will denote by $\mathcal{P}$ the number appearing in equation (7.1) and the problem of determining this number.

The standard method to deal with a constrained optimization problem is to write the Lagrangian formulation where we add variables for some of the constrains.

$$\mathcal{P} = \sup_{x\in\sigma} \inf_{y\in\tau}\{u(x) + v(y) - A(x, y)\}. \tag{7.2}$$

To see that the solution of this double optimization problem is $\mathcal{P}$ we fix $x \in \sigma$ such that

$$v - A(x) \notin \tau^\vee.$$

Then we can find $y \in \tau$ with $v(y) - A(x, y) < 0$. Since τ is a cone we can make $v(y) - A(x, y)$ arbitrarily small and

$$\inf_{y\in\tau}\{u(x) + v(y) - A(x, y)\} = -\infty.$$

By contrast, if $v - A(x) \in \tau^\vee$ we have $v(y) - A(x, y) \geq 0$ and the infimum over y is attained for $y = 0$. Thus we see that the problems (7.1) and (7.2) are equivalent.

The dual problem is obtained by switching the inf and the sup.

$$\mathcal{D} = \inf_{y\in\tau} \sup_{x\in\sigma}\{u(x) + v(y) - A(x, y)\} \tag{7.3}$$

By an argument similar to the previous one, this optimization problem is equivalent to

$$\mathcal{D} = \inf\{v(y) | y \in \tau,\ A^t(y) - u \in \sigma^\vee\}. \tag{7.4}$$

Since sup inf $\leq$ inf sup, we deduce that in general $\mathcal{P} \leq \mathcal{D}$. This property is called *weak duality.* If the stronger condition $\mathcal{P} = \mathcal{D}$ holds, then we say that the optimization problem satisfies *strong duality.*

We will say that $\mathcal{P}$ is *feasible* if there is an $x \in \sigma$ such that $v - A(x) \in \tau^\vee$. This means that there is at least a candidate for the problem $\mathcal{P}$ and thus $\mathcal{P} > -\infty$. The set of such x's is called the *feasibility set* of $\mathcal{P}$.

We will say that $\mathcal{P}$ is *bounded* if $\mathcal{P} < \infty$. Similarly we say that $\mathcal{D}$ is *feasible* if there is a $y \in \tau$ with $A^t(y) - u \in \sigma^\vee$, which implies $\mathcal{D} < \infty$ and *bounded* if $\mathcal{D} > -\infty$. Again such y's form the feasibility set of $\mathcal{D}$. Clearly if $\mathcal{P}$ is feasible, then $\mathcal{D}$ is bounded and if $\mathcal{D}$ is feasible, then $\mathcal{P}$ is bounded. On the other hand, the fact that one of the problems is bounded does not imply that the other one is feasible.

The main result of finite dimensional linear programming is the following.

Theorem 7.5. *Assume that E and F are finite dimensional and that the convex cones σ and τ are closed. Then the following are equivalent.*

1. *$\mathcal{P}$ is feasible and bounded.*

2. *$\mathcal{D}$ is feasible and bounded.*

And in this case, strong duality $\mathcal{P} = \mathcal{D}$ holds.

Proof. Assume that $\mathcal{P}$ is feasible and bounded. Then $\mathcal{D}$ is bounded. In the space $\mathbb{R} \times F^\vee$ we define the subsets.

$$V = \{(u(x), A(x)) | x \in \sigma\}$$

and for $\lambda \in \mathbb{R}$,

$$W_\lambda = \{(t, w) | t \geq \lambda, v - w \in \tau^\vee\}.$$

Since $\mathcal{P}$ is feasible, there exists $x \in \sigma$ such that $v - A(x) \in \tau^\vee$. Putting $w = A(x)$, $t = u(x)$ and choosing $\lambda_1 \leq t$, we see that

$$(t, w) \in V \cap W_{\lambda_1} \neq \emptyset.$$

Since $\mathcal{P}$ is bounded, $\mathcal{P} < \infty$. Let $\lambda_2 > \mathcal{P}$. Then

$$V \cap W_{\lambda_2} = \emptyset.$$

So, for big λ the sets V and W_λ are disjoint while for small λ they intersect. We single out the boundary point

$$\lambda_0 = \inf\{\lambda | V \cap W_\lambda = \emptyset\} = \sup\{\lambda | V \cap W_\lambda \neq \emptyset\}.$$

Let $\varepsilon > 0$. Since $V \cap W_{\lambda_0 - \varepsilon} \neq \emptyset$, there exists $x \in \sigma$ such that $u(x) \geq \lambda_0 - \varepsilon$ and $v - A(x) \in \tau^\vee$. Hence $\lambda_0 - \varepsilon \leq \mathcal{P}$ and, being ε arbitrary, we deduce $\mathcal{P} \geq \lambda_0$.

Put now $\lambda = \lambda_0 + \varepsilon > \lambda_0$. Therefore $V \cap W_\lambda = \emptyset$. Since V is a convex closed cone and W_λ is the translated of a closed convex cone, there is a hyperplane h such that $h|_V \geq 0$ and $h|_{W_\lambda} < 0$. This separation statement follows from [22, Theorem 11.1]. Write $h = (b, y)$. The conditions on h are

$$\begin{aligned} bu(x) + A(x, y) &\geq 0, \quad \text{for all } x \in \sigma, \\ bt + v(y) - v'(y) &< 0, \quad \text{for all } v' \in \tau^\vee. \end{aligned}$$

Taking $v' = v - A(x) \in \tau^\vee$ for some $x \in \sigma$ that exists because $\mathcal{P}$ is feasible, we deduce

$$bu(x) + A(x, y) \geq 0 > bt + v(y) - v'(y) = bt + A(x, y),$$

hence $bu(x) > bt$ for all $\lambda \leq t$. Since t can be arbitrarily big, this implies that $b < 0$ and after rescaling we can assume $b = -1$. The condition

$$-t + v(y) - v'(y) < 0, \quad \text{for all } v' \in \tau^\vee, t \geq \lambda$$

implies that $y \in \tau$. While the condition

$$-u(x) + A(x, y) \geq 0, \quad \text{for all } x \in \sigma$$

implies that $A^t(y) - u \in \sigma^\vee$. Thus y belongs to the feasibility set of $\mathcal{D}$ and $\mathcal{D}$ is feasible. Taking now $v' = 0$ and $t = \lambda = \lambda_0 + \varepsilon$, we obtain $v(y) < \lambda_0 + \varepsilon$. Hence $\mathcal{D} < \lambda_0 + \varepsilon$. Since ε is arbitrary, we obtain $\mathcal{D} \leq \lambda_0 \leq \mathcal{P}$. Combining this with weak duality $\mathcal{P} \leq \mathcal{D}$ we obtain the strong duality $\mathcal{P} = \mathcal{D}$. □

8 The essential minimum as a linear programming problem

The set of all irreducible non constant polynomials with integer coefficients is countable. Assume that we have enumerated them as $Q_1, Q_2, \dots, Q_n, \dots$.

Thanks to Theorem 4.4, the essential minimum is the solution of the optimization problem

$$\mathrm{ess}(h_g) = \sup_{\substack{n\in\mathbb{N} \\ a_1,\dots,a_n\geq 0}} \inf_{z\in\mathbb{C}} \left(g(z) - \sum_{i=1}^{n} a_i \log|Q_i(z)| \right).$$

This is an optimization problem but does not look like a linear programming problem. We need to massage it a little. First, instead of looking at the infimum of a function we can look at the value needed to make it positive, so we can write

$$\mathrm{ess}(h_g) = \sup_{\substack{n\in\mathbb{N} \\ a_1,\dots,a_n\geq 0 \\ \lambda\in\mathbb{R}}} \left\{ \lambda \,\middle|\, g(z) - \sum_{i=1}^{n} a_i \log|Q_i(z)| - \lambda \geq 0 \right\}.$$

This problem looks more like a linear program problem. We have a vector space $E = \mathbb{R} \oplus \bigoplus_{\mathbb{N}} \mathbb{R}$, a linear map that we want to optimize $u(\lambda, a_1, a_2, \dots) = \lambda$, a condition with the shape of a cone $\sigma = \{a_i \geq 0\}$ and another condition

$$g(z) - \sum_{i=1}^{n} a_i \log|Q_i(z)| - \lambda \geq 0. \tag{8.1}$$

We need to put the condition (8.1) as some kind of duality. The idea is that functions are in duality with measures and that a function is positive if and only if the integral of the function with respect to any positive measure is positive. To make this precise we define $\mathcal{P}'_{\log}(\mathbb{C})$ the space of all probability Radon measures μ such that

$$\int \log^+ |z| \, \mathrm{d}\mu < \infty, \qquad \int \log|Q_i(z)| \, \mathrm{d}\mu > -\infty, \ \forall i \geq 1.$$

These are the probability measures that can integrate the functions of the form $\log|Q_i(z)|$. We will also write $\mathcal{M}^+_{\log}(\mathbb{C}) = \mathbb{R}_{\geq 0}\mathcal{P}'_{\log}(\mathbb{C})$. This is the cone of positive measures that can integrate $\log|Q_i(z)|$ and finally $\mathcal{M}_{\log}(\mathbb{C}) = \mathcal{M}^+_{\log}(\mathbb{C}) - \mathcal{M}^+_{\log}(\mathbb{C})$ is the vector space of signed Radon measures that can integrate $\log|Q_i(z)|$. Now our optimization problem reads

$$\mathcal{P} = \mathrm{ess}(h_g) = \sup_{\substack{n\in\mathbb{N} \\ a_1,\dots,a_n\geq 0 \\ \lambda\in\mathbb{R}}} \left\{ \lambda \,\middle|\, \int g(z) - \sum_{i=1}^{n} a_i \log|Q_i(z)| - \lambda \, \mathrm{d}\mu \geq 0, \ \forall \mu \in \mathcal{M}^+_{\log}(\mathbb{C}) \right\}.$$

Therefore, the second vector space is $F = \mathcal{M}_{\log}(\mathbb{C})$, the cone in $\tau \subset F$ is $\mathcal{M}^+_{\log}(\mathbb{C})$, the pairing A is given by

$$A((\lambda, a_1, \dots), \mu) = \int \sum_{i=1}^{n} a_i \log|Q_i(z)| + \lambda \, \mathrm{d}\mu$$

and the linear function u is

$$u(\mu) = \int g \, \mathrm{d}\mu.$$

One can check that with these definitions, the linear programming scheme of the previous section is exactly the problem of finding the essential minimum of the height function. Now lets look for the dual problem!

First we need to write the Laplacian formulation

$$\mathcal{P} = \sup_{\substack{n\in\mathbb{N} \\ a_1,\dots,a_n\geq 0 \\ \lambda\in\mathbb{R}}} \inf_{\mu\in\mathcal{M}^+_{\log}(\mathbb{C})} \left\{ \lambda + \int g \,\mathrm{d}\mu - \int \lambda + \sum_{i=1}^n a_i \log|Q_i(z)| \,\mathrm{d}\mu \right\}.$$

Then we switch the inf and the sup

$$\mathcal{D} = \inf_{\mu\in\mathcal{M}^+_{\log}(\mathbb{C})} \sup_{\substack{n\in\mathbb{N} \\ a_1,\dots,a_n\geq 0 \\ \lambda\in\mathbb{R}}} \left\{ \lambda + \int g \,\mathrm{d}\mu - \int \lambda + \sum_{i=1}^n a_i \log|Q_i(z)| \,\mathrm{d}\mu \right\}.$$

As we discussed in the previous section, we can simplify the dual problem as

$$\mathcal{D} = \inf \left\{ \int g \,\mathrm{d}\mu \;\middle|\; \begin{array}{c} \mu \in \mathcal{M}^+_{\log}(\mathbb{C}) \\ \int \lambda + \sum_{i=1}^n a_i \log|Q_i| \,\mathrm{d}\mu - \lambda \geq 0, \ \forall a_i \geq 0, \lambda \in \mathbb{R} \end{array} \right\}.$$

The condition

$$\int \lambda + \sum_{i=1}^n a_i \log|Q_i| \,\mathrm{d}\mu - \lambda \geq 0, \ \forall a_i \geq 0, \lambda \in \mathbb{R}$$

forces that

$$\lambda \left(\int \,\mathrm{d}\mu - 1 \right) \geq 0, \forall \lambda \in \mathbb{R},$$

which is only possible if $\int \,\mathrm{d}\mu - 1 = 0$, that is, $\mu \in \mathcal{P}'_{\log}(\mathbb{C})$; and that

$$\int \log|Q_i(z)| \,\mathrm{d}\mu \geq 0,$$

which is exactly the condition in Smith and Orloski–Sardari theorem (Theorem 6.5). In conclusion

$$\mathcal{D} = \inf \left\{ \int g \,\mathrm{d}\mu \;\middle|\; \mu \in \mathcal{P}'_{\log}(\mathbb{C}), \ \int \log|Q_i(z)| \,\mathrm{d}\mu \geq 0, \forall i \right\}.$$

Since the vector spaces are infinite dimensional, in principle we do not know whether strong duality holds. Nevertheless it turns out that strong duality is satisfied in this case.

Theorem 8.2 (—, Menares, Qu, Sombra [4]). *The problems $\mathcal{P}$ and $\mathcal{D}$ satisfy the strong duality principle and therefore $\mathcal{D} = \mathcal{P}$.*

We derive two corollaries of this result.

Corollary 8.3. *If g is a computable function, then $\mathrm{ess}(h_g)$ is a computable real number.*

Corollary 8.4. *The essential minimum of h_g can be approximated by algebraic integers. In other words*

$$\mathrm{ess}(h_g) = \mathrm{ess}_{\overline{\mathbb{Z}}}(h_g).$$

9 The Grassmannian $\mathrm{Gr}(2,4)$

Let $\mathrm{Gr}(2,4)$ be the Grassmannian manifold that parameterizes 2-dimensional planes in a 4-dimensional complex vector space. The Plücker coordinates give an embedding $\mathrm{Gr}(2,4) \to \mathbb{P}^5$. If we write the homogeneous coordinates of $\mathbb{P}^5$ as $(x_{12} : x_{13} : x_{14} : x_{23} : x_{24} : x_{34})$, then the image of $\mathrm{Gr}(2,4)$ is the hypersurface of equation $x_{12}x_{34} - x_{13}x_{24} + x_{14}x_{23} = 0$.

We consider the complex vector space

$$R_n = \mathrm{H}^0(\mathrm{Gr}(2,4), \mathcal{O}(n)) = \left\{ \begin{matrix} \text{homogeneous polynomials of degree } n \\ \text{in the variables } x_{12}, \dots, x_{34} \end{matrix} \right\} \Big/ (x_{13}x_{24} = x_{12}x_{34} + x_{14}x_{23})$$

This vector space is an irreducible representation of $U(4)$ of highest weight $(n, n, 0, 0)$.

Since it is irreducible, it has a unique hermitian metric, that is invariant under the action of $U(4)$ and normalized by the condition $\|x_{12}^n\| = 1$. Moreover it has a natural lattice $R_{\mathbb{Z},n} = \mathrm{H}^0(\mathrm{Gr}(2,4), \mathcal{O}(n))_{\mathbb{Z}}$: the homogeneous polynomials with integer coefficients. Whenever we have a lattice and a metric, the first thing we can ask is what is the shortest vector.

Problem 1. Let $\lambda_n \neq 0$ be the shortest non-zero element in $\mathrm{H}^0(\mathrm{Gr}(2,4), \mathcal{O}(n))_{\mathbb{Z}}$. What is the length $\|\lambda_n\|$?

This problem may be too difficult, thus we can settle for the asymptotic behavior of this length.

Problem 2. Compute the number

$$A = \lim_{n\to\infty} \frac{-1}{n} \log \|\lambda_n\|. \tag{9.1}$$

Apparently this problem has nothing to do with the rest of the course. Nevertheless we will see that this problem is equivalent to determining an essential minimum of a height function in $\mathbb{P}^1$.

The invariant metric in $\mathrm{H}^0(\mathrm{Gr}(2,4), \mathcal{O}(n))$ can be interpreted as an L^2-norm. The sheaf $\mathcal{O}(n)$ has a hermitian metric, the Fubini-Study one. Given a section $s \in \mathrm{H}^0(\mathrm{Gr}(2,4), \mathcal{O}(n))$ that corresponds to a polynomial P_s of degree n, the Fubini-Study norm at a point $(x_{12} : x_{13} : x_{14} : x_{23} : x_{24} : x_{34})$ is given by

$$\|s(x_{ij})\| = \frac{|P(x_{ij})|}{\left(\sum_{i<j} |x_{ij}|^2\right)^{n/2}}.$$

This metric is invariant under the action of $U(4)$. There is also a unique volume form dVol invariant under $U(4)$ and of total volume 1. The L^2 norm is given by

$$\|s\|_{L^2} = \int_{\mathrm{Gr}(2,4)} \|s(x_{ij})\| \,\mathrm{dVol}.$$

Since the L^2 norm comes from a hermitian metric invariant under $U(4)$, it agrees with the original norm up to a normalization factor. The normalization factor is given by the dimension of R_n. That is

$$\|s\| = \dim R_n \|s\|_{L^2} = \frac{(n+1)(n+2)^2(n+3)}{12} \|s\|_{L^2}. \tag{9.2}$$

We can now introduce a new norm, the sup or L^∞-norm given by

$$\|s\|_{\sup} = \sup_{x_{ij}} \|s(x_{ij})\|.$$

It is clear that

$$\|s\|_{L^2} \leq \|s\|_{\sup}. \tag{9.3}$$

On the other hand there is an inequality, due to Gromov, in the other direction

$$\|s\|_{\sup} \leq Cn^2 \|s\|_{L^2}. \tag{9.4}$$

Here C is a constant, independent of s and n and the exponent $2 = \dim \mathrm{Gr}(2,4)/2$. Let μ_n be the shortest vector in $R_{\mathbb{Z},n}$ with respect to the sup-norm. Then the estimates (9.2), (9.3) and (9.4) yield

$$\lim_{n\to\infty} \frac{-1}{n} \log \|\mu_n\| \leq \lim_{n\to\infty} \frac{-1}{n} \log \|\lambda_n\| = \lim_{n\to\infty} \frac{-1}{n} \log \|\lambda_n\|_{\sup} \leq \lim_{n\to\infty} \frac{-1}{n} \log \|\mu_n\|_{\sup} = \lim_{n\to\infty} \frac{-1}{n} \log \|\mu_n\|.$$

Hence, the problem of finding the asymptotic of the length of the shortest lattice vector for the original metric is equivalent to the same problem for the sup norm. In other words, we have the following lemma.

Lemma 9.5. *For each n let μ_n be the shortest vector of $R_{\mathbb{Z},n}$ with respect to the sup norm. Then the number we are looking for is*

$$A = \lim_{n\to\infty} \frac{-1}{n} \log \|\mu_n\|_{\sup}.$$

We start by reviewing the structure of the spaces R_n. First if we write $R = \bigoplus R_n$, then R is a graded algebra and $\mathrm{Proj}(R) = \mathrm{Gr}(2,4)$. Consider the standard maximal torus $T(4) \subset \mathrm{GL}(4)$ consisting of diagonal matrices and the compact subtorus $S(4) = T(4) \cap U(4)$.

Each R_n can be decomposed into weights

$$R_n = \bigoplus_{a+b+c+d=2n} V_{(a,b,c,d)} \tag{9.6}$$

where $V_{(a,b,c,d)}$ is the subspace of weight a, b, c, d. That is, the subspace where S acts on $V_{(a,b,c,d)}$ with the action

$$(t_1, t_2, t_3, t_4) \cdot v = t_1^a t_2^b t_3^c t_4^d v, \quad \text{for } v \in V_{(a,b,c,d)}.$$

The decomposition (9.6) is orthogonal with respect to the L^2 norm. We will denote by $V_{(a,b,c,d),\mathbb{Z}}$ the lattice of vectors with integer coefficients. The decomposition (9.6) is compatible with the integral structure of R_n and $V_{(a,b,c,d)}$.

Thus if we want to search for the asymptotic of the length of the smallest vector, we can do it weight by weight. To this end we consider the normalized weight polytope Δ is the convex closure of the points

$$\frac{1}{n}(a,b,c,d), \quad V_{(a,b,c,d)} \neq \{0\}.$$

It is independent of n and agrees with the convex closure of

$$(1,1,0,0),\ (1,0,1,0),\ (1,0,0,1),\ (0,1,1,0),\ (0,1,0,1),\ (0,0,1,1).$$

For each rational point $p = (\alpha, \beta, \gamma, \delta) \in \Delta$, we consider the function

$$\varphi(p) = \lim_{\substack{n\to\infty \\ n.p \text{ integer}}} \frac{-1}{n} \log \inf\{\|s\|_{\sup} | s \in V_{(n\alpha, n\beta, n\gamma, n\delta),\mathbb{Z}}\}.$$

It can be shown that

1. The value we are looking for is $A = \sup_{x\in\Delta} \varphi(x)$.
2. The function φ is concave.
3. The function φ is symmetric under the action of the symmetric group $\mathfrak{S}_4$.

These properties imply that $A = \varphi(1/2, 1/2, 1/2, 1/2)$. That is, the asymptotic of the length of the shortest vector can be computed using only the middle weight. In other words we have established

Lemma 9.7. *Let μ_n be the shortest vector of $V_{(n,n,n,n)} \cap R_{\mathbb{Z},2n}$, then*

$$A = \lim_{n\to\infty} \frac{-1}{2n} \log \|\mu_n\|_{\sup}.$$

For a good description of Arakelov theory and GIT quotients, the reader may consult the monograph [17] or [16].

We now look more closely to the structure of the middle weight subspaces. The direct sum

$$V = \bigoplus_n V_{(n,n,n,n)}$$

is also a graded algebra like R. It is generated by the monomials $x_{12}x_{34}$ and $x_{14}x_{23}$. Therefore $\mathrm{Proj}(V) \sim \mathbb{P}^1$. The inclusion $V \to R$ defines a birational map $\mathrm{Gr}(2,4) \dashrightarrow \mathbb{P}^1$ that can be seen as a GIT quotient. Let $T(3) = T(4) \cap \mathrm{SL}(4)$ the three dimensional torus of diagonal matrices with determinant one. Then $V = R^{T(3)}$ is the subspace of invariant functions with respect to $T(3)$. Thus $\mathbb{P}^1 = \mathrm{Gr}(2,4)//T(3)$ or also $\mathbb{P}^1 = \mathrm{Gr}(2,4)^{\mathrm{ss}}/T(3)$, where $\mathrm{Gr}(2,4)^{\mathrm{ss}}$ is the subset of semistable points. Let $\pi\colon \mathrm{Gr}(2,4)^{\mathrm{ss}} \to \mathbb{P}^1$ be the quotient map. Let $\mathcal{O}_{\mathbb{P}^1}(1)$ be the tautological line bundle on $\mathbb{P}^1 = \mathrm{Proj}(V)$. The global sections of $\mathcal{O}_{\mathbb{P}^1}(n)$ are exactly the elements of $V_{(n,n,n,n)}$. The map π identifies $\pi^*\mathcal{O}_{\mathbb{P}^1}(n) = \mathcal{O}_{\mathrm{Gr}(2,4)}(2n)$. Given a section $s \in V_{(n,n,n,n)}$, the section $\pi^* s$ is just the same element viewed in R_{2n} by the inclusion $V_{(n,n,n,n)} \hookrightarrow R_{2n}$. We can define a new metric on $\mathcal{O}_{\mathbb{P}^1}(1)$, the Kempf-Ness metric by the rule

$$\|s(p)\|_{\mathrm{KN}} = \sup_{\pi(q)=p} \|\pi^* s(q)\|.$$

This norm can be computed explicitly. Let $(z:1) \in \mathbb{P}^1$. A pre-image of this point in $\mathrm{Gr}(2,4)$ is the point $(z:1:(1-z):1:1:1))$. The torus $T(3)$ is given by $(\lambda_1,\lambda_2,\lambda_3,\lambda_4)$ with $\lambda_1\lambda_2\lambda_3\lambda_4 = 1$, and the action is given by

$$(\lambda_i)\cdot(x_{ij}) = (\lambda_i\lambda_j x_{ij}).$$

The norm on $\mathcal{O}_{\mathrm{Gr}(2,4)}(2n)$ is the Fubini-Study norm. Putting everything together, we deduce that

$$\begin{aligned}\|x_{12}x_{34}(z:1)\|_{\mathrm{KN}} &= \sup \frac{|\lambda_1\lambda_2\lambda_3\lambda_4 z|}{|\lambda_1\lambda_2 z|^2 + |\lambda_1\lambda_3|^2 + |\lambda_1\lambda_4(1-z)|^2 + |\lambda_2\lambda_3|^2 + |\lambda_2\lambda_4|^2 + |\lambda_3\lambda_4|^2} \\ &= \frac{|z|}{2(1+|z|+|1-z|)}.\end{aligned}$$

The first equality is just the definition of the Kempf-Ness norm and the second follows from the arithmetic-geometric mean inequality

$$\begin{aligned}&|\lambda_1\lambda_2 z|^2 + |\lambda_1\lambda_3|^2 + |\lambda_1\lambda_4(1-z)|^2 + |\lambda_2\lambda_3|^2 + |\lambda_2\lambda_4|^2 + |\lambda_3\lambda_4|^2 \leq \\ &\quad 2\sqrt{|\lambda_1\lambda_2\lambda_3\lambda_4 z|^2} + 2\sqrt{|\lambda_1\lambda_3\lambda_2\lambda_4|^2} + 2\sqrt{|\lambda_1\lambda_4(1-z)\lambda_2\lambda_3|^2} = \lambda_1\lambda_2\lambda_3\lambda_4(|z|+1+|1-z|).\end{aligned}$$

With equality at the points

$$\lambda_1 = \frac{1}{\sqrt[4]{|1-z||z|}},\ \lambda_2 = \sqrt[4]{\frac{|1-z|}{|z|}},\ \lambda_3 = \sqrt[4]{\frac{|z|}{|1-z|}},\ \lambda_4 = \sqrt[4]{|1-z||z|}.$$

Similarly

$$\|x_{13}x_{24}(z:1)\|_{\mathrm{KN}} = \frac{1}{2(1+|z|+|1-z|)}.$$

By the very definition of the Kempf-Ness metric we deduce that, for every $s \in H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(n))$, the sup norms

$$\sup_{p\in\mathbb{P}^1} \|s(p)\|_{\mathrm{KN}} = \sup_{q\in\mathrm{Gr}(2,4)} \|\pi^* s(q)\|.$$

In consequence

$$A = \lim_{n\to\infty} \frac{-1}{2n} \log \inf \left\{ \sup_p \|s(p)\|_{\mathrm{KN}} \middle| s \in \mathrm{H}^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(n))_{\mathbb{Z}} \right\}. \tag{9.8}$$

We write

$$g(z) = -\log \|x_{13}x_{24}(z:1)\| = \log(2(1+|z|+|1-z|)).$$

If we now combine equation (9.8) with Theorem 4.4 we obtain the next result.

Proposition 9.9. *For every n, let λ_n be the shortest non zero vector of the lattice R_n with the unique Hermitian metric normalized as $\|x_{12}^n\| = 1$. Then*

$$\lim_{n\to\infty} \frac{-1}{n} \|\lambda_n\|_{L^2} = \frac{1}{2} \operatorname{ess}(h_g).$$

We can now make a wild guess. One would expect the minimal vector to be symmetric. So we may guess that the vector

$$v_{6n} = (x_{12}x_{13}x_{14}x_{23}x_{24}x_{34})^n \in V_{(3n,3n,3n,3n)} \subset R_{6n}$$

should be close to being the shortest one. We can compute easily

$$\lim_{n\to\infty} \frac{-1}{6n} \log \|v_{6n}\| = \frac{\log 6}{2} = 0.89587\ldots.$$

Can we use our knowledge of the essential minimum of P^1 to beat this estimate and have a rough idea of the value of A?

Before estimating the essential minimum we can make a further simplification. We observe that the Kempf-Ness metric on $\mathcal{O}(1)$ is invariant under the action of the 6-elements group generated by the operations

$$z \mapsto 1-z, \qquad z \mapsto 1/z.$$

We can consider the map $\varphi\colon \mathbb{P}^1 \to \Pi^1$ given by

$$z \mapsto t = \varphi(z) = \frac{(z^2-z+1)^3}{z^2(1-z)^2}.$$

this map can be seen as the quotient of $\mathbb{P}^1$ by the action of the said group. Therefore, there exists a metric on $\mathcal{O}(1)$, that we denote by $\|\cdot\|_{\mathrm{KN},d}$ such that

$$\varphi^*(\mathcal{O}(1), \|\cdot\|_{\mathrm{KN},d}) = (\mathcal{O}_{\mathbb{P}^1}(6), \|\cdot\|_{\mathrm{KN}}). \tag{9.10}$$

Let $(x:y)$ be the coordinates in the target projective space and write

$$g_d(t) = -\log \|y(t)\|_{\mathrm{KN},d}.$$

Here the subindex d stands for "down" as it is a metric on the quotient space. The equation (9.10) implies that

$$A = \frac{1}{2}\operatorname{ess}(h_g) = \frac{1}{12}\operatorname{ess}(h_{g_d}).$$

Now we can apply the lower and upper bounds we know for the essential minimum of h_{g_d}. As a first choice, take the single polynomial $Q(t) = t$ and compute

$$L = \sup_a \inf_t g_d(t) - a\log|t|.$$

It turns out that the supremum is attained at $a = 0.06979\ldots$ and has the value

$$L = g_d(1) = 10.87406\ldots$$

Hence $A \geq L/12 = 0.90617\ldots$, which shows that our wild guess was wrong and that there should be much shorter vectors. Playing with some more polynomials one can get a better bound $A \geq 0.90753\ldots$, but it seems difficult to improve this lower bound.

With respect to the upper bounds we can start by integrating sets of capacity 1. By a computer assisted search we can find a set K of capacity 1 such that

$$\frac{1}{12}\int g_d \, \mathrm{d}\mu_K = 0.90785\ldots$$

Combining the lower and upper bounds for the essential minimum one easily gets

$$0.90753\cdots \leq A \leq 0.90785\ldots$$

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