Exact volume-law entangled eigenstates in a large class of spin models

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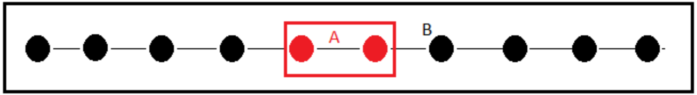
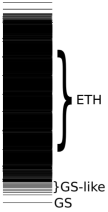
Ref.: S. Mohapatra, S. Moudgalya and Ajit C. Balram, arXiv:2410.22773

Thermalization is governed by Eigenstate thermalization hypothesis (ETH)

- For any subsystem, the rest of the system acts as a bath





$$\lim_{t\to\infty}\rho_A(t) = Tr_B(\rho^{eq}) \approx \rho_A^{eq}, \rho^{eq} = \frac{1}{Z}e^{-\beta H}$$

- expectation value of local operators matches their thermal expectation value.

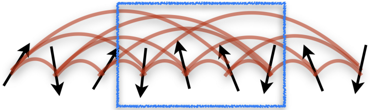
$$\langle E_\alpha|\hat{O}|E_\alpha\rangle = \frac{1}{Z}Tr(\hat{O}e^{-\beta_\alpha H})$$

Eigenstate thermalization hypothesis (ETH): every eigenstate of the thermal system is thermal

Entanglement entropy-based diagnostics of thermalization

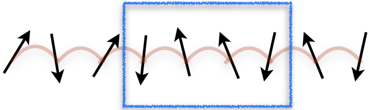
- chaotic/ergodic: volume law of EE

$$S_{ent}(A) \propto vol(A) \propto L^d$$

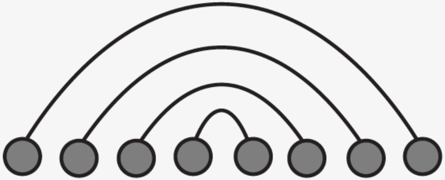


- MBL: area law of EE

$$S_{ent}(A) \propto area(A) \propto L^{d-1}$$



Examples of states with volume-law entanglement: 1) rainbow scars

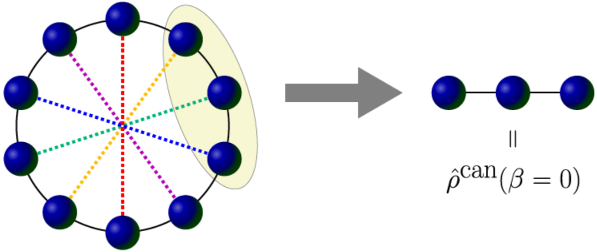


$$\frac{1}{\sqrt{2}}\left(|\uparrow,\uparrow\rangle + |\downarrow,\downarrow\rangle\right)$$

XYZ spin chains with next-nearest neighbor $z-z$ interaction

Langlett *et al.*, Phys. Rev. B **105**, L060301 (2022)

Examples of states with volume-law entanglement: 2) EAP state



$$H_1(L) = \sum_{i=1}^{L} \left(J_1 \sigma_i^x \sigma_{i+1}^y + J_2 \sigma_i^y \sigma_{i+1}^z \right)$$

dotted line is the Bell state $|\Psi^+\rangle_{i,j} = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)_{i,j}$

Phys. Rev. Lett. **133**, 246605 (2024)

Generic construction: definition of spin-chain Hamiltonians

- Consider a broad class of spin-S chains of size L with a translation invariant interaction [(with periodic boundary conditions (PBC)] of the form

$$H(L) = \sum_{\alpha,i=1}^{L} J_\alpha \hat{H}^{\alpha}_{i+j_1,\cdots,i+j_n} = \sum_{\alpha,i=1}^{L} J_\alpha \prod_{k=1}^{n} (S^{\mu}_{i+j_k})^q, \tag{1}$$

  $S^{\mu}_{j}(\mu{=}x,y,z) \rightarrow$ spin-S operator at site j and q is a non-negative integer.
  $\hat{H}^{\alpha}_{i+j_1,\cdots,i+j_n} \rightarrow$ interaction between any $n-$spins at sites $\{i{+}j_1,\cdots,i{+}j_n\}$ with $0{\leq}j_k{<}L$.
- The Hamiltonian is defined in the Hilbert space $\mathcal{H}_L$ of dimension $\mathcal{D}_L{=}(2S{+}1)^L$ on the global computational basis $|\vec{S}\rangle{=}\bigotimes_{i=1}^{L}|s_i\rangle$ ($\mathcal{H}_L{\equiv}\text{span}\{|\vec{S}\rangle\}$), where $|s_i\rangle$ defines the local Hilbert space $(2S{+}1)-$ of dimension at site i.
- for example: for a spin-1/2 Hamiltonian, the local Hilbert space at site i can be defined by eigenvectors of σ^z_i : $\{|s_i\rangle\} = \{|0\rangle, |1\rangle\}$.
  The global computational basis for 2 sites: $\mathcal{H}_L = \{|\vec{S}\rangle\} = \{|00\rangle, |01\rangle, |10\rangle, |11\rangle\}$

Generic construction: an example

$$H(L) = J_1 \sum_i (S_i^x)^2 S_{i+n}^y + J_2 \sum_i S_i^z$$

- $\hat{H}_{i,i+n}^{\alpha_1} = (S_i^x)^2 S_{i+n}^y$
- $\hat{H}_i^{\alpha_2} = S_i^z$
- $\sum_i S_i^x S_i^y$
- $\sum_i S_i^y (S_{i+1}^z S_{i+1}^x)^3$

Any spin-1/2 Hamiltonian can be recast to Eq. (1)

Does not exhaust the possibilities for higher spin-S Hamiltonians

Generic construction: criteria that needs to be met

If each $\hat{H}^{\alpha}_{i+j_1,\cdots,i+j_n}$ in the Hamiltonian $H(L)$ anti-commutes with an operator $\mathcal{C}\mathcal{K}$, where $\mathcal{C}=\prod_{i=1}^{L}\mathcal{C}_i$ with $\mathcal{C}_i$ onsite invertible operators, and $\mathcal{K}$ is the complex conjugation operator, i.e.,

$$\{\hat{H}^{\alpha}_{i+j_1,\cdots,i+j_n},\mathcal{C}\mathcal{K}\}=0 \text{ for } i\in\{1,\cdots,L\}, \tag{2}$$

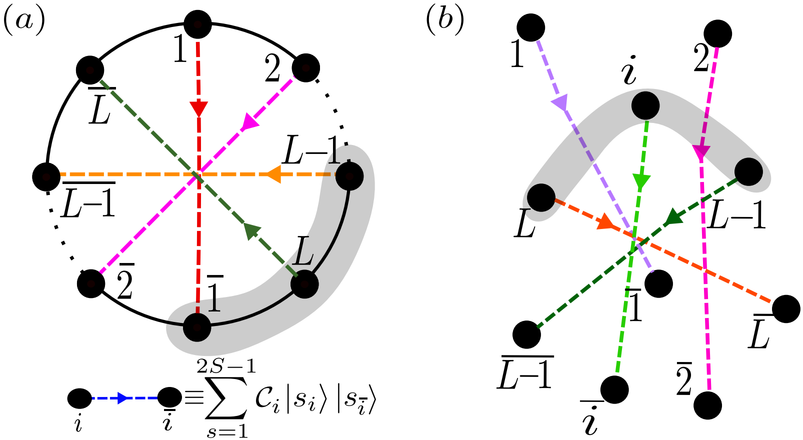
then the state

$$|\Lambda\rangle_{\mathcal{C}}=\frac{1}{\sqrt{\mathcal{D}_N}}\sum_{|\vec{S}\rangle\in\mathcal{H}_L}\mathcal{C}|\vec{S}\rangle_{1,\cdots,L}\otimes|\vec{S}\rangle_{\bar{1},\cdots,\bar{L}}=\frac{1}{\sqrt{\mathcal{D}_L}}\bigotimes_{k=1}^{L}\sum_{s=1}^{2S+1}\mathcal{C}_i|s_i\rangle|s_{\bar{i}}\rangle, \tag{3}$$

where $\bar{i}\equiv i+L$, is an exact zero-energy eigenstate of the Hamiltonian $H(2L)$ defined on a system of size $2L$ with PBC. Owes to a pair-wise cancellation $\hat{H}^{\alpha}_{i+j_1,\cdots,i+j_n}|\Lambda\rangle_{\mathcal{C}}=-\hat{H}^{\alpha}_{\overline{i+j_1},\cdots,\overline{i+j_n}}|\Lambda\rangle_{\mathcal{C}}$

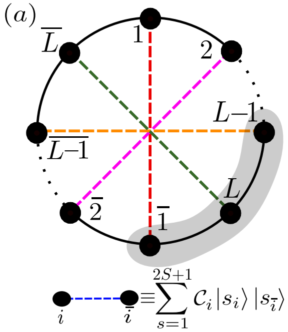
S. Mohapatra, S. Moudgalya and Ajit C. Balram, arXiv:2410.22773

Pictorial depiction of the state



S. Mohapatra, S. Moudgalya and Ajit C. Balram, arXiv: 2410.22773

State $|\Lambda\rangle_{\mathcal{C}}$ is volume-law entangled



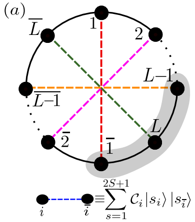
$|\Lambda\rangle_{\mathcal{C}}$ is product of “dimers” between i and $\bar{i}$

contiguous system of N sites cuts N bonds $\implies$ $\mathcal{S}_N \propto N$

S. Mohapatra, S. Moudgalya and Ajit C. Balram, arXiv: 2410.22773

State $|\Lambda\rangle_{\mathcal{C}}$ is thermal for strictly local observables

The reduced density matrix over any contiguous subsystem is just proportional to the identity matrix, which is just the infinite-temperature Gibbs density matrix, hence the state is thermal for strictly local observables.



For tailored subsystems chosen to consist solely of sites $\{i, \bar{i}\}$, $\mathcal{S}$ is strictly zero, and non-local observables with support over these subsystems exhibit highly athermal properties.

S. Mohapatra, S. Moudgalya and Ajit C. Balram, arXiv: 2410.22773

Corollary to the theorem

The state

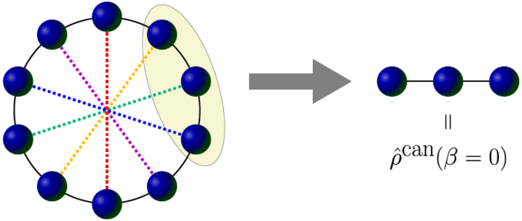
$$|\Lambda\rangle_{\mathbb{I}}=\frac{1}{\sqrt{\mathcal{D}_L}}\sum_{|\vec{S}\rangle}|\vec{S}\rangle\otimes|\vec{S}\rangle=\frac{1}{\sqrt{\mathcal{D}_L}}\bigotimes_{i=1}^{L}\sum_{s_i=1}^{2S+1}|s_i\rangle|s_{\bar{i}}\rangle \tag{4}$$

is always a zero-energy eigenstate of any Hamiltonian of the form presented defined on an even number of lattice sites ($2L$) where each $\hat{H}^{\alpha}_{i+j_1,\cdots,i+j_n}$ has a purely imaginary matrix representation in the computational basis.

This is the state $|\Lambda\rangle_{\mathcal{C}}$ with $\mathcal{C}=\mathbb{I}$ for which the anti-commutation condition always holds.

EAP state from our theorem

$H_1(L) = \sum_{i=1}^{L} \left(J_1 \sigma_i^x \sigma_{i+1}^y + J_2 \sigma_i^y \sigma_{i+1}^z\right)$ is purely imaginary in the computational basis use corollary with $\mathcal{C} = \mathbb{I}$ to get EAP state



dotted line is the Bell state $|\Psi^+\rangle_{i,j} = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)_{i,j}$

$|\Lambda(S=1/2)\rangle_{\mathbb{I}}$ is a zero-energy eigenstate of *any* spin-1/2 Hamiltonian with even number of sites that has an *odd* number of σ_y's in each $\hat{H}^{\alpha}_{i+j_1,\cdots,i+j_n}$

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Many other examples (some hold even for $S>1/2$)

- transverse-field Ising model,
- spin-1/2 model with three-spin interactions Udupa *et. al*, Phys. Rev. B **108**, 214430 (2023),
- PXP model Andrew N. Ivanov and Olexei I. Motrunich, Phys. Rev. Lett. **134**, 050403(2025),
- spin-S XY model,
- spin-S Kitaev chain
- $\vdots$

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Summary and future outlook

- Constructed a large class of spin chain Hamiltonians with exact volume-law entangled excited eigenstates
- Extensions to higher dimensions.
- Generalization to fermionic and bosonic systems.
- Algorithms to find such states given the Hamiltonian (analogous to ScarFinder)

  Petrova *et al.*, arXiv:2504.12472

  Ren *et al.*, arXiv:2504.12383

Acknowledgments

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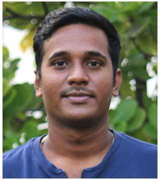
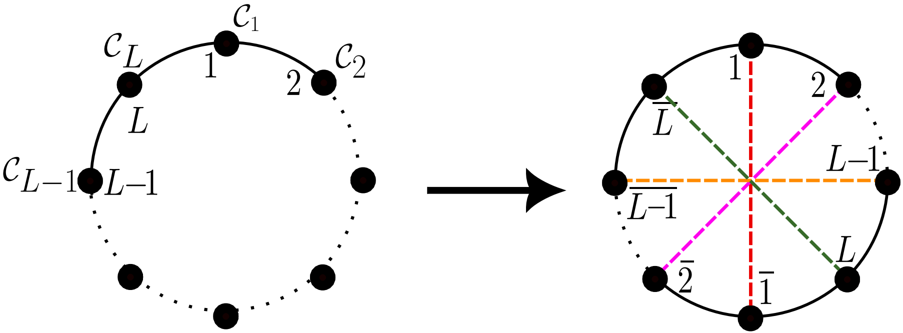
Sashikanta Mohapatra and Sanjay Moudgalya




- Supported by:

Thanks!

Pictorial depiction of the state



$$det(\mathcal{C}_i) \neq 0$$

$$\mathcal{C}(\hat{H}^{\alpha}_{i+j_1,\cdots,i+j_n})^* = -\hat{H}^{\alpha}_{i+j_1,\cdots,i+j_n}\mathcal{C}$$

$$i \text{ --- } \bar{i} \equiv \sum_{s=1}^{2S-1} \mathcal{C}_i \left|s_i\right\rangle \left|s_{\bar{i}}\right\rangle$$

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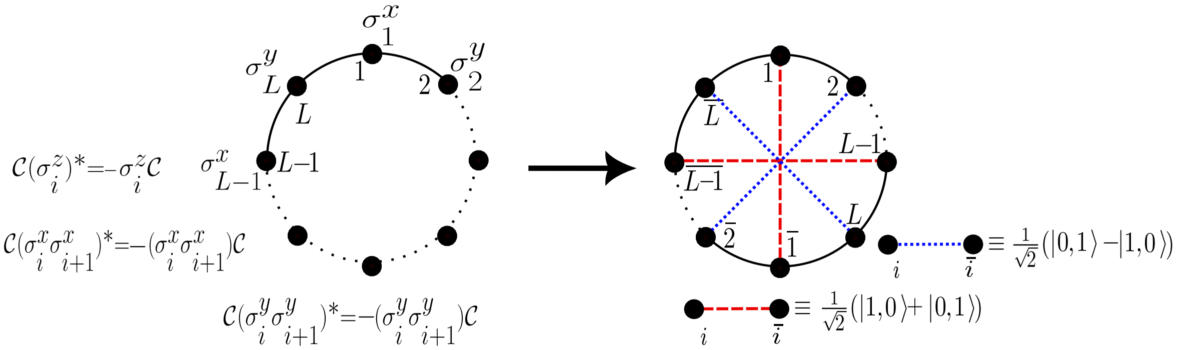
Simple example: spin-1/2 XY chain

- $H_{XY}(L) = \sum_{i=1}^{L} \left(J_1 \sigma_i^x \sigma_{i+1}^x + J_2 \sigma_i^y \sigma_{i+1}^y + h \sigma_i^z \right)$, $\{\hat{H}^\alpha\} = \{\{\sigma_i^x \sigma_{i+1}^x\}, \{\sigma_i^y \sigma_{i+1}^y\}, \{\sigma_i^z\}\}$
- When $L = even$, $\mathcal{C}_{XY} = \prod_{i=1}^{L/2} \sigma_{2i-1}^x \sigma_{2i}^y$ satisfies Eq. (2)

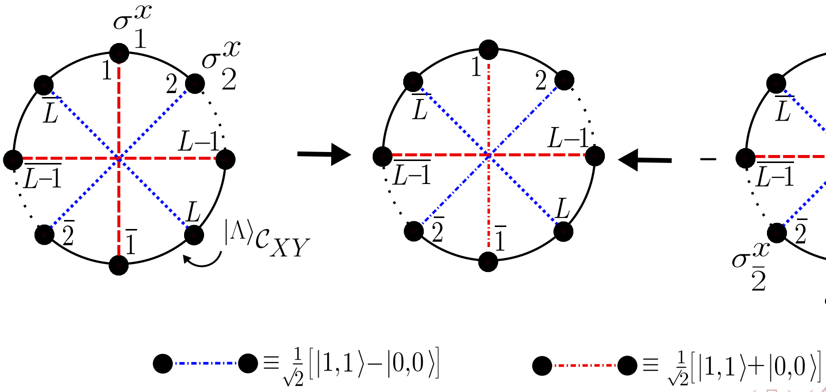
$$\implies |\Lambda\rangle_{\mathcal{C}_{XY}} = \bigotimes_{i=1}^{L-1} |\phi^+\rangle_{i,\bar{i}} |\phi^-\rangle_{i+1,\overline{i+1}} \text{ where } |\Phi^\pm\rangle_{i,j} = \frac{(|01\rangle \pm |10\rangle)_{i,j}}{\sqrt{2}}$$



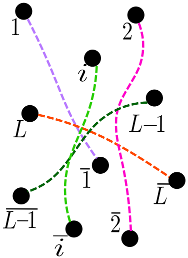
Proof

- Hamiltonian for the 2L-site with PBC: $H(2L) = \sum_{\alpha,i=1}^{L} (\hat{H}^{\alpha}_{i+j_1,\cdots,i+j_n} + \hat{H}^{\alpha}_{\overline{i+j_1},\cdots,\overline{i+j_n}})$
- For all i: $\hat{H}^{\alpha}_{i+j_1,\cdots,i+j_n} |\Lambda\rangle_{\mathcal{C}} = -\hat{H}^{\alpha}_{\overline{i+j_1},\cdots,\overline{i+j_n}} |\Lambda\rangle_{\mathcal{C}}$

E.g.



Generalization to arbitrary dimensions

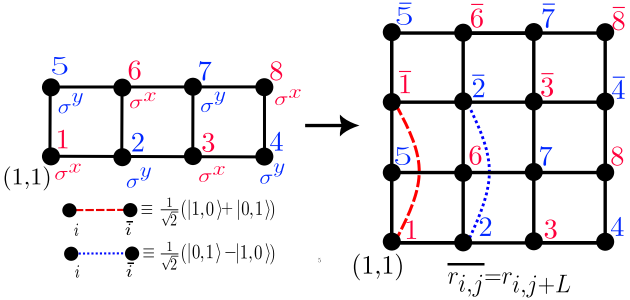


- $\{\hat{H}^{\alpha}_{j_1,\cdots,j_n}, \mathcal{C}\mathcal{K}\}=0$ (with $\mathcal{C}_i=\mathcal{C}_{\bar{i}}$) $\Longrightarrow (\hat{H}^{\alpha}_{j_1,\cdots,j_n} + \hat{H}^{\alpha}_{\bar{j}_1,\cdots,\bar{j}_n})|\Lambda\rangle_{\mathcal{C}}=0$ (where $j_k \in \{1, \bar{1}, \cdots, L, \bar{L}\}$ and $j_k \neq \bar{j}_{k'}$ for all k, k')
- All Hamiltonians of the form $H=\sum_{\alpha,\{j\}} J_\alpha(\hat{H}^{\alpha}_{j_1,\cdots,j_n}+\hat{H}^{\alpha}_{\bar{j}_1,\cdots,\bar{j}_n})$ with arbitrary coefficients J_α, host $|\Lambda\rangle_{\mathcal{C}}$ as a zero energy eigenstate.

S. Mohapatra, S. Moudgalya and Ajit C. Balram, arXiv:2410.22773

Example: 2D spin-1/2 XY model

- $H^{2D}_{XY}(M\times L) = \sum_{\eta,i,j} J_\eta(\sigma^\eta_{r_{i,j}}\sigma^\eta_{r_{i+1,j}} + \sigma^\eta_{r_{i,j}}\sigma^\eta_{r_{i,j+1}}) + h\sum_{i,j}\sigma^z_{r_{i,j}}$
  $\eta\in\{x,y\}$ and $r_{i,j}=i+M\times(j-1)$ with $i\in\{1,\cdots,M\}$ (horizontal) and $j\in\{1,\cdots,L\}$ (vertical)



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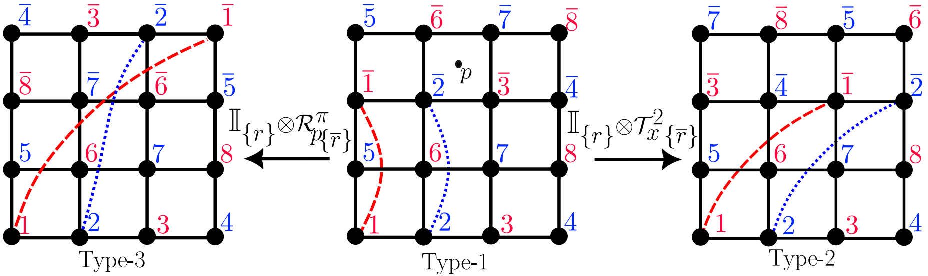
Additional classes of zero energy using lattice symmetry



The operator $\mathcal{T}^{x,M/2}_{\{\bar{r}\}}$ translates all the $\{\bar{r}\}$ sites by $M/2$ units in the horizontal direction and the operator $\mathcal{R}^{p,\pi}_{\{\bar{r}\}}$ rotates all the $\{\bar{r}\}$ sites around the point $p{=}[(M{+}1)/2,(3L{+}1)/2]$ by an angle π.

More states using full translations and rotations of each type.

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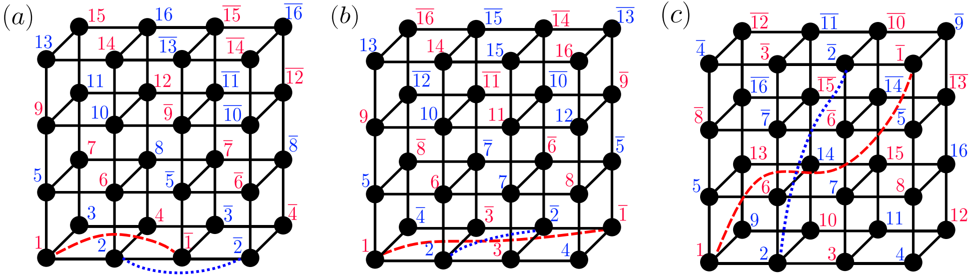
- The number of such zero energy eigenstates scales with the volume of the system $\mathcal{O}(ML)$.
- Similar construction for other geometries



Unlike the one-dimensional case, in higher dimensions, there can be special contiguous bipartitions for which the entanglement is not a volume law, although it is for most of them.

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