Collinear gluon field:

The relevant two-point function to consider is:

general cov. gauge

$$\langle 0 | T\{ A_c^{\mu,a}(x)\, A_c^{\nu,b}(0) \} | 0 \rangle = \int \underbrace{\frac{d^4p}{(2\pi)^4}}_{\lambda^4}\, e^{-ip\cdot x}\, \underbrace{\frac{i\,\delta_{ab}}{p^2+i\epsilon}}_{\lambda^{-2}} \Big[\underbrace{-g^{\mu\nu}}_{1} + (1-\xi)\, \frac{p^\mu p^\nu}{p^2} \Big]$$

various scalings

It would be incorrect to work in Feynman gauge ($\xi = 1$) and conclude that $A_a^\mu \sim \lambda$ (as suggested by NDA).

Rather, we need to decompose:

$$A_c^\mu = n\cdot A_c \,\frac{\bar{n}^\mu}{2} + \bar{n}\cdot A_c\, \frac{n^\mu}{2} + A_{c,\perp}^\mu$$

It follows from above that:

$$\underbrace{\langle 0 | T\{ n\cdot A_c(x)\; n\cdot A_c(0) \} | 0 \rangle}_{\boxed{\sim \lambda^4}} = \int \underbrace{\frac{d^4p}{(2\pi)^4}}_{\lambda^4}\, e^{-ip\cdot x}\, \underbrace{\frac{i\,\delta_{ab}}{p^2+i\epsilon}}_{\lambda^{-2}} \Big[0 + (1-\xi)\, \underbrace{\frac{(n\cdot p)^2}{p^2}}_{\frac{\lambda^4}{\lambda^2}} \Big]$$

$$\underbrace{\langle 0 | T\{ \bar{n}\cdot A_c(x)\; \bar{n}\cdot A_c(0) \} | 0 \rangle}_{\boxed{\sim \lambda^0}} = \int \underbrace{\frac{d^4p}{(2\pi)^4}}_{\lambda^4}\, e^{-ip\cdot x}\, \underbrace{\frac{i\,\delta_{ab}}{p^2+i\epsilon}}_{\lambda^{-2}} \Big[0 + (1-\xi)\, \underbrace{\frac{(\bar{n}\cdot p)^2}{p^2}}_{\frac{\lambda^0}{\lambda^2}} \Big]$$

$$\underbrace{\langle 0 | T\{ A_{c,\perp}^{\mu,a}(x)\; A_{c,\perp}^{\nu,b}(0) \} | 0 \rangle}_{\boxed{\sim \lambda^2}} = \int \underbrace{\frac{d^4p}{(2\pi)^4}}_{\lambda^4}\, e^{-ip\cdot x}\, \underbrace{\frac{i\,\delta_{ab}}{p^2+i\epsilon}}_{\lambda^{-2}} \Big[\underbrace{-g_\perp^{\mu\nu}}_{1} + (1-\xi)\, \underbrace{\frac{p_\perp^\mu p_\perp^\nu}{p^2}}_{\frac{\lambda^2}{\lambda^2}} \Big]$$

same

$$\Rightarrow \quad \boxed{\; n\cdot A_c \sim \lambda^2 \,, \qquad \bar{n}\cdot A_c \sim 1 \,, \qquad A_{c,\perp}^\mu \sim \lambda \;}$$

This also works for the mixed terms, e.g.:

$$\underbrace{\langle 0|T\{n\cdot A_c(x)\,\bar{n}\cdot A_c(0)\}|0\rangle}_{\sim\lambda^2} = \int \underbrace{\frac{d^4p}{(2\pi)^4}}_{\lambda^4}\, e^{-ip\cdot x}\, \underbrace{\frac{i\delta_{ab}}{p^2+i\epsilon}}_{\lambda^{-2}} \times \Big[\underbrace{-\,n\cdot\bar{n}}_{1} + (1-\xi)\underbrace{\frac{n\cdot p\;\bar{n}\cdot p}{p^2}}_{\frac{\lambda^2\cdot 1}{\lambda^2}} \Big]$$

etc.

We observe that the different components of the collinear gluon field scale like the components of a collinear momentum, such that the covariant collinear derivative

$$\boxed{iD_c^\mu = i\partial^\mu + g_s A_c^{\mu,a} t^a \sim (\lambda^2, 1, \lambda)}$$

has homogeneous power counting in λ !

Important note:

The presence of a field with unsuppressed power counting, $\bar{n}\cdot A_c \sim 1$, is worrisome, since this will lead to an infinite number of operators with the same power counting in the EFT. We will see the implications of this observation later.

Ultra-soft quark field:

We have:

$$\langle 0 | T\{ q_{us}(x)\, \bar{q}_{us}(0) \} | 0 \rangle = \int \underbrace{\frac{d^4k}{(2\pi)^4}}_{\lambda^8} \underbrace{e^{-ik\cdot x}}_{1} \underbrace{\frac{i \not{k}}{k^2 + i\epsilon}}_{\frac{\lambda^2}{\lambda^4}} \sim \lambda^6$$

This gives:

$$\boxed{q_{us} \sim \lambda^3}$$

There are no large or small components in this case.

Ultra-soft gluon fields:

We find:

$$\langle 0 | T\{ A_{us}^{\mu,a}(x), A_{us}^{\nu,b}(0) \} | 0 \rangle = \int \underbrace{\frac{d^4k}{(2\pi)^4}}_{\lambda^8} e^{-ik\cdot x} \underbrace{\frac{i\delta_{ab}}{k^2 + i\epsilon}}_{\lambda^{-4}} \Big[\underbrace{-g^{\mu\nu}}_{1} + (1-\xi) \underbrace{\frac{k^\mu k^\nu}{k^2}}_{\lambda^0} \Big]$$

$$\sim \lambda^4$$

This gives:

$$\boxed{A_{us}^{\mu,a} \sim \lambda^2}$$

In the following we will derive the effective Lagrangian of collinear and ultra-soft fields, focussing on the Dirac Lagrangian. The Yang-Mills Lagrangian can be discussed in an analogous way.

Introducing the decompositions

$$\psi \;\to\; \psi_c + q_{us} \;=\; \xi_n + \eta_n + q_{us}$$

$$A^\mu \;\to\; A^\mu_c + A^\mu_{us}$$

in the Dirac Lagrangian, we obtain:

$$\bar\psi\, i\not{D}\,\psi \;\to\; \big(\overset{\lambda}{\bar\xi_n} + \overset{\lambda^2}{\bar\eta_n} + \overset{\lambda^3}{\bar q_{us}}\big)\; i\not{D}_{c+us}\; \big(\overset{\lambda}{\xi_n} + \overset{\lambda^2}{\eta_n} + \overset{\lambda^3}{q_{us}}\big)$$

where:

$$i\not{D}_{c+us} = \big(\overset{\lambda^2}{i n\cdot D_c} + \overset{\lambda^2}{g_s\, n\cdot A_{us}}\big)\frac{\not{\bar n}}{2} + \big(\overset{1}{i\bar n\cdot D_c} + \overset{\lambda^2}{g_s\,\bar n\cdot A_{us}}\big)\frac{\not{n}}{2}$$

$$+\; \overset{\lambda}{i\not{D}_c^\perp} + \overset{\lambda^2}{g_s\,\not{A}^\perp_{us}}$$

$\mathcal{O}(\lambda^4)$:

$$\bar\xi_n \frac{\not{\bar n}}{2}\big(i n\cdot D_c + g_s\, n\cdot A_{us}\big)\xi_n + \bar\eta_n \frac{\not{n}}{2}\, i\bar n\cdot D_c\, \eta_n$$

$$+\; \bar\xi_n\, i\not{D}_c^\perp\, \eta_n + \bar\eta_n\, i\not{D}_c^\perp\, \xi_n$$

$\mathcal{O}(\lambda^5)$:

$$\bar\xi_n\, g_s \not{A}^\perp_{us}\, \eta_n + \bar q_{us}\, g_s\not{A}^\perp_c\, \xi_n + \bar q_{us}\frac{\not{n}}{2} g_s\,\bar n\cdot A_c\, \eta_n + \text{h.c.}$$

In the $\mathcal{O}(\lambda^5)$ Lagrangian we have used that terms involving only a single collinear field are not allowed by momentum conservation. Note that <u>all</u> terms of $\mathcal{O}(\lambda^5)$ and higher contain at least one ultra-soft field.

From now on we focus on the leading-order SCET Lagrangian:

$$\mathcal{L}_{SCET} = \mathcal{L}_c + \mathcal{L}_{us} + \mathcal{L}_{c+us}$$

with:

$$\mathcal{L}_c = \bar{\xi}_n \frac{\bar{n}\!\!\!/}{2}\, i n\cdot D_c\, \xi_n + \bar{\eta}_n \frac{n\!\!\!/}{2}\, i \bar{n}\cdot D_c\, \eta_n$$
$$+ \bar{\xi}_n\, i \not{D}_c^{\perp}\, \eta_n + \bar{\eta}_n\, i\not{D}_c^{\perp}\, \xi_n + (\text{pure glue terms}) \sim \lambda^4$$

$$\mathcal{L}_{c+us} = \bar{\xi}_n \frac{\bar{n}\!\!\!/}{2}\, g_s\, n\cdot A_{us}\, \xi_n + (\text{pure glue terms}) \sim \lambda^4$$

$$\mathcal{L}_{us} = \bar{q}_{us}\, i\not{D}_{us}\, q_{us} + (\text{pure glue terms}) \sim \lambda^8$$

The leading-order action is:

$$S_{SCET} = \underset{\lambda^{-4}}{\int d^4x} \left(\underbrace{\mathcal{L}_c + \mathcal{L}_{c+us}}_{\lambda^4} \right) + \underset{\lambda^{-8}}{\int d^4x}\, \underset{\lambda^{8}}{\mathcal{L}_{us}} \sim \lambda^0$$

Since the field ξ_n contains the large components of the collinear spinor field, we can use it to describe collinear quarks and integrate out the power-suppressed field η_n in the generating functional. Like in HQET, the functional determinant is just an irrelevant (divergent) constant. The resulting Lagrangian is obtained by using the solution of the classical equation of motion:

$$\frac{\delta \mathcal{L}_c}{\delta \bar{\eta}_n} = 0 \quad \Rightarrow \quad \frac{\not{n}}{2}\, i\, \underset{1}{\bar{n}\cdot D_c}\, \underset{\lambda^3}{\eta_n} + i\, \underset{\lambda}{\not{D}_c^{\perp}}\, \underset{\lambda^2}{\xi_n} = 0$$

To solve this equation for η_n we introduce an auxilliary regulator $i\delta$ to obtain:

$$\underbrace{\frac{\not{\bar{n}}\not{n}}{4}}_{\text{replace by: } P_{\bar{n}} + P_n = 1}\left(i\bar{n}\cdot D_c + i\delta\right)\eta_n = -\frac{\not{\bar{n}}}{2}\, i\not{D}_c^{\perp}\, \xi_n$$

P_n ← vanishes when acting on η_n

$$\Rightarrow \quad \boxed{\eta_n = -\frac{1}{\underbrace{i\bar{n}\cdot D_c}_{\mathcal{O}(\lambda^0)} + i\delta}\, \frac{\not{\bar{n}}}{2}\, i\not{D}_c^{\perp}\, \xi_n}$$

$i\delta$ ← arbitrary sign, since pole is unphysical

It is instructive to compare this to the corresponding expression in HQET:

$$H_v = \frac{1}{\underset{1}{2m_Q} + \underset{\lambda}{i v\cdot D_s}}\; i\not{D}_s^{\perp}\, h_v$$

In that case, the inverse differential operator could be expanded in powers of $iv\cdot D_s/m_Q = \mathcal{O}(\lambda)$. In the case of SCET, such an expansion parameter is lacking.

Inserting the above solution into our expression for $\mathcal{L}_c$, we find:

$$\mathcal{L}_c = \bar{\xi}_n \frac{\not{\bar{n}}}{2}\, i n\cdot D_c\, \xi_n - \bar{\xi}_n\, i \not{D}_c^{\perp} \frac{\not{\bar{n}}}{2} \frac{1}{i\bar{n}\cdot D_c + i\delta}\, i\not{D}_c^{\perp}\, \xi_n$$

$$+ \text{(pure glue terms)}$$

The inverse of a derivative is an integral, but what is the inverse of a covariant derivative?

To define the above expression properly, we introduce the <u>collinear Wilson line</u>:

large component $\bar{n}\cdot A_c \sim \lambda^0$ light-like direction $\bar{n}^\mu$

$$\boxed{W_c(x) = \mathbb{P}\exp\left(i g_s \int_{-\infty}^{0} dt\ \bar{n}\cdot A_c(x + t\bar{n})\right)}$$

This is analogous to the definition of the soft Wilson line in HQET (see lecture 2, p. 6). It follows that:

$$\left[i\bar{n}\cdot D_c\, W_c(x)\right] = 0$$

$$\Rightarrow\quad i\bar{n}\cdot D_c\, W_c(x)\, \phi_c(x) = W_c(x)\, i\bar{n}\cdot\partial\, \phi_c(x)$$

arb. function of collinear fields

As a differential operator, it follows that:

$$\boxed{W_c^\dagger(x)\ i\bar{n}\cdot D_c\ W_c(x)\ =\ i\bar{n}\cdot\partial}$$

This in turn implies:

$$\frac{1}{i\bar{n}\cdot D_c + i\delta} = W_c \frac{1}{i\bar{n}\cdot\partial + i\delta} W_c^\dagger$$

↳ proof: apply $W_c^\dagger\ i\bar{n}\cdot D_c \ldots W_c$ on both sides

The second term in the Lagrangian can now be written in the form:

$$\bar{\xi}_n\ i\not{D}_c^\perp\ \frac{\not{\bar{n}}}{2}\ \frac{1}{i\bar{n}\cdot D_c + i\delta}\ i\not{D}_c^\perp\ \xi_n(x)$$

$$= \bar{\xi}_n\ i\not{D}_c^\perp\ W_c\ \frac{\not{\bar{n}}}{2}\ \frac{1}{i\bar{n}\cdot\partial + i\delta}\ W_c^\dagger\ i\not{D}_c^\perp\ \xi_n(x)$$

$$= \left(\bar{\xi}_n\ i\not{D}_c^\perp\ W_c\right)(x)\ \frac{\not{\bar{n}}}{2}\ (-i)\int_{-\infty}^{0} dt\ \left(W_c^\dagger\ i\not{D}_c^\perp\ \xi_n\right)(x+t\bar{n})$$

↳ check:

$$i\bar{n}\cdot\partial_x\ (-i)\int_{-\infty}^{0} dt\ \phi(x+t\bar{n}) = \bar{n}\cdot\partial_x \int_{-\infty}^{\frac{n\cdot x}{2}} dt'\ \phi\!\left(\frac{\bar{n}\cdot x}{2} n + x_\perp + t'\bar{n}\right)$$

$$= \frac{\bar{n}\cdot n}{2}\ \phi(x) = \phi(x)$$

Note that the lower integration limit "$-\infty$" is appropriate for our choice of the "$i\delta$" regulator. If the collinear fields $(W_c^\dagger i\not{D}_c^\perp \xi_n)(x+t\bar{n})$ carry total momentum p_c, then the t-integral gives:

$$(-i)\int_{-\infty}^{0} dt\; e^{-ip_c\cdot(x+t\bar{n})}$$

regulator to ensure convergence at $t\to-\infty$

$$= (-i)\, e^{-ip_c\cdot x}\int_{-\infty}^{0} dt\; e^{-i(\bar{n}\cdot p_c + i\delta)\,t} = \frac{e^{-ip_c\cdot x}}{\bar{n}\cdot p_c + i\delta}$$

This indeed corresponds to the action of the inverse differential operator:

$$\frac{1}{i\bar{n}\cdot\partial + i\delta}\; e^{-ip_c\cdot x}$$

This leads to the final form of the leading-order SCET Lagrangian:

$$\boxed{\begin{aligned}\mathcal{L}_c(x) &= \bar{\xi}_n \frac{\not{\bar{n}}}{2}\, in\cdot D_c\, \xi_n(x)\\ &\quad + (\bar{\xi}_n\, i\not{D}_c^\perp W_c)(x)\,\frac{\not{\bar{n}}}{2}\, i\int_{-\infty}^{0} dt\, (W_c^\dagger\, i\not{D}_c^\perp\, \xi_n)(x+t\bar{n})\\ &\quad + (\text{pure glue terms})\end{aligned}}$$

non-local !