Detection Principles of Gravitational Wave Observatories

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Contents

Acknowledgment

- *I gratefully acknowledge the discussions and lecture notes of Profs. Gerhard Schäfer, Claus Lämmerzahl, Norbert Straumann, Bernad Schutz* $++$

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  However, any lack of clarity that I may impose on you is entirely my own.

Introduction to GW Polarization States

Linearized Field Equations: I

- It was in 1916 that Einstein predicted the existence of GWs, based on his linearized vacuum field equations

- Let our spacetime metric $g_{\mu\nu}$ be

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu},$$

  where

$$\eta_{\mu\nu} = \text{diag}(-1, 1, 1, 1),$$

- We need $h_{\mu\nu}$ to be so small that all expressions can be linearized with respect to $h_{\mu\nu}$ and their derivatives $\partial_\sigma h_{\mu\nu}$

  In other words, the spacetime is very close to the special relativity spacetime

Linearized Field Equations: II

- We may introduce

$$\gamma_{\mu\nu} = h_{\mu\nu} - \frac{h}{2}\eta_{\mu\nu}.$$

$$\gamma := \eta^{\mu\nu}\gamma_{\mu\nu} = h - \frac{1}{2}4h = -h,$$

$$\Rightarrow h_{\mu\nu} = \gamma_{\mu\nu} - \frac{\gamma}{2}\eta_{\mu\nu}.$$

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$$\Rightarrow h_{\mu\nu} = \gamma_{\mu\nu} - \frac{\gamma}{2}\eta_{\mu\nu}.$$

- EFEs $\Rightarrow$

$$\Box\gamma_{\mu\nu} = 2\kappa T_{\mu\nu},$$

where $\kappa = 8\,\pi\,G/c^4$, $\Box = \partial^\mu\partial_\mu$

- Further, $\gamma_{\mu\nu}$ have to satisfy the additional condition

$$\partial^\mu\gamma_{\mu\nu} = 0$$

This is the *Hilbert gauge*. Its popular names include *Einstein gauge*, the *de Donder gauge*, or the *Fock gauge*.)

- In particular, these Eqs are invariant under Lorentz transformations.

Table 2. The gauge freedom of linearized gravitation is analogous to that of ordinary electromagnetism in flat spacetime.

	Electromagnetism	Linearized gravity
Generator	χ	ξ_a
Potential	A_a	h_{ab}
Gauge transfo.	$A_a \to A_a + \partial_a \chi$	$h_{ab} \to h_{ab} + 2\partial_{(a}\xi_{b)}$
Gauge invariant	$F_{ab} = \partial_{[a} A_{b]}$	$R_{abcd} = -\partial_c \partial_{[a} h_{b]d} + \partial_d \partial_{[a} h_{b]c}$
Lorenz gauge cond.	$\partial^a A_a = 0$	$\partial^a \bar{h}_{ab} = 0$
Conservation law	$\partial^a j_a = 0$	$\partial^a T_{ab} = 0$
Wave equation	$\Box A_a = -\mu_0 \, j_a$	$\Box \bar{h}_{ab} = -16\pi G \, T_{ab}$

Figure: $\Box = \eta^{\mu\,\nu} \partial_\mu \partial_\nu = \partial^\nu \partial_\nu$

GWs: I

- Influenced by ED, the general solution to the linearized EFEs could be a superposition of plane harmonic waves

$$\gamma_{\mu\nu}(x) = \mathrm{Re}\{A_{\mu\nu} e^{ik_\rho x^\rho}\}$$

k_ρ: a real covariant vector a complex amplitude $A_{\mu\nu} = A_{\nu\mu}$.

- $\Rightarrow$

$$0 = \eta^{\sigma\tau}\partial_\sigma\partial_\tau\gamma_{\mu\nu}(x) = \mathrm{Re}\{\eta^{\sigma\tau} A_{\mu\nu} ik_\sigma ik_\tau e^{ik_\rho x^\rho}\}.$$

This should hold for all x, with $A_{\mu\nu} \neq 0$, if and only if

$$\eta^{\sigma\tau} k_\sigma k_\tau = 0.$$

- In other words, (k_0, k_1, k_2, k_3) has to be a lightlike covariant vector with respect to the Minkowskian metric.

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GWs propagate on Minkowski spacetime at the speed c, just as electromagnetic waves in vacuum

GWs: II

- $\partial^\mu \gamma_{\mu\nu} = 0 \Rightarrow$

$$0 = \eta^{\mu\tau} \partial_\tau \gamma_{\mu\nu}(x) = \mathrm{Re}\{\eta^{\mu\tau} A_{\mu\nu} i k_\tau e^{i k_\rho x^\rho}\}$$

  which is true, for all $x = (x^0, x^1, x^2, x^3)$, if and only if

$$k^\mu A_{\mu\nu} = 0 \quad \text{(H)}.$$

- In other words, the Hilbert gauge condition restricts the possible values of amplitudes $A_{\mu\nu}$
- It restricts the possible GW polarization states

GWs: III

- Clearly, $A_{\mu\nu}$ is a (4×4) matrix with 16 entries.
  $A_{\mu\nu} = A_{\nu\mu} \rightarrow$ only 10 of them are independent
  the Hilbert gauge condition (H) consists of 4 scalar equations
  Are there six independent polarization states?

GWs: III

- Clearly, $A_{\mu\nu}$ is a (4×4) matrix with 16 entries.
  $A_{\mu\nu} = A_{\nu\mu} \rightarrow$ only 10 of them are independent
  the Hilbert gauge condition (H) consists of 4 scalar equations
  Are there six independent polarization states?
- Introduce u^μ: a constant four-velocity vector, $\eta_{\mu\nu} u^\mu u^\nu = -c^2$.
  It is possible to make a coordinate transformation such that the Hilbert gauge condition is preserved &

$$u^\mu A_{\mu\nu} = 0, \quad \text{(T1)}$$

$$\eta^{\mu\nu} A_{\mu\nu} = 0, \quad \text{(T2)}$$

  in the new coordinates
- In the (*TT gauge, transverse-traceless gauge*), there are ONLY TWO GW polarization states

Existence of TT Gauge: I

- Plan is to perform a coordinate transformation

$$x^\mu \mapsto x^\mu + f^\mu(x), \quad f^\mu(x) = \mathrm{Re}\{iC^\mu e^{ik_\rho x^\rho}\}$$

  with the wave covariant vector (k_ρ) from our plane-harmonic-wave solution & certain complex coefficients C^μ.

- We want to choose the C^μ such that in the new coordinates (T1) and (T2) hold true

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- We want to choose the C^μ such that in the new coordinates (T1) and (T2) hold true
- Is it possible?
-

$$\gamma_{\mu\nu} \mapsto \gamma_{\mu\nu} - \partial_\nu f_\mu - \partial_\mu f_\nu + \eta_{\mu\nu}\partial_\rho f^\rho,$$

  $\Rightarrow$

$$\mathrm{Re}\{A_{\mu\nu}e^{ik_\rho x^\rho}\} \mapsto \mathrm{Re}\left\{ (A_{\mu\nu} - iik_\nu C_\mu - iik_\mu C_\nu + \eta_{\mu\nu} iik_\rho C^\rho)\, e^{ik_\rho x^\rho}\right\},$$

Existence of TT Gauge: II

- We know that $\gamma_{\mu\nu}$ should transform as

$$\gamma_{\mu\nu} \mapsto \gamma_{\mu\nu} - \partial_\nu f_\mu - \partial_\mu f_\nu + \eta_{\mu\nu}\partial_\rho f^\rho,$$

$\Rightarrow$

$$\mathrm{Re}\{A_{\mu\nu}e^{ik_\rho x^\rho}\} \mapsto \mathrm{Re}\left\{ \left(A_{\mu\nu} - iik_\nu C_\mu - iik_\mu C_\nu + \eta_{\mu\nu} iik_\rho C^\rho\right) e^{ik_\rho x^\rho}\right\},$$

$$A_{\mu\nu} \mapsto A_{\mu\nu} + k_\mu C_\nu + k_\nu C_\mu - \eta_{\mu\nu}k_\rho C^\rho.$$

- We want to choose the C_μ such that Eqs (T1) & (T2) should hold $\Rightarrow$

$$0 = u^\mu \left(A_{\mu\nu} + k_\mu C_\nu + k_\nu C_\mu - \eta_{\mu\nu}k_\rho C^\rho\right), \quad \text{(T1)}$$

$$0 = \eta^{\mu\nu} \left(A_{\mu\nu} + k_\mu C_\nu + k_\nu C_\mu - \eta_{\mu\nu}k_\rho C^\rho\right) = \eta^{\mu\nu}A_{\mu\nu} - 2k_\rho C^\rho, \quad \text{(T2)}$$

Components of C_μ : I

- Let our constant four-velocity vector

$$(u^\mu) = (c, 0, 0, 0)$$

- (T1) for $\nu = j$:

$$A_{0j} + k_0 C_j + k_j C_0 = 0 \quad \Longleftrightarrow \quad C_j = -k_0^{-1}(A_{0j} + k_j C_0).$$

$\Rightarrow C_j$ are determined by C_0.

- (T1) for $\nu = 0$:

$$A_{00} + 2k_0 C_0 + \eta^{\rho\sigma} k_\rho C_\sigma = 0 \quad \Longleftrightarrow$$

$$A_{00} + 2k_0 C_0 - k_0 C_0 + \eta^{ij} k_i C_j = 0 \quad \Longleftrightarrow$$

$$-k_0 A_{00} + \eta^{ij} k_i A_{0j} = 0 \quad \Longleftrightarrow$$

$$\eta^{\mu\nu} k_\mu A_{\nu\sigma} = 0.$$

& this is the Hilbert gauge condition $k^\mu A_{\mu\nu} = 0$.

Components of C_μ : I

- Can we fix C_0 using T2? $\left(\eta^{\mu\nu} A_{\mu\nu} - 2k_\rho C^\rho = 0\right)$
- $$\eta^{\mu\nu} A_{\mu\nu} + 2k_0 C_0 - 2\eta^{ij} k_i C_j = 0 \quad \Longleftrightarrow$$
$$A^{\mu\nu} + 2k_0 C_0 + 2\eta^{ij} k_i k_0^{-1} (A_{0j} + k_j C_0) = 0 \quad \Longleftrightarrow$$
$$A^{\mu\nu} + 2k_0 C_0 + 2\eta^{ij} k_i k_0^{-1} A_{0j} - 2\eta^{00} k_0 k_0^{-1} k_0 C_0 = 0 \quad \Longleftrightarrow$$
$$A^{\mu\nu} + 4k_0 C_0 + 2\eta^{ij} k_i k_0^{-1} A_{0j} = 0 \quad \Longleftrightarrow$$
$$C_0 = \frac{-A^{\mu\nu} - 2\eta^{ij} k_i A_{0j}}{4k_0^2}.$$
- $\Rightarrow$ We should be able to 'construct/choose' C_0 which leads to C_j & this ensures that both (T1) and (T2) are indeed satisfied in the new coordinates.

GW Polarization States: I

- Use of TT gauge $\Rightarrow \gamma = 0$ and thus $h_{\mu\nu} = \gamma_{\mu\nu}$.
  $\Rightarrow$
  $$g_{\mu\nu} = \eta_{\mu\nu} + \gamma_{\mu\nu}, \quad \gamma_{\mu\nu} = \mathrm{Re}\left\{A_{\mu\nu} e^{ik_\rho x^\rho}\right\}$$
- The amplitudes are restricted by the conditions
  $$k^\mu A_{\mu\nu} = 0, \quad u^\mu A_{\mu\nu} = 0, \quad \eta^{\mu\nu} A_{\mu\nu} = 0.$$
- If we choose the coordinates such that
  $$(u^\mu) = \begin{pmatrix} c \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad (k^\rho) = \begin{pmatrix} \omega/c \\ 0 \\ 0 \\ \omega/c \end{pmatrix},$$

  In General, a Lorentz transformation should allow us to choose such 4-vectors

GW Polarization States: II

- In the TT gauge, the amplitudes $A_{\mu\nu}$ must satisfy

$$\text{(H)} \quad 0 = k^\mu A_{\mu\nu} = \frac{\omega}{c}(A_{0\nu} + A_{3\nu}),$$

$$\text{(T1)} \quad 0 = u^\mu A_{\mu\nu} = cA_{0\nu},$$

$$\text{(T2)} \quad 0 = \eta^{\mu\nu} A_{\mu\nu} = -A_{00} + A_{11} + A_{22} + A_{33},$$

- In this representation, there are only two non-zero components of $A_{\mu\nu}$,

$$A_{11} = -A_{22} \coloneqq A_+ = |A_+|e^{i\varphi},$$

$$A_{12} = A_{21} \coloneqq A_\times = |A_\times|e^{i\psi}.$$

The fact that only the 11 and the 12 components are non-zero demonstrates that GWs are transverse.
These two independent polarization states are
the plus mode $(+)$ & the cross mode $(\times)$.

It's explanation later

+ and × GW polarization states

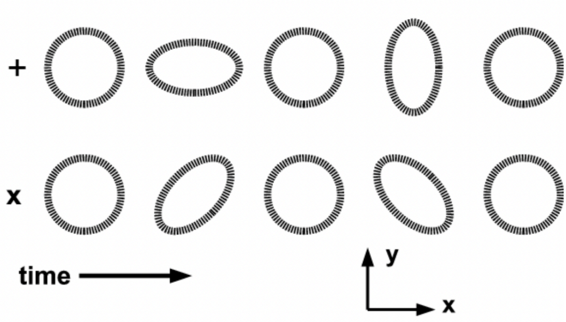

Figure: GW propagates along the z-axis

GWI: I

- We need to show that the x^0-lines, namely the worldlines $x^\mu(\tau)$ with $\dot{x}^\mu = u^\mu$, are geodesics.
- From $\dot{x}^\mu(\tau) = u^\mu$, we find $\ddot{x}^\mu(\tau) = 0$
  The Christoffel symbols read

$$\Gamma^\mu_{\nu\sigma} = \frac{1}{2} g^{\mu\tau} \left(\partial_\sigma g_{\tau\nu} + \partial_\nu g_{\tau\sigma} - \partial_\tau g_{\nu\sigma} \right)$$

$$= \frac{1}{2} \eta^{\mu\tau} \left(\partial_\sigma \gamma_{\tau\nu} + \partial_\nu \gamma_{\tau\sigma} - \partial_\tau \gamma_{\nu\sigma} \right)$$

$$= \frac{1}{2} \eta^{\mu\tau} \text{Re} \left\{ \left(ik_\sigma A_{\tau\nu} + ik_\nu A_{\tau\sigma} - ik_\tau A_{\nu\sigma} \right) e^{ik_\rho x^\rho} \right\}.$$

- $$\ddot{x}^\mu + \Gamma^\mu_{\nu\sigma} \dot{x}^\nu \dot{x}^\sigma$$

$$= 0 + \frac{1}{2} \eta^{\mu\tau} \text{Re} \left\{ \left(ik_\sigma A_{\tau\nu} u^\nu u^\sigma + ik_\nu A_{\tau\sigma} u^\nu u^\sigma - ik_\tau A_{\nu\sigma} u^\nu u^\sigma \right) e^{ik_\rho x^\rho} \right\}.$$

$\Rightarrow$

$$\ddot{x}^\mu + \Gamma^\mu_{\nu\sigma} \dot{x}^\nu \dot{x}^\sigma = 0.$$

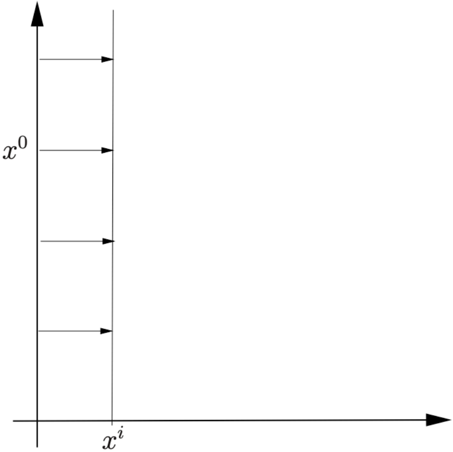

Figure: The x^0-lines are the worldlines of freely falling particles. For any such particle the (x^1, x^2, x^3)-coordinates remain constant. This does not mean that GWs have no effect on freely falling particles. The distance, as measured with the metric, between neighboring x^0-lines is not at all constant.

GWI: II

- Let us calculate the square of the distance between the x^0-line at the spatial origin $(0,0,0)$ and a neighboring x^0-line at (x^1, x^2, x^3)
- Imagine that the x^i are so small that the metric can be viewed as constant between 0 and x^i.

$$g_{ij}(x^0,0,0,0)(x^i-0)(x^j-0) =$$

$$\left(\eta_{ij} + \gamma_{ij}(x^0,0,0,0)\right) x^i x^j = \delta_{ij} x^i x^j + \text{Re}\left\{A_{ij} x^i x^j e^{ik_0 x^0}\right\}$$

$$= \delta_{ij} x^i x^j + \text{Re}\left\{A_+ \left((x^1)^2 - (x^2)^2\right) e^{-i\omega t}\right\} + \text{Re}\left\{2A_\times x^1 x^2 e^{-i\omega t}\right\}$$

- $\Rightarrow$

$$g_{ij}(x^0,0,0,0)(x^i-0)(x^j-0)$$

$$= \delta_{ij} x^i x^j + |A_+| \left((x^1)^2 - (x^2)^2\right) \cos(\varphi - \omega t) + 2|A_\times| x^1 x^2 \cos(\psi - \omega t).$$

$\Rightarrow$

$$g_{ij}(x^0,0,0,0)(x^i-0)(x^j-0)$$
$$=\delta_{ij}x^ix^j+|A_+|\left((x^1)^2-(x^2)^2\right)\cos(\varphi-\omega t)+2|A_\times|x^1x^2\cos(\psi-\omega t).$$

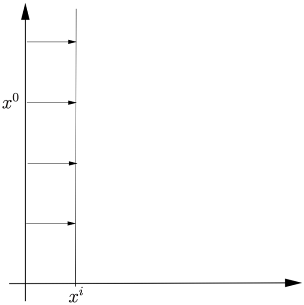

Figure: Imagine that you have particles that are arranged on a small spherical shell and then released to free-fall
Both plus & cross modes of GWs produce a time-periodic elliptic deformation in the plane perpendicular to the propagation direction.

Interferometric Hecto-Hz GW Observatories

Michelson Interferometer: I

The idea of employing MI for detecting GWs was given in M. Gerstenshtein and V. Pustovoit ["On the detection of low-frequency gravitational waves" (in Russian), Sov. Phys. JETP 16, 433 (1962)].

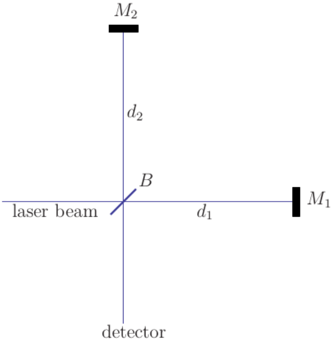

Figure: Influenced by V. Braginsky, J. Forward actually built a small model detector in the mid-1970s. The construction of LIGO, GEO600, VIRGO, and TAMA started in the 1990s after Joe Taylor's Nobel Prize!! Many people were critically instrumental, including R. Weiss and R. Drever & K. Thorne.

Michelson Interferometer: II

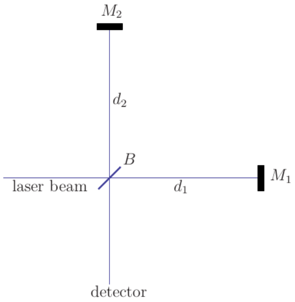

Figure: If MI is operated in vacuo, the phase difference is

$$\delta\phi = (2\,\pi/\lambda) \times (d_1 - d_2) \times 2$$

Typical (sophisticated) MIs should measure phase differences down to 10^{-5}.
$\Rightarrow$ You could detect changes in $d_1 - d_2$ that are are considerably smaller than the laser wavelength λ(600 nanometers) with the help of interference patterns

Michelson Interferometer: III

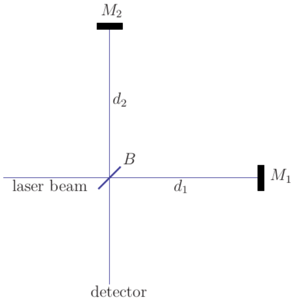

Figure: Imagine the beam splitter B, the mirrors M_1 & M_2 as being suspended so that they can move freely in the plane of the MI.
Under the influence of a passing GW whose propagation direction is orthogonal to the MI plane, the mirrors will move!
We may identify B with the particle at the center of the coordinate system, while M_1 & M_2 with particles on the x & y-axes

Michelson Interferometer: IV

Aim is to determine the time-dependence of the distances d_1 and d_2,

- Recall that the distance from the origin to a particle with coordinates x^i should be

$$\delta_{k\ell} y^k(t) y^\ell(t) = \delta_{k\ell} x^k x^\ell + \gamma_{k\ell} x^k x^\ell = \delta_{k\ell} x^k x^\ell + \mathrm{Re}\left\{ A_k^\ell e^{-i\omega t} \right\} \delta_{k\ell} x^j x^\ell .$$

- Let's restrict to a pure + mode:

$$\delta_{k\ell} y^k(t) y^\ell(t) = \delta_{k\ell} x^k x^\ell + \mathrm{Re}\left\{ A_+ \left((x^1)^2 - (x^2)^2 \right) e^{-i\omega t} \right\}$$

- Let $A_+ = |A_+| e^{i\varphi} \Rightarrow$

$$\delta_{k\ell} y^k(t) y^\ell(t) = \delta_{k\ell} x^k x^\ell + |A_+| \left((x^1)^2 - (x^2)^2 \right) \cos(\omega t - \varphi).$$

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- Let $A_+ = |A_+| e^{i\varphi} \Rightarrow$

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- For the mirror M_1, we have $x^2 = x^3 = 0 \Rightarrow$

$$d_1(t)^2 = (x^1)^2 + |A_+| (x^1)^2 \cos(\omega t - \varphi)$$

Michelson Interferometer: V

- For the mirror M_1, we have $x^2 = x^3 = 0 \Rightarrow$

$$d_1(t)^2 = (x^1)^2 + |A_+|(x^1)^2 \cos(\omega t - \varphi)$$

  & for the mirror M_2, we have $x^1 = x^3 = 0 \Rightarrow$

$$d_2(t)^2 = (x^2)^2 - |A_+|(x^2)^2 \cos(\omega t - \varphi).$$

- Let the unperturbed length of both arms be d_0: $\Rightarrow$

$$d_1(t)^2 = d_0^2 \left(1 + |A_+| \cos(\omega t - \varphi)\right),$$

$$d_2(t)^2 = d_0^2 \left(1 - |A_+| \cos(\omega t - \varphi)\right).$$

Michelson Interferometer: VI

- The induced phase difference reads

$$\Delta\phi(t) = \frac{4\pi}{\lambda}\left(d_1(t) - d_2(t)\right)$$

$$= \frac{4\pi}{\lambda} d_0 \left(\sqrt{1 + |A_+|\cos(\omega t - \varphi)} - \sqrt{1 - |A_+|\cos(\omega t - \varphi)}\right)$$

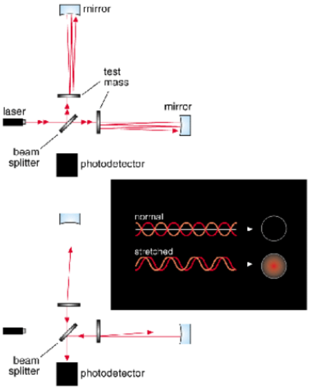
- We should linearise all expressions while dealing with our GWs

$$\Delta\phi(t) = \frac{4\pi}{\lambda} d_0 \left(1 + \frac{1}{2}|A_+|\cos(\omega t - \varphi) - 1 + \frac{1}{2}|A_+|\cos(\omega t - \varphi) + \dots\right)$$

$$= \frac{4\pi}{\lambda} d_0 |A_+|\cos(\omega t - \varphi).$$

- $\Delta\phi \propto A_+, d_0$ & inversely $\propto$ to the wavelength λ of the laser. This is why GWIs should have long armlength
- The GW frequency ω only provides certain periodicity with which the interference pattern changes.

Slides from Alan Weinstein



The effects of gravitational waves appear as a deviation in the phase differences between two orthogonal light paths of an interferometer.

For expected signal strengths, The effect is *tiny*:

Phase shift of ~10^{-10} radians

The longer the light path, the larger the phase shift…

Make the light path as long as possible!

LIGO-G000164-00-R *AJW, Caltech, LIGO Project* 6

Figure: Mirrors are essentially suspended pendulums

Optimal L values?: I

- Using a pendulum of length $l = 50\,\mathrm{cm}$,

$$f_0 = \frac{1}{2\pi}\sqrt{\frac{g}{l}} \approx 0.7\,\mathrm{Hz},$$

  $f_0 \sim 1\,\mathrm{Hz}$, so mass/mirror is “free” above $\sim 100\,\mathrm{Hz}$.

- A GW with $f_g \sim 100\,\mathrm{Hz} \implies \lambda_g \sim 3000\,\mathrm{km}$ produces a tiny strain $h = \Delta L/L$!

- We measure

$$\Delta\phi = 4\pi\frac{\Delta L}{\lambda_{\text{laser}}} = 4\pi\frac{Lh}{\lambda_{\text{laser}}},$$

  For measuring small h, we need large L.

- **But not too large!** If $L > \lambda_g/4$, the GW changes sign while the laser light is still in arms, canceling the effect on $\Delta\phi$.
- Optimal: $L \sim \lambda_g/4 \sim 750\,\mathrm{km}$!!!

Optimal L values?: II

- For more practical length ($L \sim 4\,\mathrm{km}$), increase phase sensitivity by increasing number of times light beam hits mirror!!

$$\Delta\phi = 4\pi \frac{\Delta L}{\lambda_{\text{laser}}} \implies \Delta\phi = N \left(4\pi \frac{\Delta L}{\lambda_{\text{laser}}} \right),$$

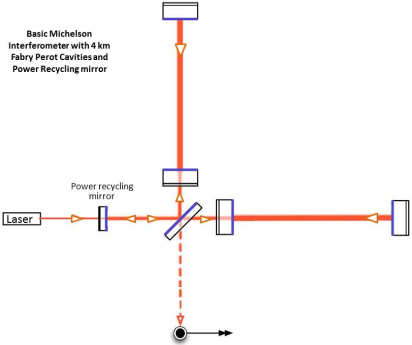


Figure: light is phase-shifted N times the single-pass length difference ΔL

Optimum L Values?: III

- One should NOT make the light path arbitrarily long!
- Time for the light wavefront to leave BS (Beam Splitter) & return:

$$\tau_{\text{stor}} \sim \frac{2NL}{c}, \quad N \text{ is the number of round trips in arms.}$$

- During that time, a GW may reverse sign, canceling the effect on the light phase:

$$\tau_{\text{rev}} \sim \frac{T_{\text{period}}}{2} = \frac{1}{2f}.$$

- For Hecto-Hz GW observatories like LIGO,

$$f_{\text{pole}} \sim 100\,\text{Hz}\,(\text{“knee”}).$$

- $\Rightarrow$

$$\tau_{\text{stor}} < \frac{1}{2f_{\text{pole}}}, \quad \text{or } N < \frac{c}{4Lf_{\text{pole}}} \approx 200 \text{ for LIGO.}$$

Some Interesting Points: B. Schutz

- It's natural to describe how MI interacts with a passing GW in terms of time-delays
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- It is rather incorrect to imagine that MI essentially measures the number of wavelengths of light that "fit" along an arm and then looks for changes in that number as the mirrors swing.
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- If this were the case, then the frequency of the light would have to be stable to parts in 10^{21}, the same accuracy as that of the "length" measurement. *There are no lasers of such stability*
- It could be argued that GWs will affect the speed of light and thus corrupt or negate the measured time-delay.
  In our TT coordinates the free particles remain at fixed locations & indeed the coordinate speed of light changes.

Interferometric Milli-Hz GW Observatories

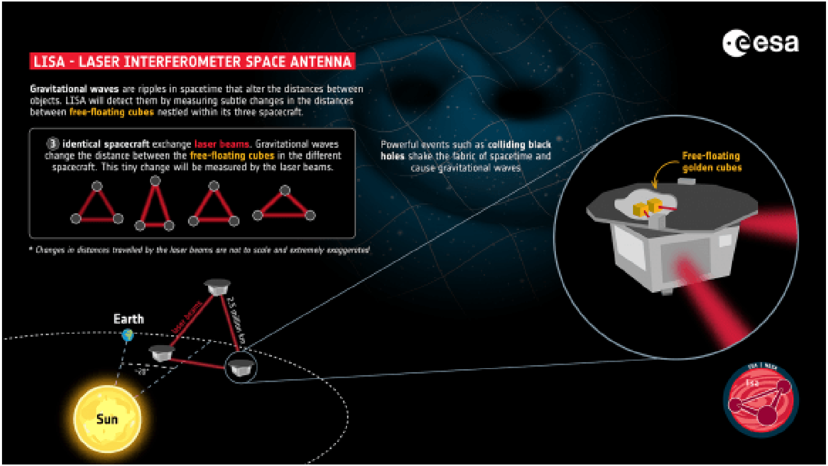
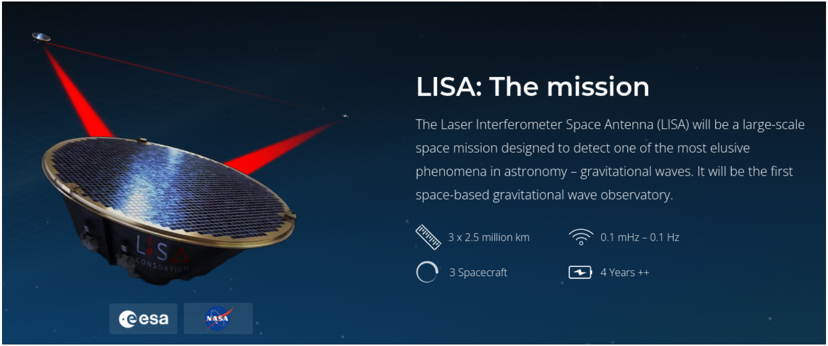


Figure: Caption

It's an amazing GW observatory(ies) between $1 \rightarrow 10$ milli-Hz


LISA: The mission
The Laser Interferometer Space Antenna (LISA) will be a large-scale space mission designed to detect one of the most elusive phenomena in astronomy – gravitational waves. It will be the first space-based gravitational wave observatory.
3 x 2.5 million km
0.1 mHz – 0.1 Hz
3 Spacecraft
4 Years ++
LISA
CONSORTIUM
esa
NASA

LISA

- LISA consists of 3 independent spacecrafts arrayed as an equilateral triangle, with laser beams along each of the arms.
  Spacecrafts are essentially point particles following geodesics,
- Given its 3 arms, LISA can be regarded as having three independent L-shaped interferometers, made from the arms that join at any two of the three vertices.
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- These interferometers sense two different GW polarization states (without requiring another space interferometer)
- Any two detectors share a common arm, their instrumental noise in the two detectors is not independent.
  $\Rightarrow$ It is NOT advisable to do a cross-correlation experiment between its two detectors in order to improve its sensitivity to a stochastic GW background!!.

How Does LISA Work?: I

- Let one arm of an interferometer is along the x-direction our plane GW is moving in the z-direction
  Restrict to $+$ polarization & $h_+(t)$ doesn't depend on x

- In this case, the null geodesic moving in the x-direction satisfies

$$ds^2 = -dt^2 + (1 + h_+)dx^2 + (1 - h_+)dy^2 + dz^2 = 0,$$

- We extract an effective speed

$$\left(\frac{dx}{dt}\right)^2 = \frac{1}{1 + h_+}.$$

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$$\left(\frac{dx}{dt}\right)^2 = \frac{1}{1 + h_+}.$$

- This is a *coordinate speed*, with no contradiction to special relativity.

How Does LISA Work?: II

- A photon emitted at time t from the origin reaches the other end, at a fixed coordinate position $x = L$, at the coordinate time

$$t_{\mathsf{far}} = t + \int_0^L \left[1 + h_+(t(x))\right]^{1/2} dx,$$

  where $t(x)$ denotes the fact that one must know the time to reach position x in order to calculate the wave field.

- In our linearized EFEqs, $t(x) = t + x$ and we should expand the square root

$$t_{\mathsf{far}} = t + L + \frac{1}{2}\int_0^L h_+(t + x)dx.$$

- In a Space interferometer, the light is reflected back $\Rightarrow$

$$t_{\mathsf{return}} = t + 2L + \frac{1}{2}\left[\int_0^L h_+(t + x)dx + \int_0^L h_+(t + L + x)dx\right].$$

How Does LISA Work?: III

- For a space-based GW observatory, one essentially monitors changes in the time for the return trip as a function of time at the origin!

$$\frac{dt_{\text{return}}}{dt} = 1 + \frac{1}{2}\left[h_+(t+2L) - h_+(t)\right].$$

Interestingly, it does not involve the wave amplitude at the other end.

- The wave amplitude at the other end does get involved if the wave travels at an angle θ to the z-axis in the $x - z$ plane.

$$\frac{dt_{\text{return}}}{dt} = 1 + \frac{1}{2}\Big\{(1-\sin\theta)h_+(t+2L) - (1+\sin\theta)h_+(t)$$
$$+2\sin\theta h_+\left[t + L(1-\sin\theta)\right]\Big\}.$$

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We need to allow the phase of the wave to depend on x in the appropriate way to incorporate the fact that h_{xx} is comparatively reduced

It's a bit more complicated!!

C. LISA's response function

With its three arms, LISA functions as a pair of two-arm detectors, outputting two orthogonal signals. Let l_1^i, l_2^i, l_3^i be unit vectors, each along one of LISA's three arms, and let L be LISA's average arm length. Let also $L_i(t)$ be the length of the i'th arm when LISA measures an incident GW, and denote $\delta L_1(t) \equiv L_i(t) - L$. We refer to the two-arm detector formed by arms 1 and 2 as "detector I." The strain amplitude in this detector is given by

$$h_I(t) \equiv [\delta L_1(t) - \delta L_2(t)]/L = \frac{1}{2} h_{ij}(t)(l_1^i l_1^j - l_2^i l_2^j). \tag{11}$$

The second, orthogonal signal is then given by [10]

$$h_{II}(t) = 3^{-1/2}[\delta L_1(t) + \delta L_2(t) - 2\,\delta L_3(t)]/L$$
$$= \frac{1}{2\sqrt{3}} h_{ij}(t)(l_1^i l_1^j + l_2^i l_2^j - 2 l_3^i l_3^j). \tag{12}$$

For GW wavelengths much larger than the LISA arm length, $h_I(t)$ and $h_{II}(t)$ coincide with the two "Michelson variables" [26], describing the responses of a pair of two-arm/90° detectors. We can then write $h_I(t)$ and $h_{II}(t)$ as a sum over n-harmonic contributions,

$$h_\alpha(t) = \sum_n h_{\alpha,n}(t) \quad (\alpha = I, II), \tag{13}$$

$$h_{\alpha,n}(t) = \frac{1}{D}\frac{\sqrt{3}}{2}[F_{\alpha}^{+}(t)A_{n}^{+}(t) + F_{\alpha}^{\times}(t)A_{n}^{\times}(t)]. \qquad (14)$$

Here $A_n^{+,\times}(t)$ are the two polarization coefficients [given, in our model, by Eq. (10) above], the factor $\sqrt{3}/2$ accounts for the fact that the actual angle between LISA arms is 60° rather than 90°, and $F_\alpha^{+,\times}$ are the "antenna pattern" functions, reading [23,27]

$$F_I^{+} = \frac{1}{2}(1+\cos^2\theta)\cos(2\phi)\cos(2\psi) - \cos\theta\sin(2\phi)\sin(2\psi),$$

$$F_I^{\times} = \frac{1}{2}(1+\cos^2\theta)\cos(2\phi)\sin(2\psi) + \cos\theta\sin(2\phi)\cos(2\psi), \qquad (15a)$$

$$F_{II}^{+} = \frac{1}{2}(1+\cos^2\theta)\sin(2\phi)\cos(2\psi) + \cos\theta\cos(2\phi)\sin(2\psi),$$

$$F_{II}^{\times} = \frac{1}{2}(1+\cos^2\theta)\sin(2\phi)\sin(2\psi)$$

In these expressions, (θ,ϕ) is the source's sky location in a detector-based coordinate system and ψ is the "polarization angle" describing the orientation of the "apparent ellipse" drawn by the projection of the orbit on the sky—see Fig. 1 in Ref. [23] and the explicit relation (17) given below.

It is more convenient to express the above response function in terms of angles defined not in the rotating, detector-based system, but rather in a fixed, ecliptic-based coordinate system. The angles θ,ϕ are related to θ_S,ϕ_S—the source location in an ecliptic-based system—through

$$\cos\theta(t)=\frac{1}{2}\cos\theta_S-\frac{\sqrt{3}}{2}\sin\theta_S\cos[\bar{\phi}_0+2\pi(t/T)-\phi_S],$$

$$\phi(t)=\bar{\alpha}_0+2\pi(t/T)+\tan^{-1}\left[\frac{\sqrt{3}\cos\theta_S+\sin\theta_S\cos[\bar{\phi}_0+2\pi(t/T)-\phi_S]}{2\sin\theta_S\sin[\bar{\phi}_0+2\pi(t/T)-\phi_S]}\right], \tag{16}$$

where $T=1$ year and $\bar{\phi}_0,\bar{\alpha}_0$ are constant angles specifying, respectively, the orbital and rotational phase of the detector at $t=0$. (See Cutler [10] for a complete definition of these angles; note, though, that the angle $\bar{\alpha}_0=0$ in this paper is referred to as $\alpha_0=0$ in Cutler [10].)

Figure: Papers by Curt Cutler

LISA & TDI

- In the $1 - 10$ mHz regime, the fact that the signals from the three interferometers are not independent leads to a noise veto:
- There exists a linear combination of the three TDI output signals that cancels out all GW signals

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- There exists a linear combination of the three TDI output signals that cancels out all GW signals

  An amazing way to measure the instrumental noise!!

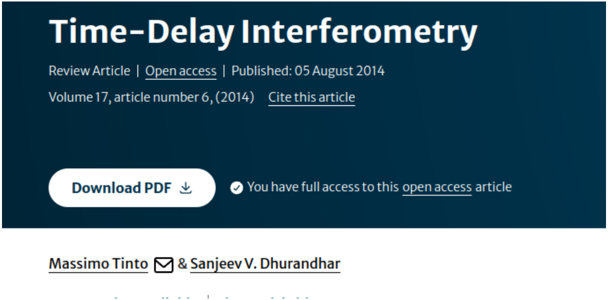

Figure: It should be possible to adapt it for PTAs!!

nano-Hz GW Observatories

PTA: Introduction

- A millisecond pulsar (MSP) emits radio pulses at a rate that is highly stable Stability of their rotational frequency is comparable to the stablity of Atomic Clocks!!
- However, pulses from MSPs does NOT arrive a constant frequency here on our Radio Telescopes like uGMRT!!
- Pulse Times-of-Arrival (TOAs) are affected by many many effects like the relative motion of the pulsar and the Earth, by the influence of the gravitational field of the Sun and of other masses the pulses might pass, and by the interstellar medium $++++$
- We have developed deterministic Timing Formulae to ensure that many such delays are taken into account by regularly monitoring MSPs for years
- In other words, we have approaches to predict 'TOAs' from close to 100 MSPs

- The time-series that arise from the difference between the predicted & observed TOAs ⇒ Pulsar Timing Residuals
- A passing GW should lead to such Timing Residuals
- The idea of searching for GWs in the Pulsar timing residuals of pulsars are due to
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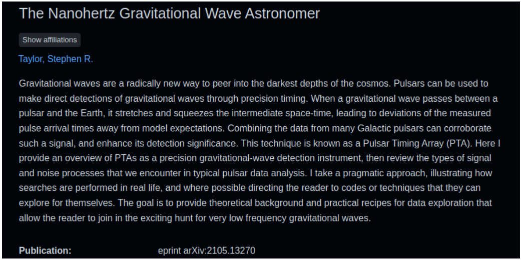
The Nanohertz Gravitational Wave Astronomer

Show affiliations

Taylor, Stephen R.

Gravitational waves are a radically new way to peer into the darkest depths of the cosmos. Pulsars can be used to make direct detections of gravitational waves through precision timing. When a gravitational wave passes between a pulsar and the Earth, it stretches and squeezes the intermediate space-time, leading to deviations of the measured pulse arrival times away from model expectations. Combining the data from many Galactic pulsars can corroborate such a signal, and enhance its detection significance. This technique is known as a Pulsar Timing Array (PTA). Here I provide an overview of PTAs as a precision gravitational-wave detection instrument, then review the types of signal and noise processes that we encounter in typical pulsar data analysis. I take a pragmatic approach, illustrating how searches are performed in real life, and where possible directing the reader to codes or techniques that they can explore for themselves. The goal is to provide theoretical background and practical recipes for data exploration that allow the reader to join in the exciting hunt for very low frequency gravitational waves.

Publication: eprint arXiv:2105.13270

Figure: arXiv:2105.13270

Frame Title

- Let the pulsar and the Earth be at rest in a Minkowski background such that

  $$g_{\mu\nu}(x) = \eta_{\mu\nu} + h_{\mu\nu}(x),$$

  and we employ the TT gauge such that
  A) $h_{0\mu} = 0$ & B) the curves $(x^\mu(t)) = (ct, \vec{r}_0)$ with a constant $\vec{r}_0$, are geodesics.
  (the worldlines of constant spatial coordinates are geodesics)

- Our $h_{\mu\nu}$ is an arbitrary superposition of GWs & Our metric $\rightarrow$

  $$g_{\mu\nu}(x)dx^\mu dx^\nu = -c^2dt^2 + \delta_{ij}dx^i dx^j + h_{ij}(x)x^i dx^j$$

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- Along the wordlines of Pulsar & Earth, the time coordinate t coincides with proper time.
  Therefore, we can identify frequencies with respect to the time coordinate t with frequencies with respect to proper time of the pulsar or of the Earth.

We want to figure out $\nu_E(t_E)$ when linearized GWs are around

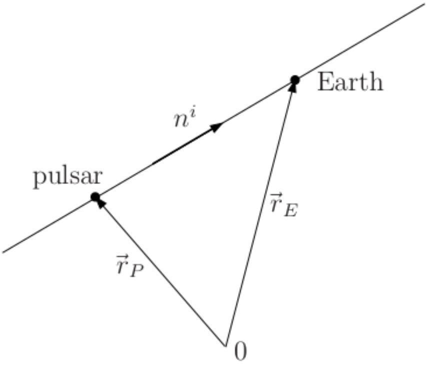

Figure: Pulsar emits signals at a fixed frequency ν_P. These pulses arrive at the Earth with a frequency $\nu_E(t_E)$

Frame Title

- Along a light ray from the pulsar to the Earth, we must have

$$0 = -c^2 dt^2 + \delta_{ij} dx^i dx^j + h_{ij}(x) dx^i dx^j$$

$\Rightarrow$

$$c^2 \left(\frac{dt}{d\ell} \right)^2 = 1 + h_{ij}(x) \frac{dx^i}{d\ell} \frac{dx^j}{d\ell},$$

- We have the arclength with respect to the flat background metric ℓ

$$d\ell^2 = \delta_{ij} dx^i dx^j.$$

- When a GW is around

$$\frac{dx^i}{d\ell} = n^i + O(h),$$

$$\Rightarrow c \frac{dt}{d\ell} = \sqrt{1 + h_{ij}(x) n^i n^j + O(h^2)},$$

Frame Title

- Integrating over the path of the light ray, from its emission time t_P to the arrival time t_E $\rightarrow$

$$\int_{t_P}^{t_E} \left(1 - \frac{1}{2} h_{ij}(x) n^i n^j\right) dt = \frac{L}{c},$$

  where L is the distance from the pulsar to the Earth measured in the flat background.
  We have a way to obtain t_E as a function of t_P.

- Differentiate with respect to t_P yields

$$\frac{dt_E}{dt_P}\left(1 - \frac{1}{2} h_{ij}(ct_E, \vec{r}_E) n^i n^j\right) - \left(1 - \frac{1}{2} h_{kl}(ct_P, \vec{r}_P) n^k n^\ell\right) = 0,$$

-

$$\frac{dt_P}{dt_E} = \frac{\left(1 - \frac{1}{2} h_{kl}(ct_E, \vec{r}_E) n^k n^\ell\right)}{\left(1 - \frac{1}{2} h_{ij}(ct_P, \vec{r}_P) n^i n^j\right)} = 1 - \frac{1}{2} h_{kl}(ct_E, \vec{r}_E) n^k n^\ell + \frac{1}{2} h_{ij}(ct_P, \vec{r}_P) n^i n^j + \ldots$$

-

Frame Title

- Pulses, which are emitted with a constant frequency ν_P, arrive on Earth with a frequency $\nu_E(t_E)$

$$\frac{\nu_E(t_E) - \nu_P}{\nu_P} = \frac{dt_P}{dt_E} - 1 = \frac{n^i n^j}{2} \left(h_{ij}(ct_P, \vec{r}_P) - h_{ij}(ct_E, \vec{r}_E) \right).$$

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- Clearly, the frequency shift depends on the projection of the wave amplitude h_{ij} onto n^i (both at the pulsar and at the Earth) .
- If our GW is propagating in the x^3 direction, n^i should have a non-vanishing components in the $x^1 - x^2$ plane

How GWs affect TOAs?

- If a GW propagates along the positive z direction

$$z(t) = \frac{\nu_P - \nu(t)}{\nu_P} = \frac{\alpha^2 - \beta^2}{2(1+\gamma)} \Delta h_+(t) + \frac{\alpha\,\beta}{1+\gamma} \Delta h_\times(t)\,,$$

  α, β, and γ provide a pulsar's direction cosines w.r.t the x, y, and z axes (Detweiler 1979)
  Where $\Delta h_{+,\times} = h^P_{+,\times} - h^E_{+,\times}$

- The observed/measured TOAs are compared with their predictions, based on an appropriate pulsar timing model
  The resulting differences are usually referred to as the 'Pulsar Timing Residuals'

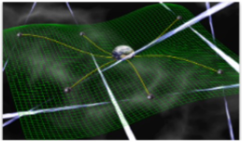
- Observable timing residuals, induced by the GWs, are given by the integral over time of the above Equation

$$R(t) = \int_0^t \frac{\nu_0 - \nu(t')}{\nu_0} \delta t'$$

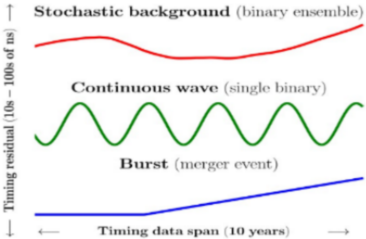
Pulsar Timing Array's GW Sources

- Observe a bunch of MSPs at different locations in the sky using the ***best radio telescopes*** of our times.

- Search for the correlated timing residual for detecting nHz GWs.



Courtesy: Web



Stochastic GWB from massive BH binaries ?: I

Cosmological population of MBH binaries is expected to provide a diffusive GW background for PTAs !!
We need to compute # of sources in a frequency interval $\Delta f = 1/T_{\rm obs}$

- $$\Delta N = \left(\frac{dN}{df}\right) \Delta f \; = \frac{dN}{dt} \left(\frac{df}{dt}\right)^{-1} \Delta f \qquad (1)$$
- $$\Delta N \propto \frac{dN}{dt} \left(\mathcal{M}_c{}^{-5/3} f_{\rm GW}^{-11/3}\right) \Delta f \qquad (2)$$
- There are some 10^{11} galaxies in our Universe and each galaxy is likely to experience one merger with another galaxy in Hubble time (10^{10} year)
- $$\rightarrow \frac{dN}{dt} \sim 10 \text{ mergers/year}$$

Stochastic GWB from massive BH binaries ?: I

- $\rightarrow$ a rough estimate for the number of binary BH sources in a frequency interval $\Delta f = 1/T_{\rm obs}$
-

$$\Delta N \sim 3 \times 10^{12} \left(\frac{\mathcal{M}}{10^9\, M_\odot}\right)^{-5/3} \left(\frac{f_{\rm GW}}{10^{-8}\, Hz}\right)^{-11/3} \left(\frac{T_{\rm obs}}{10\, yr}\right)^{-1} \times \left(\frac{dN/dt}{10 {\rm merger/yr}}\right)$$

- This is clearly $>> 1$

  This ensures a diffuse GW background in the PTA GW frequency window from merging massive BHs in the universe

- It is NOT very difficult to show that its characteristic strain spectrum is given by $h_c(f) = A_{1yr} \times (f/{\rm yr}^{-1})^{-2/3}$

Thank you!