1. Show that a 2×2 $SU(2)$ matrix U ($\det U = 1$) can be written as

$$U = u^0 \mathbb{1} + i\boldsymbol{u} \cdot \boldsymbol{\tau}$$

in terms of 4 parameters u^0 and $\boldsymbol{u}$ which are real, and $(u^0)^2 + \boldsymbol{u} \cdot \boldsymbol{u} = 1$. Thus $SU(2)$ is isomorphic to S^3. This construction does not generalize to $SU(n)$, $n > 2$ because the d-symbol does not vanish. Show that

(a) Under a $SU(2)_L$ transformation $L = e^{i\epsilon_L^a T^a}$, $T^a = \tau^a/2$, the infinitesimal transformation $U \to LU$ is

$$\delta u^0 = -\frac{1}{2}\boldsymbol{\epsilon}_L \cdot \boldsymbol{u}$$
$$\delta u^a = \frac{1}{2}u^0 \epsilon_L^a - \frac{1}{2}\epsilon^{abc}\epsilon_L^b u^c$$

(b) Under a $SU(2)_R$ transformation $R = e^{i\epsilon_R^a T^a}$, $T^a = \tau^a/2$, the infinitesimal transformation $U \to UR^\dagger$ is

$$\delta u^0 = \frac{1}{2}\boldsymbol{\epsilon}_R \cdot \boldsymbol{u}$$
$$\delta u^a = -\frac{1}{2}u^0 \epsilon_R^a - \frac{1}{2}\epsilon^{abc}\epsilon_R^b u^c$$

(c) Under a $SU(2)_V$ transformation $h = e^{i\epsilon^a T^a}$, $T^a = \tau^a/2$, the infinitesimal transformation $U \to hUh^\dagger$ is

$$\delta u^0 = 0$$
$$\delta u^a = -\epsilon^{abc}\epsilon^b u^c$$

2. Show for a 2×2 matrix A that

$$\langle A \rangle^3 - 3\langle A \rangle \langle A^2 \rangle + 2\langle A^3 \rangle = 0$$

(a) Differentiate w.r.t. A to get

$$\langle A \rangle^2 \mathbb{1} - \langle A^2 \rangle \mathbb{1} - 2\langle A \rangle A + 2A^2 = 0$$

(b) Let $A \to A + B$ and pick out the AB terms to get the identity

$$\langle A \rangle \langle B \rangle \mathbb{1} - \langle AB \rangle \mathbb{1} - \langle A \rangle B - \langle B \rangle A + AB + BA = 0$$

3. Show for a 3×3 matrix A that

$$6 \left\langle A^4 \right\rangle - 8 \langle A \rangle \left\langle A^3 \right\rangle - 3 \left\langle A^2 \right\rangle^2 + 6 \left\langle A^2 \right\rangle \langle A \rangle^2 - \langle A \rangle^4 = 0$$

(a) Differentiate w.r.t. A to get

$$6A^3 - 2 \left\langle A^3 \right\rangle \mathbb{1} - 6 \langle A \rangle A^2 - 3 \left\langle A^2 \right\rangle A + 3 \langle A \rangle^2 A + 3 \left\langle A^2 \right\rangle \langle A \rangle \mathbb{1} - \langle A \rangle^3 \mathbb{1} = 0$$

(b) Let $A \to A + B + C$ and pick out the ABC terms to get the identity in Bijnens, Colangelo and Ecker, JHEP 02 (1999) 020 Eq. (3.1).

4. See if you can find the analog of the first trace relation for 4×4 matrices, etc.

5. Show that if X and Y are $n \times n$ matrices,

$$e^X Y e^{-X} = Y + [X, Y] + \frac{1}{2!}[X, [X, Y]] + \ldots = e^{\mathrm{Ad}_X} Y$$

where $\mathrm{Ad}_X\, Y \equiv [X, Y]$.

6. Show that

$$\begin{aligned} e^{-X} \partial_\mu e^X &= \int_0^1 \mathrm{d}t\ e^{-tX} \partial_\mu X e^{tX} = \frac{1 - e^{-\mathrm{Ad}_X}}{\mathrm{Ad}_X} \partial_\mu X \\ &= \partial_\mu X - \frac{1}{2}[X, \partial_\mu X] + \frac{1}{6}[X, [X, \partial_\mu X]] \\ &= \sum_{k=0}^{\infty} \frac{(-1)^k}{(k+1)!} \mathrm{Ad}_X^k\, \partial_\mu X \end{aligned}$$

7. Work out the $SU(2)_R$ current from the leading order chiral Lagrangian to order π^3.

8. Work out the decay rate for $\pi \to \ell \overline{\nu}$ including the phase space.

9. Expand the $SU(3)$ chiral Lagrangian mass term to order π^2, and verify the mass results, Weinberg's formula for the quark mass ratios, and the Gell-Mann–Okubo formula.

10. Show that the chiral Lagrangian term

$$L = c\left\langle [Q,\Sigma][Q,\Sigma^\dagger]\right\rangle$$

leads to a mass term for the π^+ and K^+

$$L = -\frac{4c}{f^2}\left(\pi^+\pi^- + K^+K^-\right)$$

11. Work out the $\mathcal{O}(p^2)$ $SU(2)$ chiral Lagrangian up to 4 pions for both the kinetic and mass term.

12. From the previous result, work out the $\pi\pi \to \pi\pi$ scattering amplitude to order p^2.

13. Work out the baryon masses from the order M symmetry breaking term and verify the Gell-Mann–Okubo formula for baryons.

14. Find the $BB^*\pi$ and $B^*B^*\pi$ couplings from

$$L = g \mathrm{Tr}\, \overline{H}_v H_v \gamma^\mu \gamma_5 A_\mu$$

15. Define

$$u_{R\mu} = u^\dagger(\partial_\mu - ir_\mu)u \qquad\qquad u_{L\mu} = u(\partial_\mu - il_\mu)u^\dagger$$

and find their transformations under a *local* $SU(3)_L \times SU(3)_R$ transformation. From this, get the transformations for $u_\mu = i\left[u_{R\mu} - u_{L\mu}\right]$ and $\Gamma_\mu = \frac{1}{2}\left[u_{R\mu} + u_{L\mu}\right]$. Show that $u_{R\mu}$ and $u_{L\mu}$ are anti-hermitian.

16. Show that

$$\nabla^\mu u^\nu - \nabla^\nu u^\mu = f_-^{\mu\nu}$$

17. Show that

$$[\nabla_\mu, \nabla_\nu] \equiv \Gamma_{\mu\nu} = \frac{1}{4}\left[u_\mu, u_\nu\right] - \frac{i}{2} f_{+\mu\nu}$$