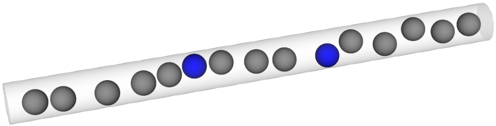
Tagged particle(s) in single-file systems

In and out of equilibrium

Alexis Poncet

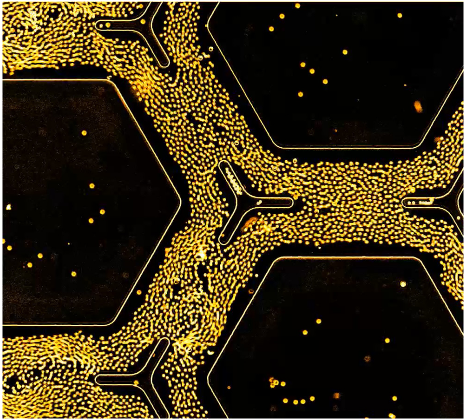
ICTS / CNRS meeting – Bangalore
December 2024



cnrs
Lab PHYS ENS de LYON
ENS DE LYON

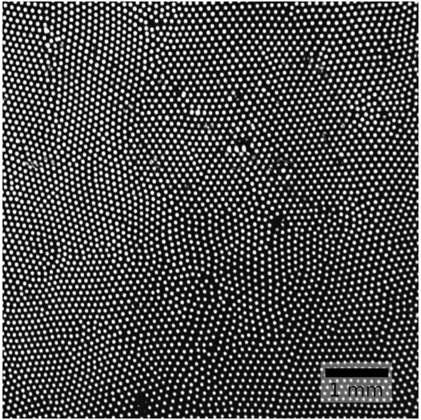
I CHOSE A TOPIC BUT...

Active hydraulics in trivalent networks



[Jorge, Chardac, **Poncet** & Bartolo, Nat. Phys. 2024]

Melting of non-reciprocal hydrodynamic crystals



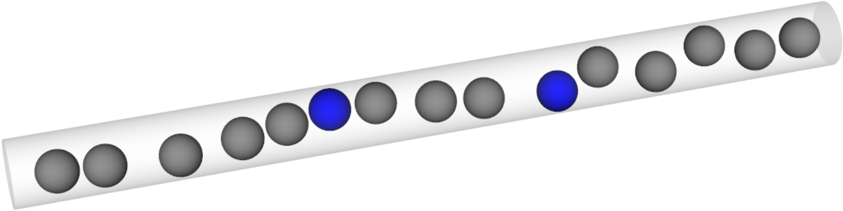
[Guillet, **Poncet**, Bartolo et al., under review]

Single-file systems

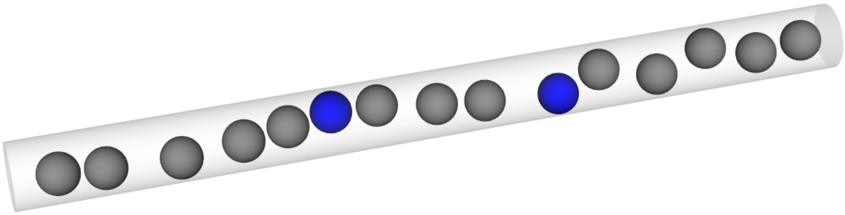
Olivier Bénichou

Aurélien Grabsch

Pierre Illien

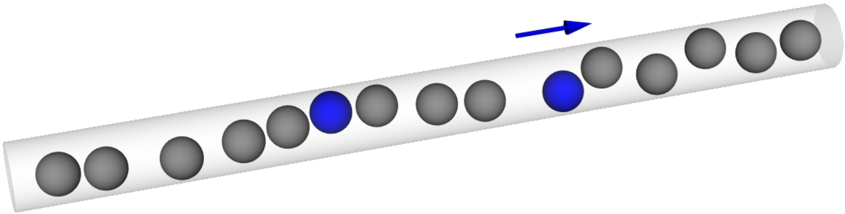


Diffusive particles that cannot pass one another.



Diffusive particles that cannot pass one another.

Major impact of confinement: **subdiffusion**, $\langle X(t)^2 \rangle \propto t^{1/2}$



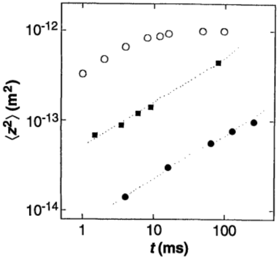
Diffusive particles that cannot pass one another.

Major impact of confinement: **subdiffusion**, $\langle X(t)^2 \rangle \propto t^{1/2}$

For a driven particle: **sub-ballistic motion**, $\langle X(t) \rangle \propto t^{1/2}$

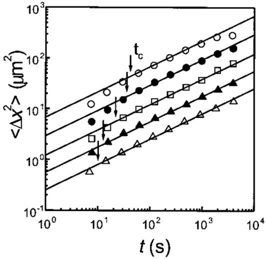
Zeolites

[Kukla et al., Science (1996)]



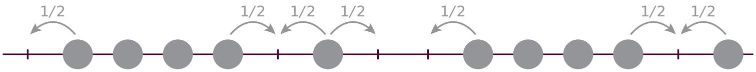
Colloids

[Wei et al., Science (2000)]



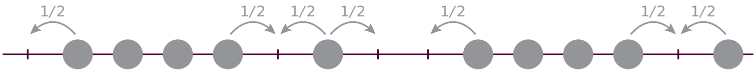
Symmetric exclusion process (SEP)

Paradigmatic model at equilibrium (and out-of-equilibrium if forced)



Exponential clock and hard core exclusion

Paradigmatic model at equilibrium (and out-of-equilibrium if forced)



Exponential clock and hard core exclusion

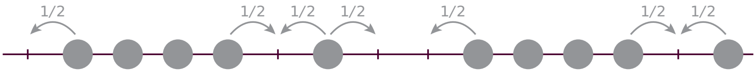
Long-history in physics and mathematics

1983 $\langle X^2(t)\rangle = \frac{1-\rho}{\rho}\sqrt{\frac{2t}{\pi}}$ [Arratia, Ann. Probab.]

2009 Current at the origin [Derrida & Gershenfeld, J. Stat. Phys.]

2017 Full probability law of $X(t)$ [Imamura, Sasamoto & Mallick, PRL]

Paradigmatic model at equilibrium (and out-of-equilibrium if forced)



Exponential clock and hard core exclusion

Long-history in physics and mathematics

1983 $\langle X^2(t)\rangle = \frac{1-\rho}{\rho}\sqrt{\frac{2t}{\pi}}$ [Arratia, Ann. Probab.]

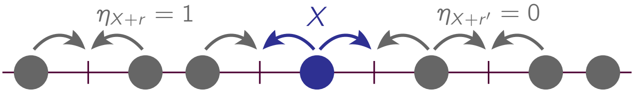
2009 Current at the origin [Derrida & Gershenfeld, J. Stat. Phys.]

2017 Full probability law of $X(t)$ [Imamura, Sasamoto & Mallick, PRL]

Connection with larger issues

- Mapping on XXX quantum spin chain [Alexander & Holstein 1978; de Gier & Essler 2006]
- Test for Macroscopic Fluctuation Theory [Bertini, RMP 2015]

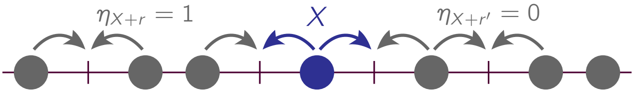
1. Tagged particle at equilibrium



How to close the hierarchy of density-displacement correlations ?

Method: Generalized density profiles

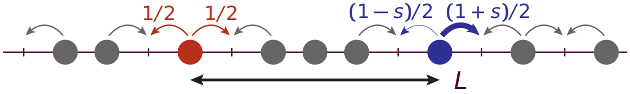
1. Tagged particle at equilibrium



How to close the hierarchy of density-displacement correlations ?

Method: Generalized density profiles

2. Out-of-equilibrium effects at high density

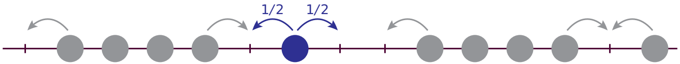


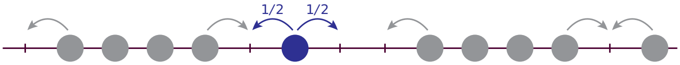
How to probe the effects of a driven particle?

Method: Vacancy motion at high density

1. Tagged particle at equilibrium

One particle in the SEP

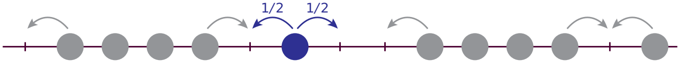




1. The variance

$$\langle X(t)^2 \rangle \underset{t\to\infty}{\sim} \frac{1-\rho}{\rho}\sqrt{\frac{2t}{\pi}}$$ [Arratia 1983]

$$\langle X(t)^2 \rangle \underset{t\to\infty}{\sim} \frac{S(\rho)}{\rho}\sqrt{\frac{4D(\rho)t}{\pi}}$$ [Kollmann, PRL 2003]



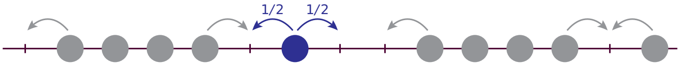
1. The variance

$$\langle X(t)^2 \rangle \underset{t\to\infty}{\sim} \frac{1-\rho}{\rho}\sqrt{\frac{2t}{\pi}}$$ [Arratia 1983]

$$\langle X(t)^2 \rangle \underset{t\to\infty}{\sim} \frac{S(\rho)}{\rho}\sqrt{\frac{4D(\rho)t}{\pi}}$$ [Kollmann, PRL 2003]

2. Full probability law in the dilute limit

[Krapivsky, Mallick & Sadhu, PRL 2014; Hedge, Sabahpandit & Dhar, PRL 2014; Sadhu & Derrida, J. Stat. Mech. 2015]



1. The variance

$$\langle X(t)^2 \rangle \underset{t\to\infty}{\sim} \frac{1-\rho}{\rho}\sqrt{\frac{2t}{\pi}}$$ [Arratia 1983]

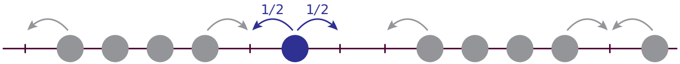
$$\langle X(t)^2 \rangle \underset{t\to\infty}{\sim} \frac{S(\rho)}{\rho}\sqrt{\frac{4D(\rho)t}{\pi}}$$ [Kollmann, PRL 2003]

2. Full probability law in the dilute limit

[Krapivsky, Mallick & Sadhu, PRL 2014; Hedge, Sabahpandit & Dhar, PRL 2014; Sadhu & Derrida, J. Stat. Mech. 2015]

3. Full probability law at arbitrary density

[Imamura, Sasamoto & Mallick, PRL 2017; Comm. Math. Phys. 2021]



1. The variance

$$\langle X(t)^2\rangle \underset{t\to\infty}{\sim} \frac{1-\rho}{\rho}\sqrt{\frac{2t}{\pi}}$$ [Arratia 1983]

$$\langle X(t)^2\rangle \underset{t\to\infty}{\sim} \frac{S(\rho)}{\rho}\sqrt{\frac{4D(\rho)t}{\pi}}$$ [Kollmann, PRL 2003]

2. Full probability law in the dilute limit

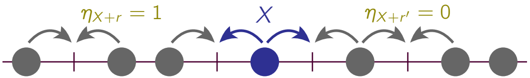
[Krapivsky, Mallick & Sadhu, PRL 2014; Hedge, Sabahpandit & Dhar, PRL 2014; Sadhu & Derrida, J. Stat. Mech. 2015]

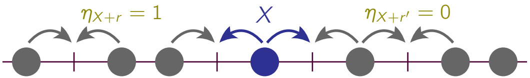
3. Full probability law at arbitrary density

[Imamura, Sasamoto & Mallick, PRL 2017; Comm. Math. Phys. 2021]

But what is the underlying structure?
Can we quantify density-displacement correlations?

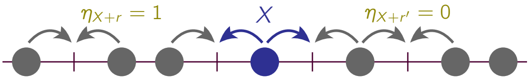
Generalized density profiles framework





Sublinear cumulants

$$\langle X^{2n}(t)\rangle_c = \kappa_{2n}\sqrt{2t}$$



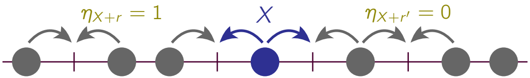
Sublinear cumulants

$$\langle X^{2n}(t)\rangle_c = \kappa_{2n}\sqrt{2t}$$

Non-stationary "generalized density profiles"

$$\langle \eta_{X+r} X^n \rangle_c = \Phi_n\left(\frac{r}{\sqrt{2t}}\right)$$

Generalized density profiles framework



Sublinear cumulants

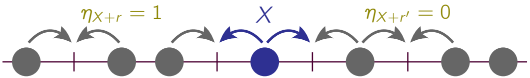
$$\langle X^{2n}(t)\rangle_c = \kappa_{2n}\sqrt{2t}$$

$$\ln\langle e^{\lambda X(t)}\rangle = \psi(\lambda)\sqrt{2t}$$

Non-stationary "generalized density profiles"

$$\langle \eta_{X+r} X^n\rangle_c = \Phi_n\left(\frac{r}{\sqrt{2t}}\right)$$

$$\frac{\langle e^{\lambda X(t)}\rangle}{\langle e^{\lambda \eta_{X+r} X(t)}\rangle} = \Phi(\lambda, v = \frac{r}{\sqrt{2t}})$$



Sublinear cumulants

$$\langle X^{2n}(t)\rangle_c = \kappa_{2n}\sqrt{2t}$$

$$\ln\langle e^{\lambda X(t)}\rangle = \psi(\lambda)\sqrt{2t}$$

Non-stationary "generalized density profiles"

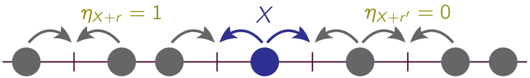
$$\langle \eta_{X+r} X^n\rangle_c = \Phi_n\left(\frac{r}{\sqrt{2t}}\right)$$

$$\frac{\langle e^{\lambda X(t)}\rangle}{\langle e^{\lambda}\eta_{X+r}{}^{X(t)}\rangle} = \Phi(\lambda, v = \frac{r}{\sqrt{2t}})$$

Exact boundary equations

$$\Phi'(\lambda, 0^{\pm}) + \frac{\psi(\lambda)}{e^{\pm\lambda}-1}\Phi(\lambda, 0^{\pm}) = 0 \qquad \frac{1-\Phi(\lambda, 0^{+})}{1-\Phi(\lambda, 0^{-})} = e^{\lambda}$$

Generalized density profiles framework



Sublinear cumulants

$$\langle X^{2n}(t)\rangle_c = \kappa_{2n}\sqrt{2t}$$

$$\ln\langle e^{\lambda X(t)}\rangle = \psi(\lambda)\sqrt{2t}$$

Non-stationary “generalized density profiles”

$$\langle \eta_{X+r} X^n\rangle_c = \Phi_n\left(\frac{r}{\sqrt{2t}}\right)$$

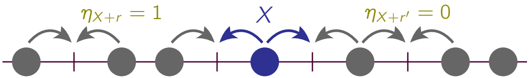
$$\frac{\langle e^{\lambda X(t)}\rangle}{\langle e^{\lambda \eta_{X+r} X(t)}\rangle} = \Phi\left(\lambda, v = \frac{r}{\sqrt{2t}}\right)$$

Exact boundary equations

$$\Phi'(\lambda, 0^{\pm}) + \frac{\psi(\lambda)}{e^{\pm\lambda} - 1}\Phi(\lambda, 0^{\pm}) = 0 \qquad \frac{1 - \Phi(\lambda, 0^{+})}{1 - \Phi(\lambda, 0^{-})} = e^{\lambda}$$

Bulk equation: not closed

$$\Phi''(\lambda, v) + 2\left(v + \frac{\psi(\lambda)}{e^{\pm\lambda} - 1}\right)\Phi'(\lambda, v) + C(v) = 0$$



Sublinear cumulants

$$\langle X^{2n}(t)\rangle_c = \kappa_{2n}\sqrt{2t}$$

$$\ln\langle e^{\lambda X(t)}\rangle = \psi(\lambda)\sqrt{2t}$$

Non-stationary "generalized density profiles"

$$\langle \eta_{X+r} X^n\rangle_c = \Phi_n\left(\frac{r}{\sqrt{2t}}\right)$$

$$\frac{\langle e^{\lambda X(t)}\rangle}{\langle e^{\lambda \eta_{X+r} X(t)}\rangle} = \Phi(\lambda, v = \frac{r}{\sqrt{2t}})$$

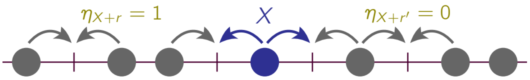
Exact boundary equations

$$\Phi'(\lambda, 0^{\pm}) + \frac{\psi(\lambda)}{e^{\pm\lambda}-1}\Phi(\lambda, 0^{\pm}) = 0 \qquad \frac{1-\Phi(\lambda,0^+)}{1-\Phi(\lambda,0^-)} = e^{\lambda}$$

Bulk equation: not closed

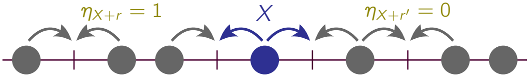
$$\Phi''(\lambda, v) + 2\left(v + \frac{\psi(\lambda)}{e^{\pm\lambda}-1}\right)\Phi'(\lambda, v) + C(v) = 0$$

Our goal is to find a closed bulk equation



At the lowest order, the bulk equation is closed!

[**Poncet**, Grabsch, Illien & Bénichou, PRL 2021]

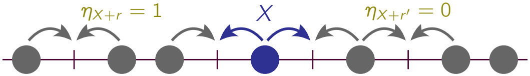


At the lowest order, the bulk equation is closed!

[**Poncet**, Grabsch, Illien & Bénichou, PRL 2021]

$$\langle X^2 \rangle = \kappa_2 \sqrt{2t}$$

$$\langle \eta_{X+r} X \rangle = \Phi_1\left(\frac{r}{\sqrt{2t}}\right)$$



At the lowest order, the bulk equation is closed!

[**Poncet**, Grabsch, Illien & Bénichou, PRL 2021]

$$\langle X^2 \rangle = \kappa_2 \sqrt{2t}$$

$$\langle \eta_{X+r} X \rangle = \Phi_1\left(\frac{r}{\sqrt{2t}}\right)$$

$$\begin{cases} \Phi_1''(v) + 2v\Phi_1'(v) = 0 \\ \Phi_1'(0^\pm) + \rho\kappa_2 = 0 \\ \Phi_1(0^+) - \Phi_1(0^-) = 1 - \rho \end{cases}$$

At the lowest order, the bulk equation is closed!

[**Poncet**, Grabsch, Illien & Bénichou, PRL 2021]

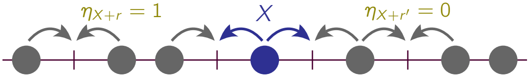
$$\langle X^2 \rangle = \kappa_2 \sqrt{2t}$$

$$\langle \eta_{X+r} X \rangle = \Phi_1\left(\frac{r}{\sqrt{2t}}\right)$$

$$\begin{cases} \Phi_1''(v) + 2v\Phi_1'(v) = 0 \\ \Phi_1'(0^\pm) + \rho\kappa_2 = 0 \\ \Phi_1(0^+) - \Phi_1(0^-) = 1 - \rho \end{cases}$$

$$\kappa_2 = \frac{1-\rho}{\rho}\frac{1}{\sqrt{\pi}}$$

$$\Phi_1(v) = \frac{1-\rho}{2}\,\mathrm{erfc}(v)$$



At the lowest order, the bulk equation is closed!

[**Poncet**, Grabsch, Illien & Bénichou, PRL 2021]

$$\langle X^2 \rangle = \kappa_2 \sqrt{2t}$$

$$\langle \eta_{X+r} X \rangle = \Phi_1\left(\frac{r}{\sqrt{2t}}\right)$$

$$\begin{cases} \Phi_1''(v) + 2v\Phi_1'(v) = 0 \\ \Phi_1'(0^\pm) + \rho\kappa_2 = 0 \\ \Phi_1(0^+) - \Phi_1(0^-) = 1 - \rho \end{cases}$$

$$\kappa_2 = \frac{1-\rho}{\rho}\frac{1}{\sqrt{\pi}}$$

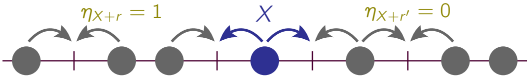
$$\Phi_1(v) = \frac{1-\rho}{2}\operatorname{erfc}(v)$$

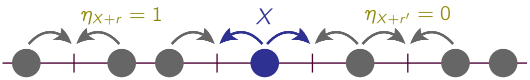
Also works for an arbitrary single-file system

structure factor $S(\rho)$ and diffusion coefficient $D(\rho)$

$$\kappa_2 = \frac{S(\rho)}{\rho}\sqrt{\frac{2D(\rho)}{\pi}} \qquad \Phi_1(v) = \frac{S(\rho)}{2}\operatorname{erfc}(v)$$

2. Low density limit ($\rho \to 0$)

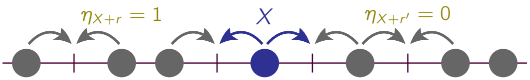




Closure relation at all orders

[**Poncet**, Grabsch, Illien & Bénichou, PRL 2021]

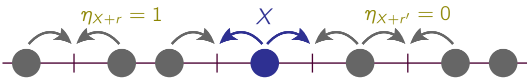
2. Low density limit ($\rho \to 0$)



Closure relation at all orders

[**Poncet**, Grabsch, Illien & Bénichou, PRL 2021]

Scalings when $\rho \to 0$: $\lambda = \rho\hat{\lambda} \qquad \Psi = \Psi/\rho \qquad \hat{\Phi} = \Phi/\rho$

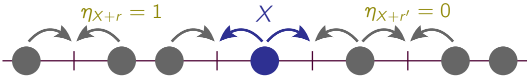


Closure relation at all orders

[**Poncet**, Grabsch, Illien & Bénichou, PRL 2021]

Scalings when $\rho \to 0$: $\lambda = \rho\hat{\lambda}$ $\quad \Psi = \Psi/\rho$ $\quad \hat{\Phi} = \Phi/\rho$

Closed bulk equation: $\hat{\Phi}''(\hat{\lambda}, v) + 2\left(v + \hat{\psi}'(\hat{\lambda})\right)\hat{\Phi}'(\hat{\lambda}, v) = 0$ + Boundary conditions



Closure relation at all orders

[**Poncet**, Grabsch, Illien & Bénichou, PRL 2021]

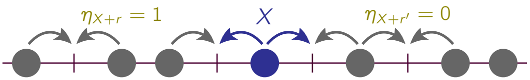
Scalings when $\rho \to 0$: $\lambda = \rho\hat{\lambda}$ $\quad \Psi = \Psi/\rho \quad$ $\hat{\Phi} = \Phi/\rho$

Closed bulk equation: $\hat{\Phi}''(\hat{\lambda}, v) + 2\left(v + \hat{\psi}'(\hat{\lambda})\right)\hat{\Phi}'(\hat{\lambda}, v) = 0$ + Boundary conditions

$$\langle X^2 \rangle_c = \frac{1}{\rho}\sqrt{\frac{2t}{\pi}}$$

$$\langle X^4 \rangle_c = \frac{1}{\rho^3}\frac{3(4-\pi)}{\pi^{3/2}}\sqrt{2t}$$

etc.



Closure relation at all orders

[**Poncet**, Grabsch, Illien & Bénichou, PRL 2021]

Scalings when $\rho \to 0$: $\lambda = \rho\hat{\lambda}$ $\quad \Psi = \Psi/\rho$ $\quad \hat{\Phi} = \Phi/\rho$

Closed bulk equation: $\hat{\Phi}''(\hat{\lambda}, v) + 2\left(v + \hat{\psi}'(\hat{\lambda})\right)\hat{\Phi}'(\hat{\lambda}, v) = 0$ $\quad$ + Boundary conditions

$$\langle X^2 \rangle_c = \frac{1}{\rho}\sqrt{\frac{2t}{\pi}}$$

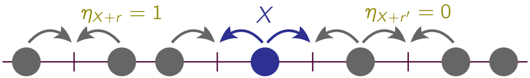
$$\langle X^4 \rangle_c = \frac{1}{\rho^3}\frac{3(4-\pi)}{\pi^{3/2}}\sqrt{2t}$$

etc.

$$\hat{\Phi}(\hat{\lambda}, v) = \frac{\beta\,\mathrm{erfc}(v+\xi)}{\pi^{-1/2}e^{-\xi^2} - \beta\,\mathrm{erfc}\,\xi}$$

with $\beta = \hat{\psi}/\hat{\lambda}$, $\xi = \psi'(\hat{\lambda})$

2. Low density limit ($\rho \to 0$)



Closure relation at all orders

[**Poncet**, Grabsch, Illien & Bénichou, PRL 2021]

Scalings when $\rho \to 0$: $\lambda = \rho\hat{\lambda}$ $\quad \Psi = \Psi/\rho$ $\quad \hat{\Phi} = \Phi/\rho$

Closed bulk equation: $\hat{\Phi}''(\hat{\lambda}, v) + 2\left(v + \hat{\psi}'(\hat{\lambda})\right)\hat{\Phi}'(\hat{\lambda}, v) = 0$ + Boundary conditions

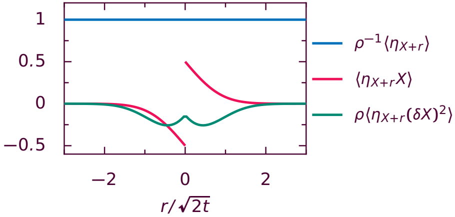
$$\langle X^2 \rangle_c = \frac{1}{\rho}\sqrt{\frac{2t}{\pi}}$$

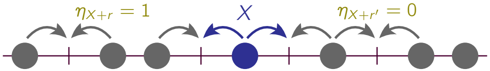
$$\langle X^4 \rangle_c = \frac{1}{\rho^3}\frac{3(4-\pi)}{\pi^{3/2}}\sqrt{2t}$$

etc.

$$\hat{\Phi}(\hat{\lambda}, v) = \frac{\beta \operatorname{erfc}(v+\xi)}{\pi^{-1/2}e^{-\xi^2} - \beta \operatorname{erfc}\xi}$$

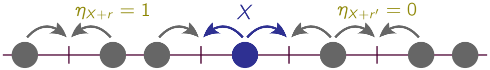
with $\beta = \hat{\psi}/\hat{\lambda}$, $\xi = \psi'(\hat{\lambda})$





Closure relation found iteratively

[Grabsch, **Poncet**, Rizkallah, Illien & Bénichou, Sci. Adv. 2021 / PRE 2023]



Closure relation found iteratively

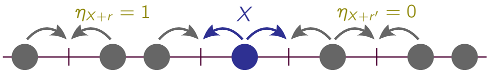
[Grabsch, **Poncet**, Rizkallah, Illien & Bénichou, Sci. Adv. 2021 / PRE 2023]

Generating function

$$\ln\langle e^{\lambda X}\rangle \sim \Psi(\lambda)\sqrt{2t}$$

Generalized density profiles

$$\frac{\langle \eta_{X+r} e^{\lambda X}\rangle}{\langle e^{\lambda X}\rangle} \sim \Phi\left(\lambda, v = \frac{r}{\sqrt{2t}}\right)$$



Closure relation found iteratively

[Grabsch, **Poncet**, Rizkallah, Illien & Bénichou, Sci. Adv. 2021 / PRE 2023]

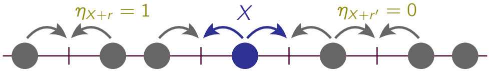
Generating function

$$\ln\langle e^{\lambda X}\rangle \sim \Psi(\lambda)\sqrt{2t}$$

Generalized density profiles

$$\frac{\langle \eta_{X+r} e^{\lambda X}\rangle}{\langle e^{\lambda X}\rangle} \sim \Phi\left(\lambda, v = \frac{r}{\sqrt{2t}}\right)$$

Coupled equations for $\Psi(\lambda)$ and $\Omega(v) = 2A\dfrac{\Phi'(v>0)}{\Phi'(0^+)}$



Closure relation found iteratively

[Grabsch, **Poncet**, Rizkallah, Illien & Bénichou, Sci. Adv. 2021 / PRE 2023]

Generating function

$\ln\langle e^{\lambda X}\rangle \sim \Psi(\lambda)\sqrt{2t}$

Generalized density profiles

$\frac{\langle \eta_{X+r} e^{\lambda X}\rangle}{\langle e^{\lambda X}\rangle} \sim \Phi\left(\lambda, v = \frac{r}{\sqrt{2t}}\right)$

Coupled equations for $\Psi(\lambda)$ and $\Omega(v) = 2A\,\frac{\Phi'(v>0)}{\Phi'(0^+)}$

Solution as Wiener-Hopf equation

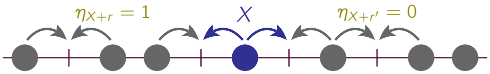
$$\Omega(v) + \int_{-\infty}^{0} dz\, K(v-z)\Omega(z) = K(v)$$

with $K(v) = \omega\,\frac{e^{(v+\Psi')^2}}{\sqrt{\pi}}$,

ω such that $\Omega(0) = 2\Psi$,

and boundary conditions.

3. Arbitrary density



Closure relation found iteratively

[Grabsch, **Poncet**, Rizkallah, Illien & Bénichou, Sci. Adv. 2021 / PRE 2023]

Generating function

$$\ln\langle e^{\lambda X}\rangle \sim \Psi(\lambda)\sqrt{2t}$$

Generalized density profiles

$$\frac{\langle \eta_{X+r} e^{\lambda X}\rangle}{\langle e^{\lambda X}\rangle} \sim \Phi\left(\lambda, v = \frac{r}{\sqrt{2t}}\right)$$

Coupled equations for $\Psi(\lambda)$ and $\Omega(v) = 2A\dfrac{\Phi'(v>0)}{\Phi'(0^+)}$

Solution as Wiener-Hopf equation

$$\Omega(v) + \int_{-\infty}^{0} dz\, K(v-z)\Omega(z) = K(v)$$

with $K(v) = \omega \frac{e^{(v+\Psi')^2}}{\sqrt{\pi}}$,

ω such that $\Omega(0) = 2\Psi$,

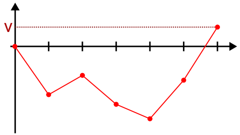
and boundary conditions.

Link with a simple random walk

$$\Omega(v) = -\sum_{n\geq 1}(-\omega)^n p_n(v)$$

$$p_n(v) = \mathbb{P}(X_0 = 0, X_1 < 0, \ldots, X_{n-1} < 0, X_n = v)$$

$$P(X_{n+1} - X_n = \eta) = \frac{e^{-(\eta+\Psi')^2}}{\sqrt{\pi}}$$



Moments

$$\langle X^2(t)\rangle \underset{t\to\infty}{\sim} \frac{1-\rho}{\rho}\sqrt{\frac{2t}{\pi}} \qquad \langle X^4(t)\rangle_c \underset{t\to\infty}{\sim} \frac{1-\rho}{\rho^3}\left(12(1-\rho)^2-\pi(2-3(4-\sqrt{2})\rho)+(8-3\sqrt{2})\rho^2\right)\sqrt{\frac{2t}{\pi^3}}$$

Analytical expressions for the moments and correlations

Moments

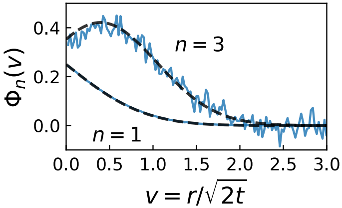
$$\langle X^2(t)\rangle \underset{t\to\infty}{\sim} \frac{1-\rho}{\rho}\sqrt{\frac{2t}{\pi}}$$

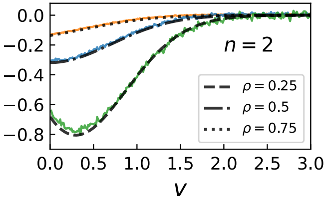
$$\langle X^4(t)\rangle_c \underset{t\to\infty}{\sim} \frac{1-\rho}{\rho^3}\left(12(1-\rho)^2-\pi(2-3(4-\sqrt{2})\rho)+(8-3\sqrt{2})\rho^2\right)\sqrt{\frac{2t}{\pi^3}}$$

Correlations

$$\langle X\eta_{X+r}\rangle(t) \underset{t\to\infty}{\sim} \operatorname{erfc}\left(\frac{r}{\sqrt{2t}}\right)$$

$$\langle X^2\eta_{X+r}\rangle(t) \underset{t\to\infty}{\sim} = \frac{(1-\rho)(1-2\rho)}{2\rho}\operatorname{erfc}\left(\frac{r}{\sqrt{2t}}\right)-\frac{2}{\pi}\frac{(1-\rho)^2}{\rho}e^{-\frac{r^2}{2t}}$$





Link with Macroscopic Fluctuation Theory

Fluctuating hydrodynamics of the SEP

$$\partial_t \rho = \partial_x \left[D(\rho)\partial_x \rho + \sqrt{\sigma(\rho)}\eta \right]$$

Fluctuating hydrodynamics of the SEP

$$\partial_t \rho = \partial_x \left[D(\rho) \partial_x \rho + \sqrt{\sigma(\rho)} \eta \right] \quad \text{with } D(\rho) = 1/2, \sigma(\rho) = \rho(1-\rho)$$

Fluctuating hydrodynamics of the SEP

$$\partial_t \rho = \partial_x \left[D(\rho)\partial_x \rho + \sqrt{\sigma(\rho)}\eta \right] \quad \text{with } D(\rho) = 1/2,\ \sigma(\rho) = \rho(1-\rho)$$

Macroscopic fluctuation theory = large deviations in time

Fluctuating hydrodynamics of the SEP

$$\partial_t \rho = \partial_x \left[D(\rho)\partial_x \rho + \sqrt{\sigma(\rho)}\eta \right] \quad \text{with } D(\rho) = 1/2, \sigma(\rho) = \rho(1-\rho)$$

Macroscopic fluctuation theory = large deviations in time

optimal path ρ^* between $t = 0$ and $t = T$

$$\partial_t \rho^* = \partial_x \left[D(\rho^*)\partial_x \rho^* - \sigma(\rho^*)\partial_x H \right] \qquad \partial_t H = -\partial_x^2 H - \frac{\sigma'(\rho^*)}{2}(\partial_x H)^2$$

Fluctuating hydrodynamics of the SEP

$$\partial_t \rho = \partial_x \left[D(\rho)\partial_x \rho + \sqrt{\sigma(\rho)}\eta \right] \quad \text{with } D(\rho) = 1/2, \sigma(\rho) = \rho(1-\rho)$$

Macroscopic fluctuation theory = large deviations in time

optimal path ρ^* between $t = 0$ and $t = T$

$$\partial_t \rho^* = \partial_x [D(\rho^*)\partial_x \rho^* - \sigma(\rho^*)\partial_x H] \qquad \partial_t H = -\partial_x^2 H - \frac{\sigma'(\rho^*)}{2}(\partial_x H)^2$$

MFT solution at final time = generalized density profile

$$\tilde{\rho}^*(x = v, t = T) = \Phi(v)$$

- **Solution of the MFT equations for the SEP**
  Non-local Cole-Hopf mapping gives AKNS integrable systems
  Wiener-Hopf equation for final profile
  [Mallick, Moriya, Sasamoto, PRL 2022]

- **Solution of the MFT equations for the SEP**
  Non-local Cole-Hopf mapping gives AKNS integrable systems
  Wiener-Hopf equation for final profile
  [Mallick, Moriya, Sasamoto, PRL 2022]

- **Fourth cumulant of an arbitrary single-file system**
  [Grabsch & Bénichou, PRL 2024]

$$\kappa_4 = \frac{3\sigma(\rho)^3\left(\rho D'(\rho)+D(\rho)\right)}{\pi^{3/2}\rho^6 D(\rho)^{7/2}} - \frac{\sigma(\rho)\left(\sigma(\rho)\sigma'(\rho)\left(\rho D'(\rho)+4D(\rho)\right)+2\sigma(\rho)^2 D'(\rho)-\rho D(\rho)\sigma'(\rho)^2\right)}{4\sqrt{\pi}\rho^5 D(\rho)^{7/2}} + \frac{3\sigma(\rho)^3\left(D'(\rho)^2-D(\rho)D''(\rho)\right)}{8\sqrt{\pi}\rho^4 D(\rho)^{9/2}}$$
$$+\frac{3\sigma(\rho)^3\left(2D(\rho)D''(\rho)-D'(\rho)^2\right)}{8\pi^{3/2}\rho^4 D(\rho)^{9/2}} + \frac{\left(3\sqrt{2}-4\right)\sigma(\rho)^2\sigma''(\rho)}{8\sqrt{\pi}\rho^4 D(\rho)^{5/2}} + \frac{3\left(\sqrt{2}\pi-2\sqrt{3}\right)\sigma(\rho)^3\left(2D(\rho)D''(\rho)-3D'(\rho)^2\right)}{16\pi^{3/2}\rho^4 D(\rho)^{9/2}}.$$

- **Solution of the MFT equations for the SEP**
  Non-local Cole-Hopf mapping gives AKNS integrable systems
  Wiener-Hopf equation for final profile
  [Mallick, Moriya, Sasamoto, PRL 2022]

- **Fourth cumulant of an arbitrary single-file system**
  [Grabsch & Bénichou, PRL 2024]

$$\kappa_4 = \frac{3\sigma(\rho)^3\left(\rho D'(\rho)+D(\rho)\right)}{\pi^{3/2}\rho^6 D(\rho)^{7/2}} - \frac{\sigma(\rho)\left(\sigma(\rho)\sigma'(\rho)\left(\rho D'(\rho)+4D(\rho)\right)+2\sigma(\rho)^2 D'(\rho)-\rho D(\rho)\sigma'(\rho)^2\right)}{4\sqrt{\pi}\rho^5 D(\rho)^{7/2}} + \frac{3\sigma(\rho)^3\left(D'(\rho)^2-D(\rho)D''(\rho)\right)}{8\sqrt{\pi}\rho^4 D(\rho)^{9/2}}$$
$$+ \frac{3\sigma(\rho)^3\left(2D(\rho)D''(\rho)-D'(\rho)^2\right)}{8\pi^{3/2}\rho^4 D(\rho)^{9/2}} + \frac{\left(3\sqrt{2}-4\right)\sigma(\rho)^2\sigma''(\rho)}{8\sqrt{\pi}\rho^4 D(\rho)^{5/2}} + \frac{3\left(\sqrt{2}\pi-2\sqrt{3}\right)\sigma(\rho)^3\left(2D(\rho)D''(\rho)-3D'(\rho)^2\right)}{16\pi^{3/2}\rho^4 D(\rho)^{9/2}}\,.$$

- **Interest of generalized density profiles beyond diffusive systems**
  run and tumble, Levy flights, etc.
  [Grabsch, Berlioz, Rizkallah, Illien & Bénichou, PRL 2024]

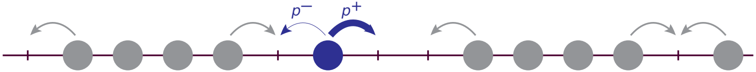
The generalized density profiles $\langle \eta_{X+r} X^n \rangle$ encode the **non-stationary structure** associated with the **anomalous dynamics** of single-file systems.

The generalized density profiles $\langle \eta_{X+r} X^n \rangle$
encode the **non-stationary structure** associated with
the **anomalous dynamics** of single-file systems.

- Exact equations giving the variance of an arbitrary single-file
- Closure relation for the SEP at all densities
- Highlights the integrable nature of the problem
- Relevant beyond the diffusive case

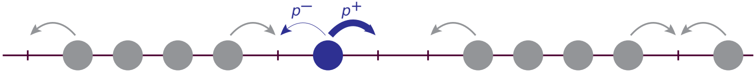
Out-of-equilibrium effects at high density

Probability law of a biased intruder



$$\langle X(t)\rangle \underset{t\to\infty}{\sim} A\sqrt{t}$$ [Burlatsky 1996; Landim 1998]

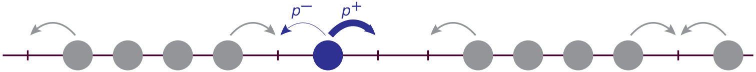
Probability law of a biased intruder



$$\langle X(t)\rangle \underset{t\to\infty}{\sim} A\sqrt{t}$$ [Burlatsky 1996; Landim 1998]

Can we write the full probability law at all times?

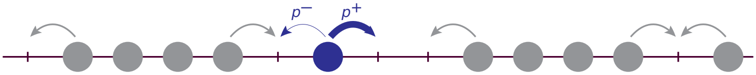
Probability law of a biased intruder



$$\langle X(t) \rangle \underset{t\to\infty}{\sim} A\sqrt{t}$$ [Burlatsky 1996; Landim 1998]

Can we write the full probability law at all times?
Can we obtain the N-time statistics at large times?

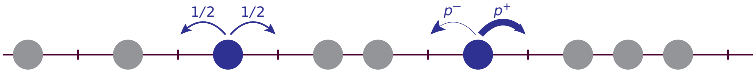
Probability law of a biased intruder



$$\langle X(t)\rangle \underset{t\to\infty}{\sim} A\sqrt{t}$$ [Burlatsky 1996; Landim 1998]

Can we write the full probability law at all times?
Can we obtain the N-time statistics at large times?

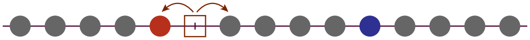
Driving effect of a biased intruder



Only simpler models: RAP [Cividini 2016], pointlike particles [Ooshida 2018]

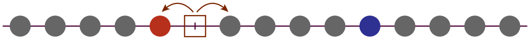
Can we obtain the 2-point statistics?

High density ($\rho \to 1$) : the vacancies are independent from one another



High density limit: vacancy-based approach

High density ($\rho \to 1$) : the vacancies are independent from one another

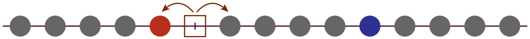


→ **The problem boils down to the case of a *single vacancy***

[Brummelhuis & Hilorst 1989; Illien *et al.* 2013]

High density limit: vacancy-based approach

High density ($\rho \to 1$) : the vacancies are independent from one another



→ **The problem boils down to the case of a *single vacancy***

[Brummelhuis & Hilorst 1989; Illien *et al.* 2013]

$$\lim_{\rho\to 1} \frac{\psi(k_1,k_2,t)}{1-\rho} = \sum_Z \left[\tilde{p}_Z(k_1,k_2,t) - 1\right]$$

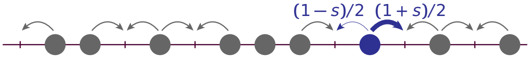
$$\psi(k_1,k_2,t) = \ln\left\langle e^{i(k_1 X_1 + k_2 X_2)}\right\rangle$$

Generating function giving $\langle X_1\rangle$, $\langle X_2\rangle$, $\langle X_1 X_2\rangle$, etc.

$$\tilde{p}_Z(k_1,k_2,t)$$

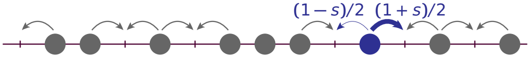
Fourier-transform of the propagator of (X_1, X_2) with a *single* vacancy initially at Z.

Biased particle: statistics at all times



[**Poncet**, Grabsch, Bénichou & Illien, PRE 2022]

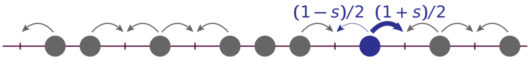
Biased particle: statistics at all times



[**Poncet**, Grabsch, Bénichou & Illien, PRE 2022]

$$\lim_{\rho\to 1} \frac{\langle X(t)^{2n+1}\rangle}{1-\rho} = s\,te^{-t}\left[I_0(t)+I_1(t)\right]$$

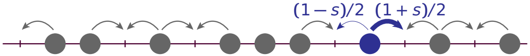
Biased particle: statistics at all times



[**Poncet**, Grabsch, Bénichou & Illien, PRE 2022]

$$\lim_{\rho\to 1}\frac{\langle X(t)^{2n+1}\rangle}{1-\rho} = s\,te^{-t}\left[I_0(t)+I_1(t)\right]$$

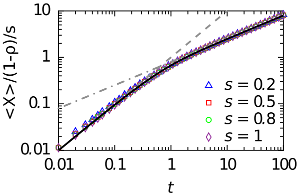
$$\sim \begin{cases} s\,t & \text{when } t\to 0 \\ s\sqrt{\frac{2t}{\pi}} & \text{when } t\to\infty \end{cases}$$

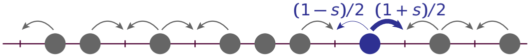


[**Poncet**, Grabsch, Bénichou & Illien, PRE 2022]

$$\lim_{\rho\to 1} \frac{\langle X(t)^{2n+1}\rangle}{1-\rho} = s\,t e^{-t}\left[I_0(t) + I_1(t)\right]$$

$$\sim \begin{cases} s\,t & \text{when } t \to 0 \\ s\sqrt{\frac{2t}{\pi}} & \text{when } t \to \infty \end{cases}$$

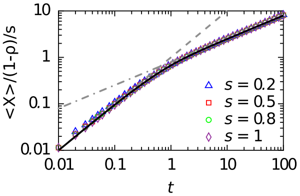




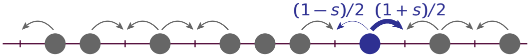
[**Poncet**, Grabsch, Bénichou & Illien, PRE 2022]

$$\lim_{\rho\to 1}\frac{\langle X(t)^{2n+1}\rangle}{1-\rho} = s\,t e^{-t}\left[I_0(t)+I_1(t)\right]$$

$$\sim \begin{cases} s\,t & \text{when } t\to 0 \\ s\sqrt{\frac{2t}{\pi}} & \text{when } t\to\infty \end{cases}$$



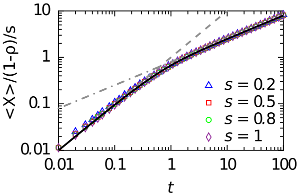
Biased particle: statistics at all times



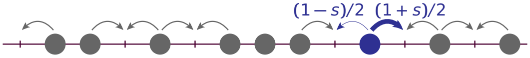
[**Poncet**, Grabsch, Bénichou & Illien, PRE 2022]

$$\lim_{\rho\to 1} \frac{\langle X(t)^{2n+1}\rangle}{1-\rho} = s\,t e^{-t}\left[I_0(t)+I_1(t)\right] \sim \begin{cases} s\,t & \text{when } t\to 0 \\ s\sqrt{\frac{2t}{\pi}} & \text{when } t\to\infty \end{cases}$$

$$\lim_{\rho\to 1} \frac{\langle X(t)^{2n}\rangle}{1-\rho} = t e^{-t}\left[I_0(t)+I_1(t)\right] \sim \begin{cases} t & \text{when } t\to 0 \\ \sqrt{\frac{2t}{\pi}} & \text{when } t\to\infty \end{cases}$$

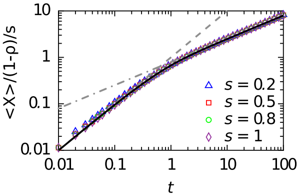


Biased particle: statistics at all times

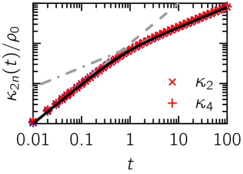


[**Poncet**, Grabsch, Bénichou & Illien, PRE 2022]

$$\lim_{\rho\to 1}\frac{\langle X(t)^{2n+1}\rangle}{1-\rho} = s\,t e^{-t}\left[I_0(t)+I_1(t)\right] \sim \begin{cases} s\,t & \text{when } t\to 0\\ s\sqrt{\frac{2t}{\pi}} & \text{when } t\to\infty\end{cases}$$

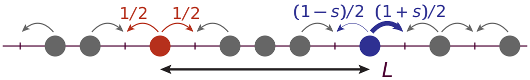


$$\lim_{\rho\to 1}\frac{\langle X(t)^{2n}\rangle}{1-\rho} = t e^{-t}\left[I_0(t)+I_1(t)\right] \sim \begin{cases} t & \text{when } t\to 0\\ \sqrt{\frac{2t}{\pi}} & \text{when } t\to\infty\end{cases}$$



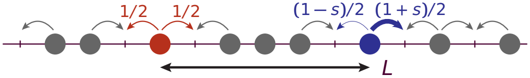
(also comb structure and 2D lattice)

Biased particle: driving of another particle



[**Poncet**, Bénichou, Démery & Oshanin, PRR 2019]

Biased particle: driving of another particle

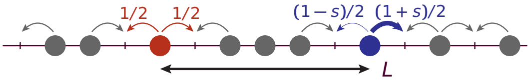


[**Poncet**, Bénichou, Démery & Oshanin, PRR 2019]

$$\lim_{\rho\to 1} \frac{\langle X_1(t)\rangle}{1-\rho} \underset{t\to\infty}{\sim} s\sqrt{\frac{2t}{\pi}}\, g\left(\frac{L}{\sqrt{2t}}\right)$$

avec $g(u) = e^{-u^2} - \sqrt{\pi}u \operatorname{erfc} u$

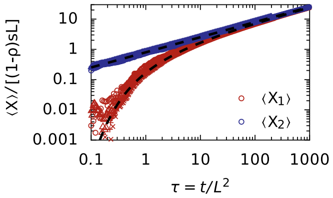
Biased particle: driving of another particle

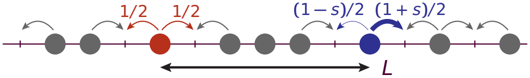


[**Poncet**, Bénichou, Démery & Oshanin, PRR 2019]

$$\lim_{\rho\to 1}\frac{\langle X_1(t)\rangle}{1-\rho}\underset{t\to\infty}{\sim} s\sqrt{\frac{2t}{\pi}}\,g\left(\frac{L}{\sqrt{2t}}\right)$$

avec $g(u) = e^{-u^2} - \sqrt{\pi}u\,\mathrm{erfc}\,u$

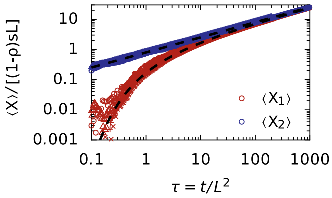




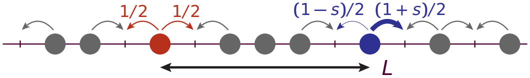
[**Poncet**, Bénichou, Démery & Oshanin, PRR 2019]

$$\lim_{\rho\to 1} \frac{\langle X_1(t)\rangle}{1-\rho} \underset{t\to\infty}{\sim} s\sqrt{\frac{2t}{\pi}}\, g\left(\frac{L}{\sqrt{2t}}\right)$$

avec $g(u) = e^{-u^2} - \sqrt{\pi} u \operatorname{erfc} u$



$$\lim_{\rho\to 1} \frac{\langle X_1(t)^n X_2(t)^m\rangle}{1-\rho} \underset{t\to\infty}{\sim} (s)\sqrt{\frac{2t}{\pi}}\, g\left(\frac{L}{\sqrt{2t}}\right)$$

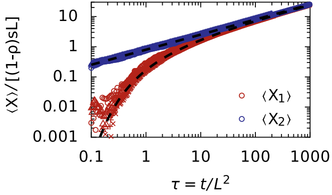


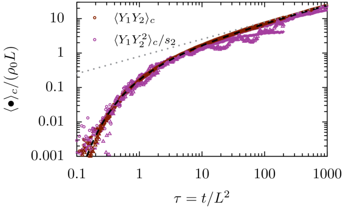
[**Poncet**, Bénichou, Démery & Oshanin, PRR 2019]

$$\lim_{\rho\to 1}\frac{\langle X_1(t)\rangle}{1-\rho} \underset{t\to\infty}{\sim} s\sqrt{\frac{2t}{\pi}}\, g\left(\frac{L}{\sqrt{2t}}\right)$$

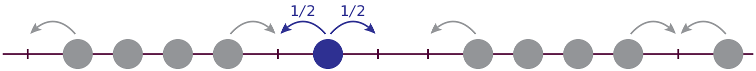
avec $g(u) = e^{-u^2} - \sqrt{\pi} u \operatorname{erfc} u$

$$\lim_{\rho\to 1}\frac{\langle X_1(t)^n X_2(t)^m\rangle}{1-\rho} \underset{t\to\infty}{\sim} (s)\sqrt{\frac{2t}{\pi}}\, g\left(\frac{L}{\sqrt{2t}}\right)$$



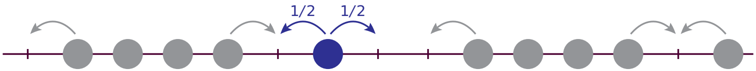


Two-time statistics of the SEP



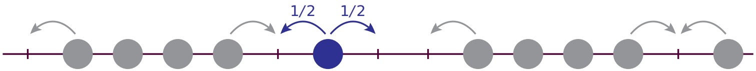
$$\langle X(t_1)X(t_2)\rangle = ?$$

Two-time statistics of the SEP



$$\langle X(t_1)X(t_2)\rangle = ?$$

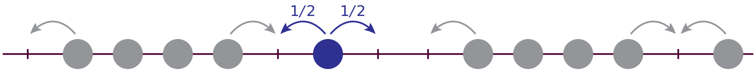
- Edwards-Wilkinson universality class [Majumdar & Barma 1991]
- Two-time statistics in dilute limit [Sadhu & Derrida 2015]
- Proof of fractional Brownian motion [Peligrad & Sethuraman 2008]



$$\langle X(t_1)X(t_2)\rangle = ?$$

- ▶ Edwards-Wilkinson universality class [Majumdar & Barma 1991]
- ▶ Two-time statistics in dilute limit [Sadhu & Derrida 2015]
- ▶ Proof of fractional Brownian motion [Peligrad & Sethuraman 2008]

$$\langle X(t_1)X(t_2)\rangle \sim \sqrt{t_1} + \sqrt{t_2} - \sqrt{|t_2 - t_1|}$$



$$\langle X(t_1)X(t_2)\rangle = ?$$

- Edwards-Wilkinson universality class [Majumdar & Barma 1991]
- Two-time statistics in dilute limit [Sadhu & Derrida 2015]
- Proof of fractional Brownian motion [Peligrad & Sethuraman 2008]

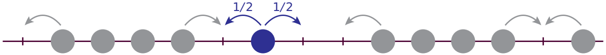
$$\langle X(t_1)X(t_2)\rangle \sim \sqrt{t_1} + \sqrt{t_2} - \sqrt{|t_2 - t_1|}$$

Can we also have such a result for a biased particle?

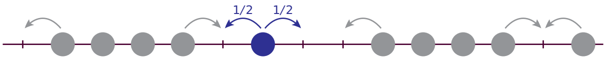
What about higher order, non-Gaussian, statistics?

(What about the dependence on initial conditions?)
[Leibovich & Barkai 2013; Sadhu & Derrida 2015]

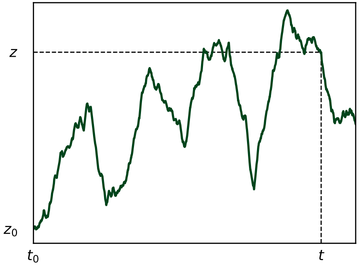
Dense limit: simple Brownian motion



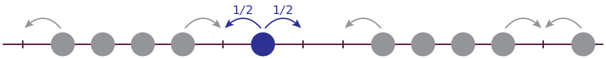
[**Poncet**, Grabsch & Bénichou, in preparation]



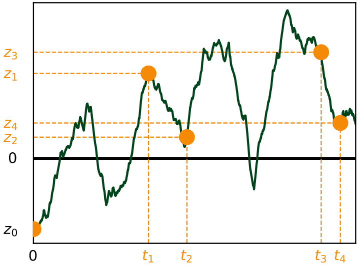
[**Poncet**, Grabsch & Bénichou, in preparation]



$$P(z,t|z_0,t_0) = \frac{e^{-\frac{(z-z_0)^2}{2(t-t_0)}}}{\sqrt{2\pi(t-t_0)}}$$

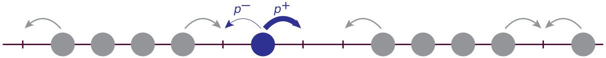


[**Poncet**, Grabsch & Bénichou, in preparation]

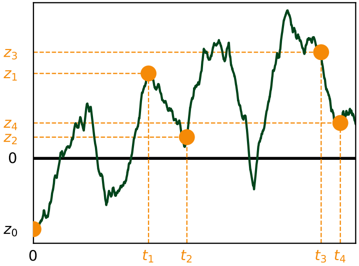


$$P(z,t|z_0,t_0) = \frac{e^{-\frac{(z-z_0)^2}{2(t-t_0)}}}{\sqrt{2\pi(t-t_0)}}$$

$$\lim_{\rho\to 1} \frac{\langle X(t_1)\dots X(t_n)\rangle_c}{1-\rho} \sim 2\sqrt{2}\int_{-\infty}^{0} dz_0\, \mathbb{P}\left(Z(t_1)>0,\dots,Z(t_n)>0 | Z(0)=z_0\right)$$



[**Poncet**, Grabsch & Bénichou, in preparation]

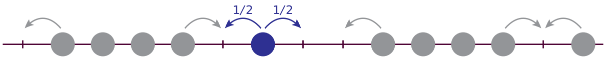


$$P(z,t|z_0,t_0) = \frac{e^{-\frac{(z-z_0)^2}{2(t-t_0)}}}{\sqrt{2\pi(t-t_0)}}$$

$$\lim_{\rho\to 1} \frac{\langle X(t_1)\dots X(t_n)\rangle_c}{1-\rho} \sim 2\sqrt{2}\int_{-\infty}^{0} dz_0\, \mathbb{P}\left(Z(t_1)>0,\dots,Z(t_n)>0|Z(0)=z_0\right)$$

If biased, we need to consider the 2^n possibilities, $Z(t_1) \gtrless 0, \dots, Z(t_n) \gtrless 0$

Dense limit: results for time correlations (no bias)



[**Poncet**, Grabsch & Bénichou, in preparation]

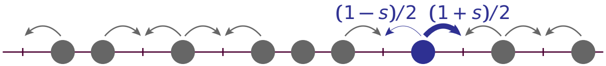
$$\lim_{\rho\to 1} \frac{\langle X(t_1)\rangle}{1-\rho} \sim 0$$

$$\lim_{\rho\to 1} \frac{\langle X(t_1)X(t_2)\rangle_c}{1-\rho} \sim \frac{1}{\sqrt{2\pi}}\left[\sqrt{t_1}+\sqrt{t_2}-\sqrt{t_2-t_1}\right]$$

$$\lim_{\rho\to 1} \frac{\langle X(t_1)X(t_2)X(t_3)\rangle_c}{1-\rho} \sim 0$$

$$\lim_{\rho\to 1} \frac{\langle X(t_1)X(t_2)X(t_3)X(t_4)\rangle_c}{1-\rho} \sim \sum_{0\le i<j\le 4} C_{ij}\sqrt{t_j-t_i}$$

Dense limit: results for time correlations (biased)



[**Poncet**, Grabsch & Bénichou, in preparation]

$$\lim_{\rho\to 1} \frac{\langle X(t_1)\rangle}{1-\rho} \sim s\sqrt{\frac{2t}{\pi}}$$

$$\lim_{\rho\to 1} \frac{\langle X(t_1)X(t_2)\rangle_c}{1-\rho} \sim \frac{1}{\sqrt{2\pi}}\left[(1+s^2)\sqrt{t_1}+(1-s^2)(\sqrt{t_2}-\sqrt{|t_2-t_1|})\right]$$

$$\lim_{\rho\to 1} \frac{\langle X(t_1)X(t_2)X(t_3)\rangle_c}{1-\rho} \sim \frac{s}{2\sqrt{2\pi}}\Big\{\left[4+(1-s^2)(a_{123}-1)\right]\sqrt{t_1}$$
$$+(1-s^2)\left[\sqrt{t_2}+(b_{0123}-1)\sqrt{t_3}-\sqrt{t_2-t_1}+\sqrt{t_3-t_1}-a_{012}\sqrt{t_3-t_2}\right]\Big\}$$

$$\lim_{\rho\to 1} \frac{\langle X(t_1)X(t_2)X(t_3)X(t_4)\rangle_c}{1-\rho} \sim \sum_{0\le i<j\le 4} C_{ij}\sqrt{t_j-t_i}$$

$$a_{ijk} = A\left(\frac{t_j-t_i}{t_k-t_j}\right), \quad b_{ijkl} = A\left(\frac{(t_j-t_i)(t_l-t_k)}{(t_l-t_i)(t_k-t_j)}\right), \quad A(u) = \arctan\sqrt{u}$$

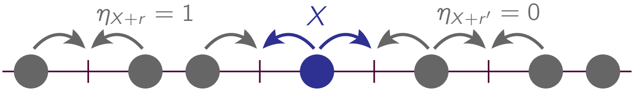
At high density,
the **complex behavior of a driven intruder** in a single-file
is encoded in the **simple dynamics of a diffusive vacancy**!

At high density,
the **complex behavior of a driven intruder** in a single-file
is encoded in the **simple dynamics of a diffusive vacancy**!

- One-time statistics at arbitrary time
- Driving effect on other particles
- Non-Gaussian *N*-time statistics

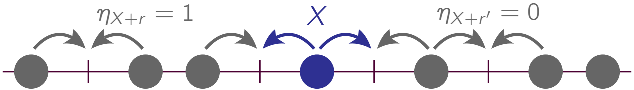
1. At equilibrium

using generalized density profiles



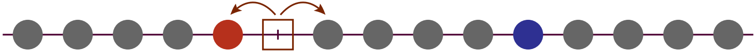
1. At equilibrium

using generalized density profiles

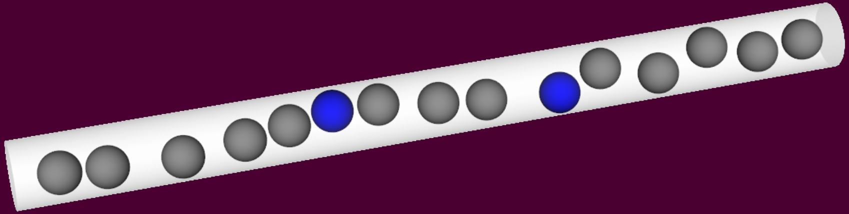


2. Out-of-equilibrium at high density

by looking at the dynamics of a single vacancy



Thank you for your attention!



Fourth annealed cumulant (no bias)

$$\lim_{\rho\to 1}\frac{\langle X(t_1)X(t_2)X(t_3)X(t_4)\rangle_c^A}{1-\rho}\sim\frac{1}{4\sqrt{2\pi}}\sum_{0\le i<j\le 4}C_{ij}^{(4)}\sqrt{t_j-t_i}$$

$$C_{01}^{(4)}=1+a_{123}+a_{124}+a_{234}\qquad C_{02}^{(3)}=1+a_{234}\qquad C_{03}^{(4)}=1+b_{0123}\qquad C_{04}^{(4)}=1+b_{0124}+b_{0134}+b_{0234}$$

$$C_{12}^{(4)}=-1-a_{234}\qquad C_{13}^{(4)}=-1\qquad C_{14}^{(4)}=-1-b_{1234}$$

$$C_{23}^{(4)}=-a_{012}\qquad C_{24}^{(4)}=-a_{012}\qquad C_{34}^{(4)}=-a_{013}-a_{023}+a_{123}$$

$$a_{ijk}=A\left(\frac{t_j-t_i}{t_k-t_j}\right),\quad b_{ijkl}=A\left(\frac{(t_j-t_i)(t_l-t_k)}{(t_l-t_i)(t_k-t_j)}\right),\quad A(u)=\arctan\sqrt{u}$$

Fourth quenched cumulant (no bias)

$$\lim_{\rho\to 1}\frac{\langle X(t_1)X(t_2)X(t_3)X(t_4)\rangle_c^Q}{1-\rho} \sim \frac{1}{4\sqrt{2\pi}}\sum_{1\le i<j\le 4}\left(D_{ij}^{(4)}\sqrt{t_j-t_i}+E_{ij}^{(4)}\sqrt{t_i+t_j}\right)$$

$$D_{ij}^{(4)} = a'_{ikl} - 2c_{ikl} + \tilde{D}_{ij}^{(4)}, \qquad E_{ij}^{(4)} = -2d_{ijkl} + 6h_{ijkl} + \tilde{E}_{ij}^{(4)},$$

where k, l are the two indices in $\{1,2,3,4\}$ that are different from i, j and such that $k<l$

$$\tilde{D}_{12}^{(4)} = -a_{234}, \qquad \tilde{D}_{13}^{(4)} = 0, \qquad \tilde{D}_{14}^{(4)} = -b_{1234},$$

$$\tilde{D}_{23}^{(4)} = 1 - a'_{421}, \qquad \tilde{D}_{24}^{(4)} = 1 - a'_{312}, \qquad \tilde{D}_{34}^{(4)} = 1 + a_{123} - a'_{123} - a'_{213},$$

$$\tilde{E}_{12}^{(4)} = a_{134} + a_{234}, \qquad \tilde{E}_{13}^{(4)} = a_{124} - 2g_{1324}, \qquad \tilde{E}_{14}^{(4)} = a_{123} + e'_{1423},$$

$$\tilde{E}_{23}^{(4)} = -2g_{2314} - 2g_{3214}, \qquad \tilde{E}_{24}^{(4)} = e''_{2413} - 2g_{4213}, \qquad \tilde{E}_{34}^{(4)} = e''_{3412} + e''_{4312}.$$

$$a_{ijk} = A\left(\frac{t_j-t_i}{t_k-t_j}\right) \qquad a'_{ijk} = A\left(\frac{t_i+t_j}{t_k-t_j}\right) \qquad b_{ijkl} = A\left(\frac{(t_j-t_i)(t_l-t_k)}{(t_k-t_j)(t_l-t_i)}\right) \qquad c_{ijk} = A\left(\frac{t_i^2}{t_it_j+t_it_k+t_jt_k}\right)$$

$$d_{ijkl} = A\left(\frac{t_it_j+t_it_k+t_jt_k}{(t_i+t_j)(t_l-t_k)}\right) \qquad e_{ijkl} = A\left(\frac{(t_j-t_l)(t_i+t_k)}{(t_l-t_k)(t_i+t_j)}\right) \qquad e'_{ijkl} = e_{ijkl} - 2g_{ijkl} - 2g_{ijlk} \qquad e''_{ijkl} = e'_{ijkl} + f_{ijkl}$$

$$f_{ijkl} = A\left(\frac{(t_k-t_i)(t_l-t_j)}{(t_i+t_j)(t_k+t_l)}\right) \qquad g_{ijkl} = A\left(\frac{t_i^2(t_j-t_k)}{(t_i+t_j)(t_it_k+t_it_l+t_kt_l)}\right) \qquad h_{ijkl} = A\left(\frac{t_i^2t_j^2}{(t_i+t_j)[t_it_j(t_k+t_l)+t_kt_l(t_i+t_j)]}\right)$$