Thermal pressure on ultra-relativistic bubble walls

Andrew J. Long
Rice University
@ ICTS workshop
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RICE

First order cosmological phase transitions

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- SUSY scale
- electroweak scale – Higgs fields gets a vev
- QCD scale – color confinement & chiral symmetry breaking
- misc: B-breaking, L-breaking, PQ-breaking, dark sectors, inflaton sector, …

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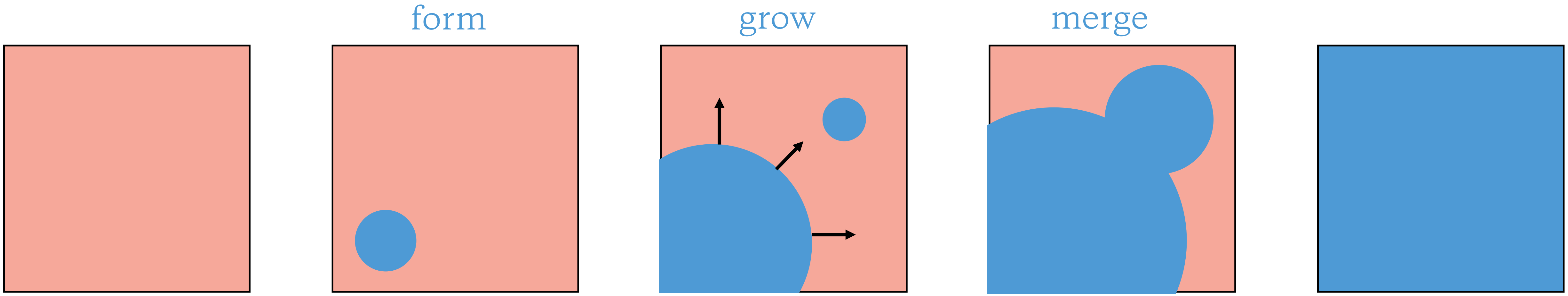
SM predicts continuous crossover for EW & QCD *(at small chemical potential)* ➔ new physics is required for a first order cosmological phase transition ➔ observational probes of 1^{st} order PT are necessarily tests of new physics

Bubble business

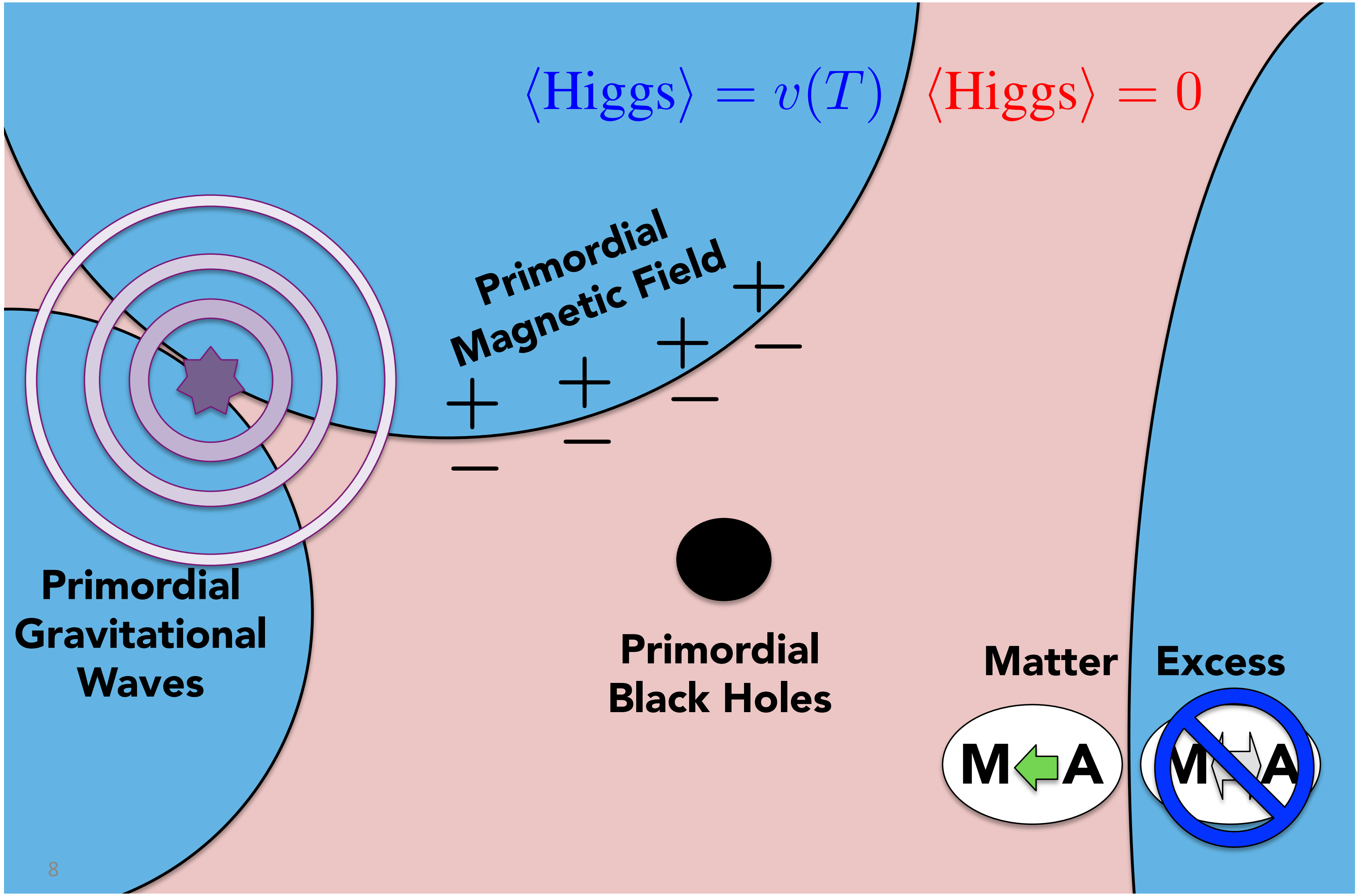
Phase coexistence means boundaries between phase domains (a.k.a., bubbles or walls)



today's talk:

How quickly do bubbles grow?
What is the speed of the bubble walls?

Why care about bubble wall speed?



Equation of motion

for a nonrelativistic planar bubble wall:

$$\vec{F} = m\vec{a} \quad \Rightarrow \quad P = \sigma \dot{v}_w$$

(P = force/area = pressure)
(σ = mass/area = tension)

pressure on the wall

$$P = P_{\text{vac}} - P_{\text{therm}}(v_w)$$

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Starting simple: bubbles in vacuum

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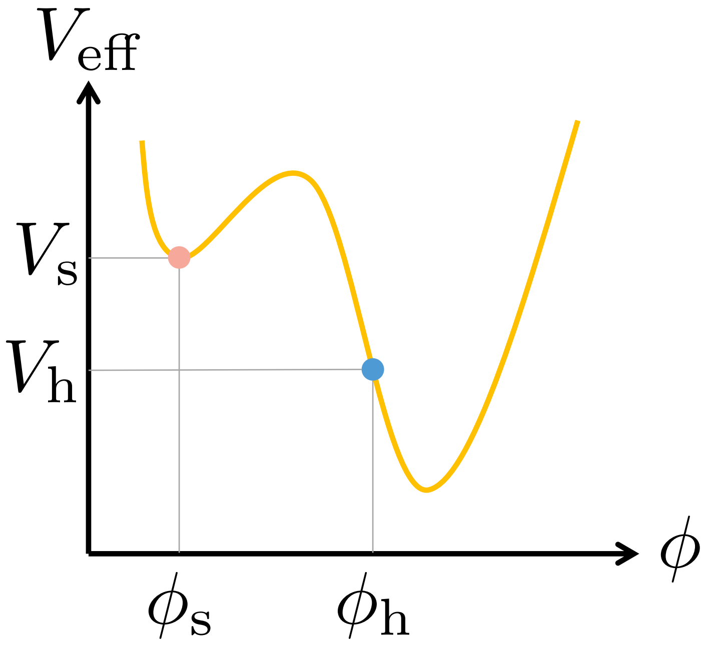
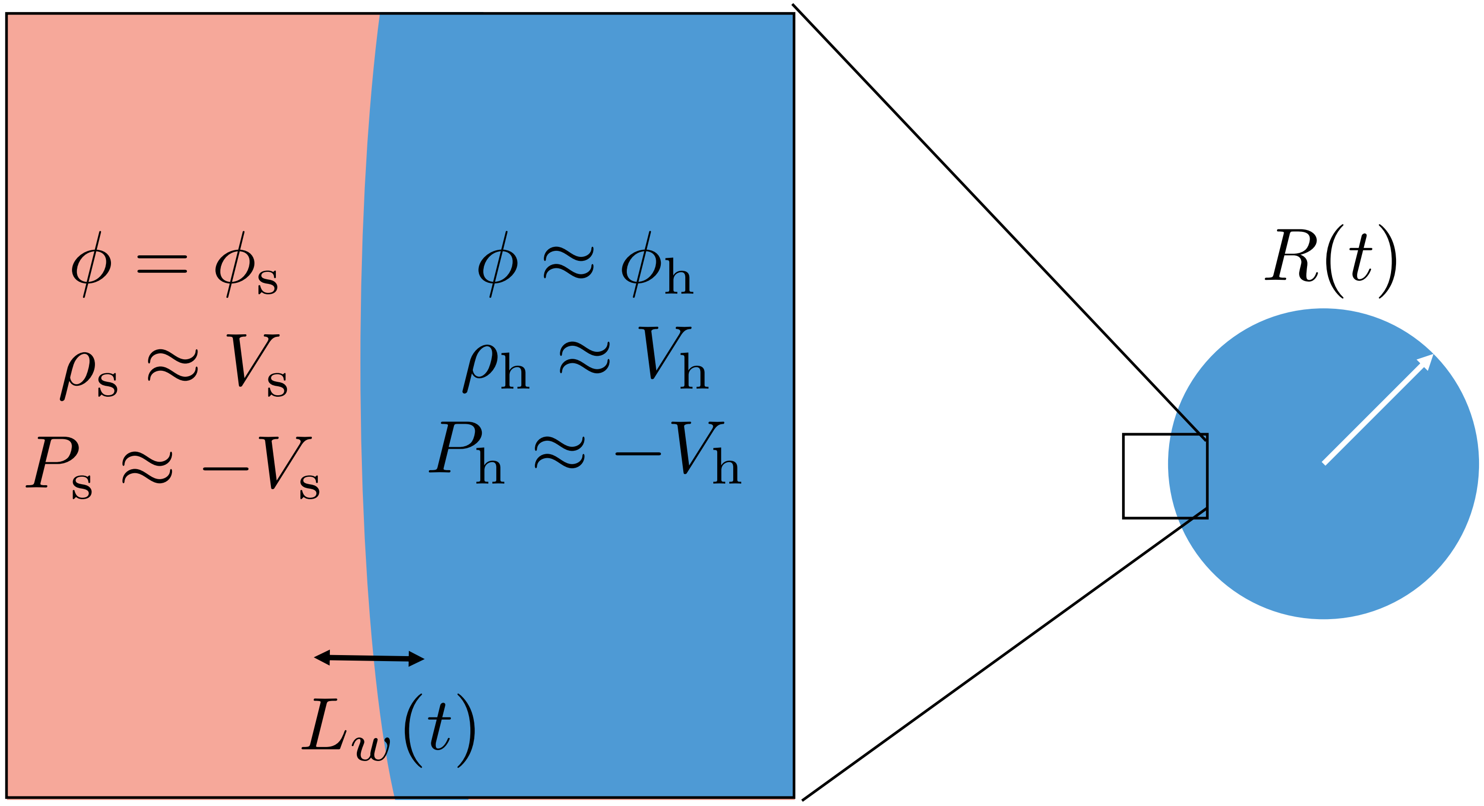
(This calculation applies more generally for a sufficiently dilute medium with small friction.)





$L_w \ll R \Leftrightarrow$ planar approx.

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equation of motion for a planar bubble wall:

$$P_{\text{vac}} = \sigma \dot{v}_w$$

differential vacuum pressure on the wall:

$$P_{\text{vac}} = P_{\text{h}} - P_{\text{s}} = V_{\text{s}} - V_{\text{h}}$$

solution for the wall speed:

$$v_w(t) = v_0 + (P_{\text{vac}}/\sigma)\,(t - t_0)$$

wall approaches the speed of light after a time:

$$t - t_0 \approx \tau = c\sigma/P_{\text{vac}}$$

this is typically a very short time compared to Hubble:

$$\sigma \sim M^3 \quad \text{and} \quad P_{\text{vac}} \sim M^4 \quad \text{and} \quad H \sim M^2/M_{\text{pl}} \quad \text{so} \quad \tau H \sim M/M_{\text{pl}} \ll 1$$

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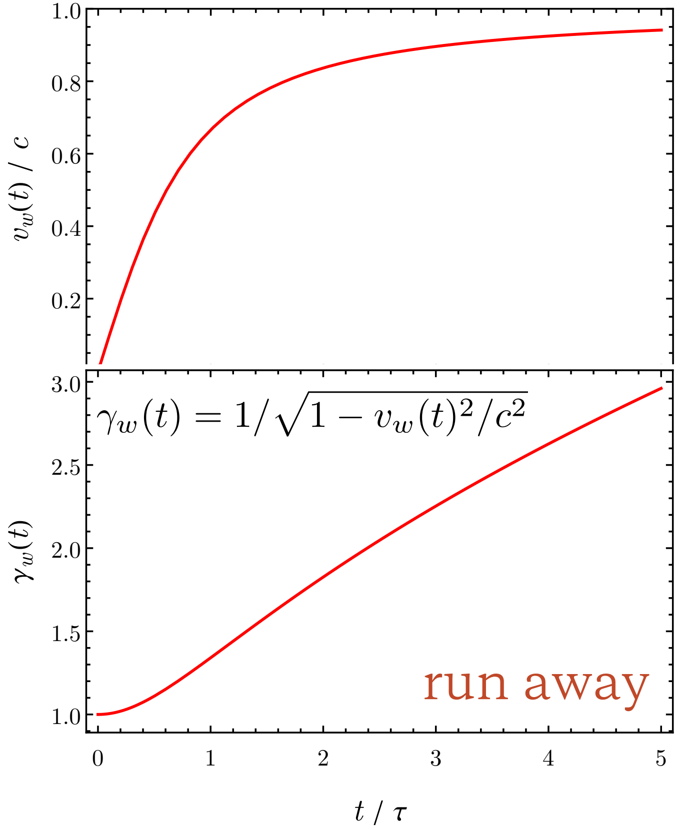
$$P_{\mathrm{vac}}/\sigma = \gamma_w \frac{\mathrm{d}}{\mathrm{d}t}\left(\gamma_w v_w\right)$$



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bubble wall rapidly accelerates as $v_w \rightarrow c$ and $\gamma_w \rightarrow$ infinity
➔ "runaway" growth

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what if it's very large?

More difficult: bubbles in a fluid

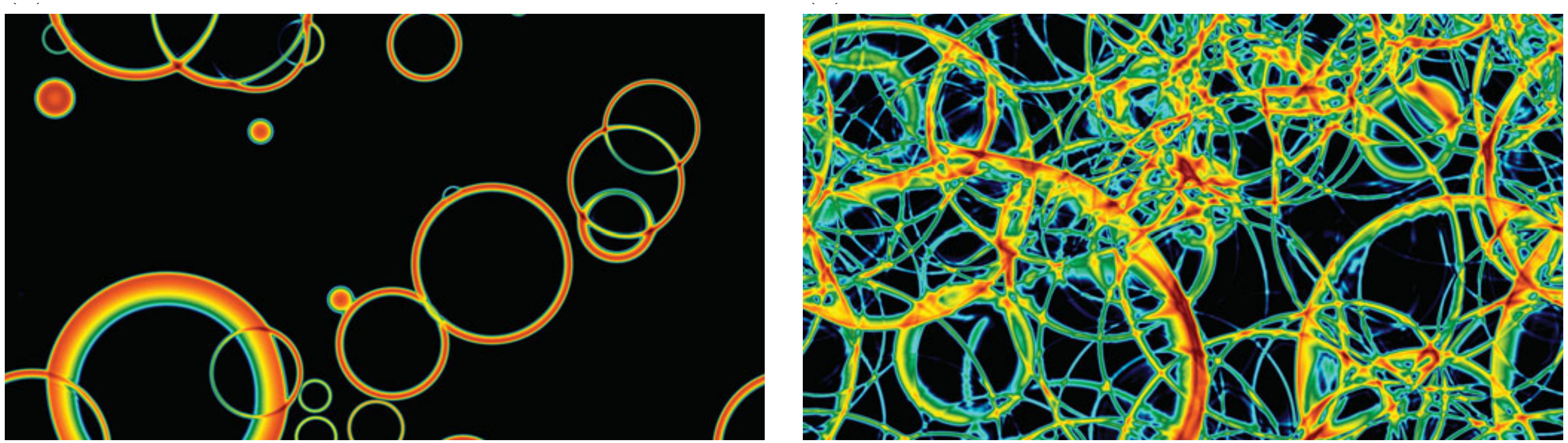
[simulation: Weir (2022)]

On the other extreme, bubbles can be tightly coupled to a medium.

The system is very nonlinear -- bubble moves fluid -- fluid drags bubble.

Often studied using numerical simulation.

Important to study for predicting gravitational waves and baryogenesis.



see talks by: Ryuske Jinno,   Hu-aike Guo,   Thomas Konstandin,   Graham White

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complicated nonlinear dynamics – study with simulation

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Is there a middle ground
where the calculation is still
analytically tractable?

Manageable middle ground: ultrarelativistic bubbles

If the bubbles are ultra-relativistic ($\gamma_w >> 1$) then they're moving much faster than the speed of sound in the medium ($c_s^2 = 1/3$). So the fluid in front of the wall doesn't "know" that the wall is coming. This makes it much easier to calculate the thermal pressure and wall speed.

$$P_{\mathrm{vac}} - P_{\mathrm{therm}} = \sigma \dot{v}_w$$

The rest of this talk is all about:
how to calculate P_{therm}

Literature

Lots of recent interest in ultra-relativistic bubbles, particularly at electroweak phase transition.

2009 -- Bodeker & Moore
2017 -- Bodeker & Moore
2020 -- Hoche, Kozaczuk, AL, Turner, & Wang

… lots of papers including but not limited to:

- 2020 – Azatov & Vanvlasselaer
- 2021 – Azatov, Vanvlasselaer, & Yin
- 2022 – Gouttenoire, Jinno, & Salas
- 2022 – De Curtis, Rose, Guiggiani, Muyor, & Panico
- 2022 – Laurent & Cline
- 2023 – Azatov, Barni, Petrossian-Byrne, & Vanvlasselaer
- …

2024 -- AL & Turner

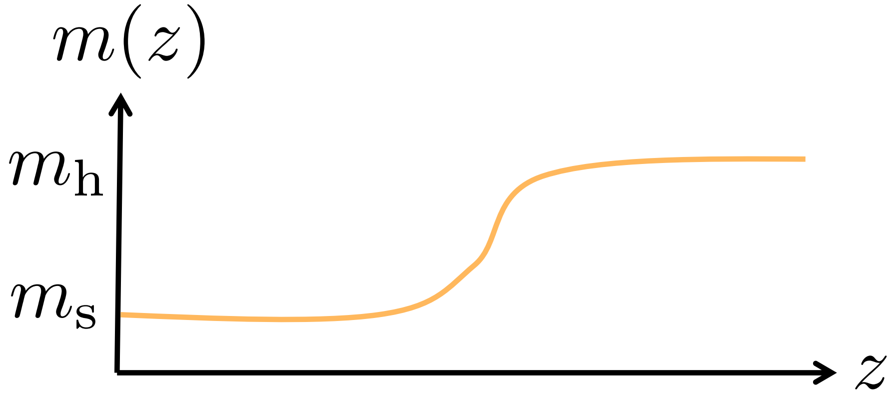
my plan: summarize BM09, BM17, & my two papers

BM09
one-to-one transitions

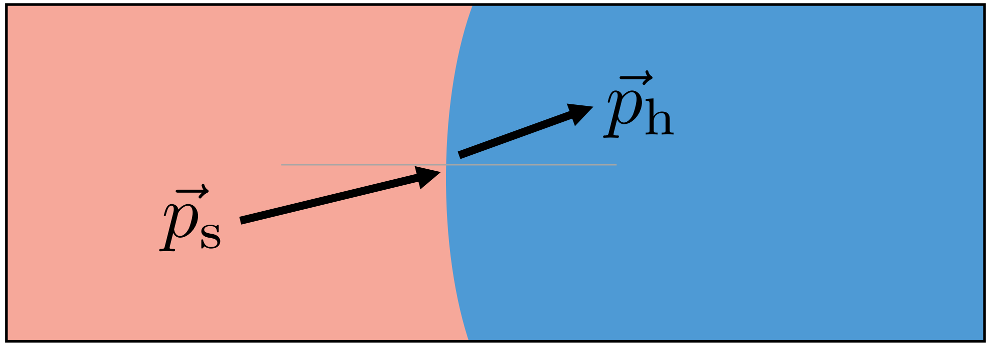
Kinematics at the wall

[Bodeker & Moore (2009)]

Suppose that there is some particle species that gains mass upon entering the bubble.



(in rest frame of the wall)



$$\Delta E = 0$$

$$\Delta \vec{p}_{\perp} = 0$$

on-shell

$\longrightarrow$

$$\begin{aligned}\Delta p_z &= p_{z,\mathrm{s}} - p_{z,\mathrm{h}} \\ &= \sqrt{E^2 - |\vec{p}_\perp|^2 - m_{\mathrm{s}}^2} - \sqrt{E^2 - |\vec{p}_\perp|^2 - m_{\mathrm{h}}^2} \\ &\approx \frac{m_{\mathrm{h}}^2 - m_{\mathrm{s}}^2}{2E} \\ &\sim \Delta m^2 / \gamma_w T\end{aligned}$$

since $E \sim \gamma_w T >> p_{\mathrm{perp}}$ & m

Pressure on the wall

[Bodeker & Moore (2009)]

The longitudinal momentum transfer induces a force (i.e., thermal pressure) on the wall:

$$P_{\mathrm{therm}} = \nu \int \frac{\mathrm{d}^3\vec{p}}{(2\pi)^3} \, f(\vec{p}) \, v_z \, \Delta p_z$$

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$$\sim \mathcal{F} \sim v_w \gamma_w T^3 \qquad \sim \Delta m^2 / \gamma_w T$$

Observe that the dependence on γ_w cancels out!

$$\Rightarrow \quad P_{\text{therm}} \sim \Delta m^2 T^2$$

How does the thermal pressure affect the motion?

$$P_{\text{vac}} - P_{\text{therm}} = \sigma \dot{v}_w$$

Runaway is still possible despite the thermal pressure:

$$\text{if} \quad P_{\text{vac}} > P_{\text{therm}} \quad \text{then} \quad \gamma_w \to \infty \quad \text{runaway}$$

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2015

Science with the space-based interferometer eLISA. II: Gravitational waves from cosmological phase transitions

Chiara Caprini[a], Mark Hindmarsh[b,c], Stephan Huber[b], Thomas Konstandin[d], Jonathan Kozaczuk[e], Germano Nardini[f], Jose Miguel No[b], Antoine Petiteau[g], Pedro Schwaller[d], Géraldine Servant[d,h], David J. Weir[i]

BM17

one-to-two transitions

Three-body kinematics

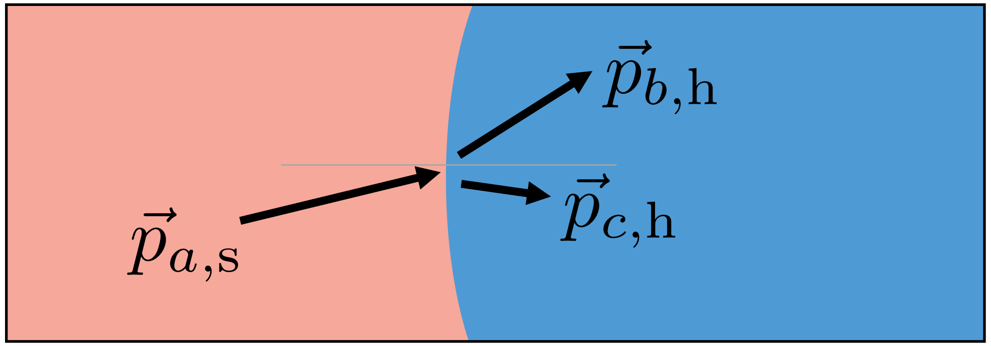
[Bodeker & Moore (2017)]

Consider a model in which a particle splits upon hitting the wall – transition radiation.

examples: $q \to q' + W$ or $e^- \to e^- + \gamma$

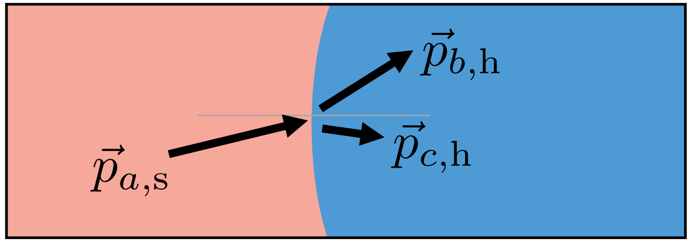
Now we add a flavor label (a,b,c) to distinguish different particle species.

(in rest frame of the wall)



Thermal pressure

[Bodeker & Moore (2017)]



Since momenta of the recoiling particles are quantum random variables, we have to calculate the *average* momentum transfer:

$$P_{\mathrm{th}} = \nu_a \int \frac{\mathrm{d}^3\vec{p}_a}{(2\pi)^3} f_a(\vec{p}_a)\, v_{a,z} \langle \Delta p_z \rangle$$

(note angled brackets)

$$\langle \Delta p_z \rangle = \int \mathrm{d}\mathbb{P}_{a\to bc}\, \Delta p_z \qquad \text{where} \qquad \Delta p_z = p_{a,z,\mathrm{s}} - p_{b,z,\mathrm{h}} - p_{c,z,\mathrm{h}}$$

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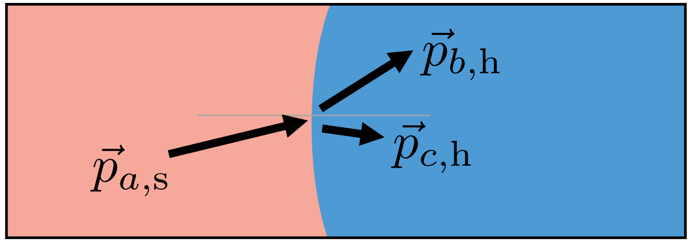

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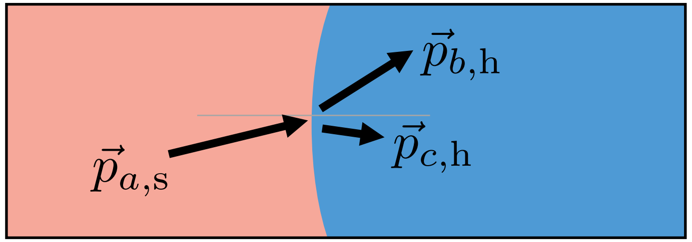
How can we calculate the differential probability?

QPS = quantum particle splitting formalism → used by BM17
SCR = semiclassical current radiation formalism → used by HKLTW20 & LT24

Quantum Particle Splitting (QPS) formalism

[Bodeker & Moore (2017)]



Evaluate the probability as an S-matrix element

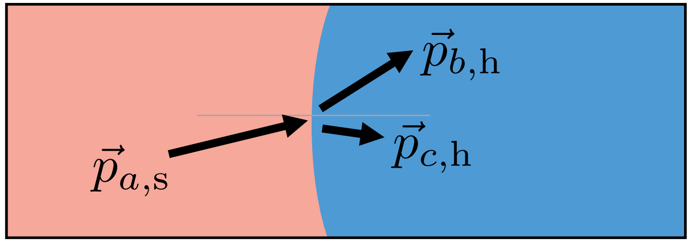
$$\mathrm{d}\mathbb{P}_{a\to bc} = \frac{1}{2E_a}\frac{\mathrm{d}^3\vec{p}_{b,\mathrm{s}}}{(2\pi)^3}\frac{1}{2E_b}\frac{\mathrm{d}^3\vec{p}_{c,\mathrm{s}}}{(2\pi)^3}\frac{1}{2E_c}(2\pi)^3\frac{p_{a,z,\mathrm{s}}}{E_a}\,\delta(\vec{p}_{a,\perp}-\vec{p}_{b,\perp}-\vec{p}_{c,\perp})\,\delta(E_a-E_b-E_c)\,|\mathcal{M}_{a\to bc}|^2$$

Use WKB approximation to calculate mode functions for particles that change mass at the wall - this affects $M_{a\text{->}bc}$

let's see
some
examples

Examples

[Bodeker & Moore (2017)]
[appendix of AL & Turner (2024)]



massless radiator / massive radiation

$$q \to q' + W$$

$m_{\mathrm{a,s}} = m_{\mathrm{a,h}} = m_{\mathrm{b,s}} = m_{\mathrm{b,h}} = 0$

$$\langle \Delta p_z \rangle \approx \frac{g^2 C_2[R]}{4\pi^2} \left(\log \frac{m_{c,\mathrm{h}}}{m_{c,\mathrm{s}}} - \frac{m_{c,\mathrm{h}}^2 - m_{c,\mathrm{s}}^2}{2m_{c,\mathrm{h}}^2} \right) \frac{m_{c,\mathrm{h}}^2}{E_{c,\mathrm{IR}}}$$

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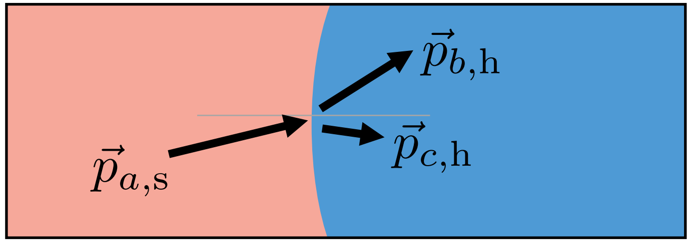

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$\propto \gamma_w^0$

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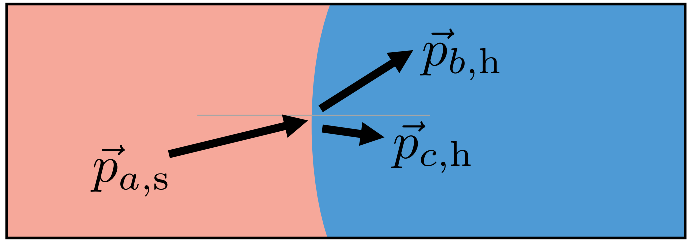

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$$\propto \gamma_w^{-1}$$

Summary of BM17

Bodeker & Moore (2017) employs the Quantum Particle Splitting (QPS) formalism to calculate the average momentum transfer to the bubble wall, which factors into the thermal pressure.

If there exists a channel where a massless radiator emits a massive (spin-1) radiation, then they find that this channel will dominate the thermal pressure, going like:

$$q \to q' + W \quad \Rightarrow \quad P_{\mathrm{therm}} \propto \gamma_w^1$$

Since the vacuum pressure does not grow with γ_w, they conclude that the thermal pressure will eventually win out, and the bubble wall reaches an (ultrarelativistic) terminal velocity.

For channels in which a massive radiator emits massless (spin-1) radiation, they find that the thermal pressure does not grow with increasing γ_w.

$$e^- \to e^- + \gamma \quad \Rightarrow \quad P_{\mathrm{therm}} \propto \gamma_w^0$$

HKLTW20 & LT24
semiclassical current

Summary of HKLTW20

[HKLTW20 = 2007.10343]

- Motivated by BM17, my collaborators and I began to study EW bubble wall velocity.
- We noted the IR sensitivity of BM17's result.
- We suspected that it could be important to re-sum multiple soft emissions.
- We didn't see how to do this using BM's formalism (QPS) for calculating matrix elements.
- So we adopted a different formalism (SCR) that treats the radiator particle as a classical current and calculates the spectrum of radiation.
- Using this SCR formalism, we re-summed soft emissions to all orders.
- We concluded that the average momentum transfer and thermal pressure scale as:

$$\langle \Delta p_z \rangle \propto \gamma_w^1 \quad \text{and} \quad P_{\text{therm}} \propto \gamma_w^2$$

regardless of whether the radiator is massive, or the radiation is massive.

- Note that we did two things differently from BM17 - we used SCR rather than QPS - and we re-summed multiple soft emissions. It wasn't clear which change led to the different result.

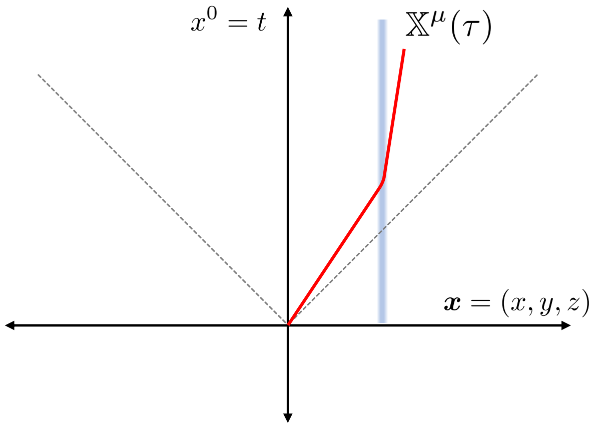
in the newer paper LT24 we clarify a subtlety about SCR formalism, which explains the different γ_w scaling

Semiclassical Current Radiation (SCR) formalism

[HKLTW20 & LT24]

Focus on channels with massive radiator / massless radiation, like: $e^- \to e^- + \gamma$

Treat the radiating particle as a classical electromagnetic current.



$$j^\mu(x) = qe \int_{-\infty}^{\infty} \mathrm{d}\tau \, \mathbb{U}^\mu(\tau) \, \delta^{(4)}\big(x - \mathbb{X}(\tau)\big)$$

Treat the electromagnetic radiation as quantum, i.e. photons.

$$\hat{A}_\mu(x) = \int \frac{\mathrm{d}^3\boldsymbol{p}}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} \sum_{s=\pm 1} \Big[\hat{a}_{\boldsymbol{p},s} \, \varepsilon_\mu(\boldsymbol{p}, s) \, \mathrm{e}^{-\mathrm{i}p\cdot x} + \hat{a}^\dagger_{\boldsymbol{p},s} \, \varepsilon^*_\mu(\boldsymbol{p}, s) \, \mathrm{e}^{\mathrm{i}p\cdot x} \Big]$$

$$\hat{H}_{\text{int}}(t) = \int \mathrm{d}^3\boldsymbol{x} \, j^\mu(t, \boldsymbol{x}) \, \hat{A}_\mu(t, \boldsymbol{x})$$

Emission probability

[HKLTW20 & LT24]

For a given current $j^\mu(x)$, calculate the probability distribution over photon momenta:

$$\mathrm{d}\mathbb{P}_{0\to 0\gamma}(\boldsymbol{p}) = \frac{\mathrm{d}^3\boldsymbol{p}}{(2\pi)^3}\,\frac{1}{2E_p}\sum_{s=\pm 1}\left|W_{0\to 0\gamma}\right|^2$$

where $$W_{0\to 0\gamma}(\boldsymbol{p},s) = \big\langle (\boldsymbol{p},s)_{\mathrm{OUT}} \big| 0_{\mathrm{IN}} \big\rangle$$

After a little work …

$$\begin{aligned} W_{0\to 0\gamma}(\boldsymbol{p},s) &= (-\mathrm{i}) \int \mathrm{d}^4x\, j(x)\cdot \varepsilon^*(\boldsymbol{p},s)\, \mathrm{e}^{\mathrm{i}p\cdot x} \\ &= (-\mathrm{i})\, \tilde{j}(p)\cdot \varepsilon^*(\boldsymbol{p},s) \end{aligned}$$

$$\mathrm{d}\mathbb{P}_{0\to 0\gamma} = -\frac{\mathrm{d}^3\boldsymbol{p}}{(2\pi)^3}\,\frac{1}{2E_p}\,\tilde{j}(p)^*\cdot \tilde{j}(p)$$

Emission probability

[HKLTW20 & LT24]

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$$= (-\mathrm{i})\,\tilde{j}(p)\cdot\varepsilon^*(\boldsymbol{p},s)$$

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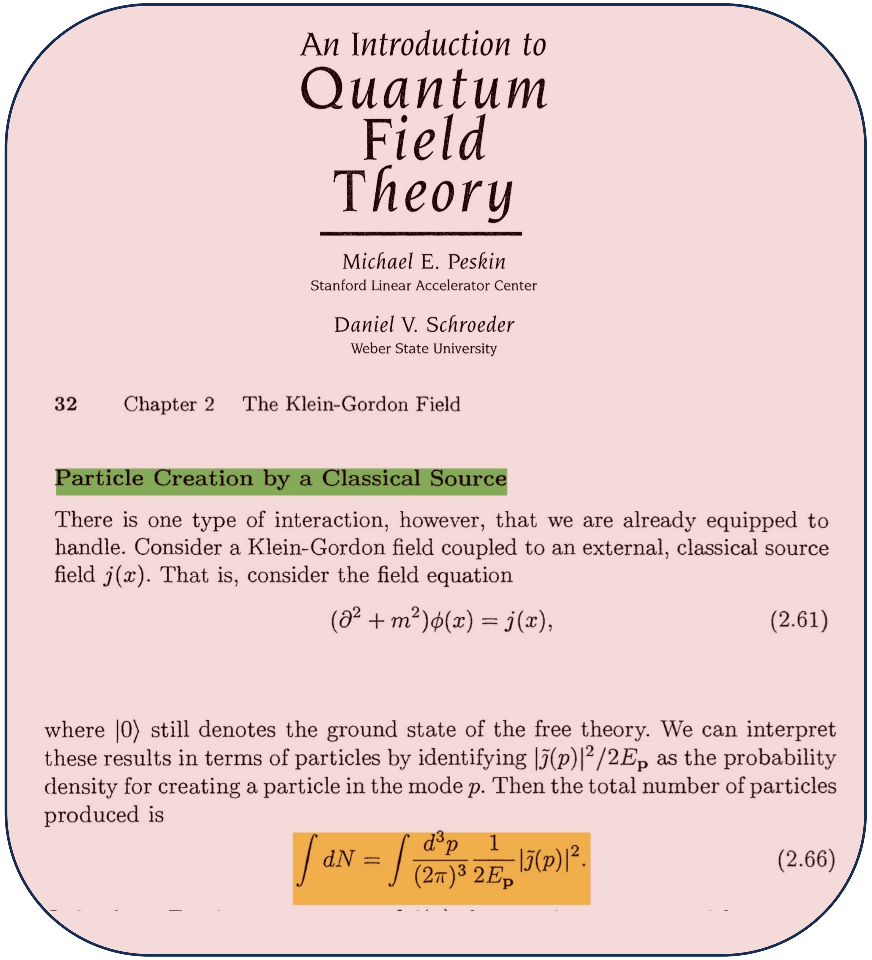


An Introduction to Quantum Field Theory

Michael E. Peskin
Stanford Linear Accelerator Center

Daniel V. Schroeder
Weber State University

32 Chapter 2 The Klein-Gordon Field

Particle Creation by a Classical Source

There is one type of interaction, however, that we are already equipped to handle. Consider a Klein-Gordon field coupled to an external, classical source field $j(x)$. That is, consider the field equation

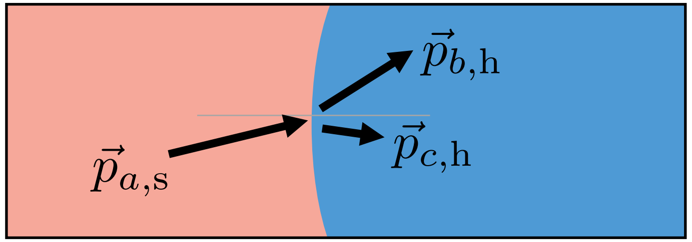
$$(\partial^2 + m^2)\phi(x) = j(x), \tag{2.61}$$

where $|0\rangle$ still denotes the ground state of the free theory. We can interpret these results in terms of particles by identifying $|\tilde{j}(p)|^2/2E_{\mathbf{p}}$ as the probability density for creating a particle in the mode p. Then the total number of particles produced is

$$\int dN = \int \frac{d^3p}{(2\pi)^3}\frac{1}{2E_{\mathbf{p}}}|\tilde{j}(p)|^2. \tag{2.66}$$

Comparison of QPS & SCR

In both formalisms, we calculate the average momentum transfer as



$$\langle \Delta p_z \rangle = \int \mathrm{d}\mathbb{P}_{a\to bc}\, \Delta p_z$$

where $\Delta p_z = p_{a,z,\mathrm{s}} - p_{b,z,\mathrm{h}} - p_{c,z,\mathrm{h}}$

QPS Formalism: splitting probability ~~ you pick p_a and calculate probability over p_b and p_c

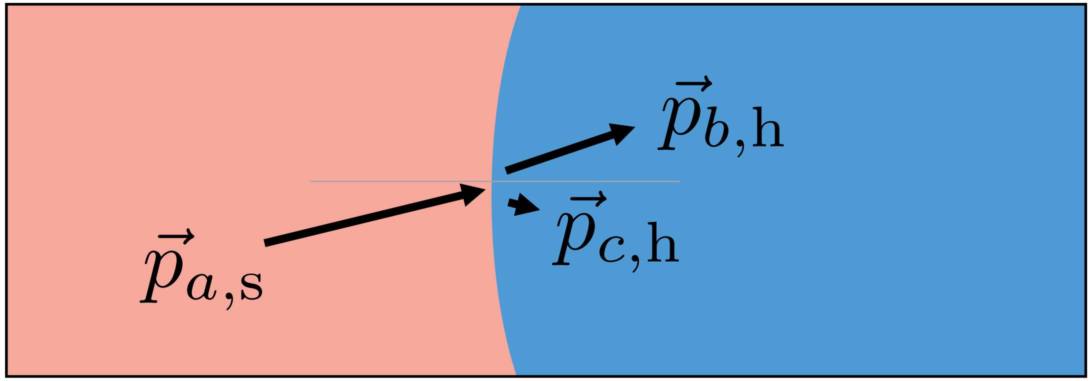
$$\mathrm{d}\mathbb{P}_{a\to bc} = \frac{1}{2E_a} \frac{\mathrm{d}^3\vec{p}_{b,\mathrm{s}}}{(2\pi)^3} \frac{1}{2E_b} \frac{\mathrm{d}^3\vec{p}_{c,\mathrm{s}}}{(2\pi)^3} \frac{1}{2E_c} (2\pi)^3 \frac{p_{a,z,\mathrm{s}}}{E_a} \delta(\vec{p}_{a,\perp} - \vec{p}_{b,\perp} - \vec{p}_{c,\perp})\, \delta(E_a - E_b - E_c)\, |\mathcal{M}_{a\to bc}|^2$$

SCR Formalism: emission probability ~~ you pick p_a and p_b and calculate probability over p_c

$$\mathrm{d}\mathbb{P}_{0\to 0\gamma} = -\frac{\mathrm{d}^3\vec{p}}{(2\pi)^3} \frac{1}{2E_p} \tilde{j}(p)^* \cdot \tilde{j}(p)$$

How to choose p_b?

If the radiation is very soft (p_c ~ small) then the kinematics are approx. same as 1-to-1 transition:



$$|\vec{p}_{c,\mathrm{h}}| \ll |\vec{p}_{b,\mathrm{h}}|$$

$$\left.\begin{aligned} E_b &= E_a \\ \vec{p}_{b,\perp} &= \vec{p}_{a,\perp} \\ &\text{on-shell} \end{aligned}\right\} \longrightarrow \langle \Delta p_z \rangle \approx \frac{g^2 C_2[R]}{4\pi^2} \left(\log \frac{m_{b,\mathrm{h}}}{m_{a,\mathrm{s}}} - \frac{m_{b,\mathrm{h}}^2 - m_{a,\mathrm{s}}^2}{m_{b,\mathrm{h}}^2 + m_{a,\mathrm{s}}^2} \right) \frac{m_{b,\mathrm{h}}^2 + m_{a,\mathrm{s}}^2}{E_a} \log \frac{E_{c,\mathrm{UV}}}{E_{c,\mathrm{IR}}}$$

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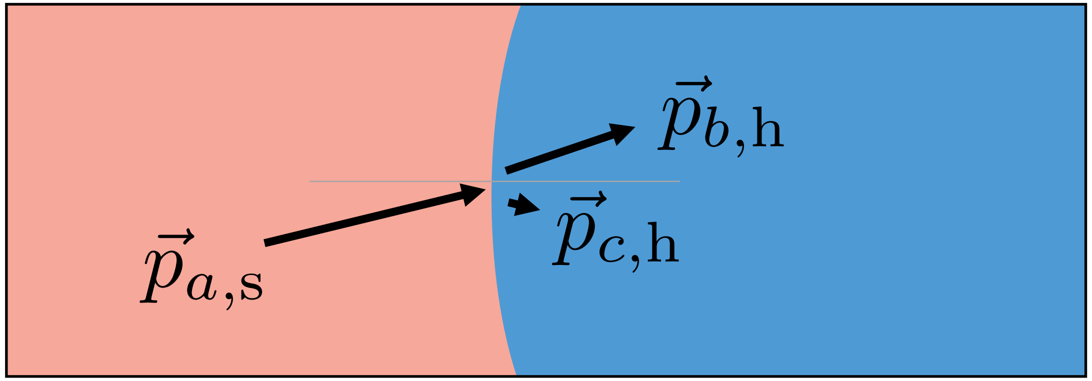


$$|\vec{p}_{c,\mathrm{h}}| \ll |\vec{p}_{b,\mathrm{h}}|$$

$$E_b = E_a$$

$$\vec{p}_{b,\perp} = \vec{p}_{a,\perp}$$

on-shell

$$\longrightarrow \langle \Delta p_z \rangle \approx \frac{g^2 C_2[R]}{4\pi^2} \left(\log \frac{m_{b,\mathrm{h}}}{m_{a,\mathrm{s}}} - \frac{m_{b,\mathrm{h}}^2 - m_{a,\mathrm{s}}^2}{m_{b,\mathrm{h}}^2 + m_{a,\mathrm{s}}^2} \right) \frac{m_{b,\mathrm{h}}^2 + m_{a,\mathrm{s}}^2}{E_a} \log \frac{E_{c,\mathrm{UV}}}{E_{c,\mathrm{IR}}}$$

SCR yields identical result as QPS formalism

(for massive radiator / massless radiation)

$$P_{\mathrm{therm}} \propto \gamma_w^0$$

How to choose p_b?

Alternatively, impose energy & transverse-momentum conservation among all 3 particles:

$$\left.\begin{array}{c} E_b = E_a - E_c \\ \vec{p}_{b,\perp} = \vec{p}_{a,\perp} - \vec{p}_{c,\perp} \\ \text{on-shell} \end{array}\right\} \longrightarrow \langle \Delta p_z \rangle = \frac{g^2}{2\pi^2} E_{c,\mathrm{UV}}$$

Now SCR yields a funny result

- does not vanish for $m_b = m_a$ (i.e., limit of no mass change)
- strong UV sensitivity

if you take $E_{c,\mathrm{UV}} \sim \gamma_w T$ then $\langle \Delta p_z \rangle \propto \gamma_w^1$ and $P_{\mathrm{therm}} \propto \gamma_w^2$

(same as HKLTW20)

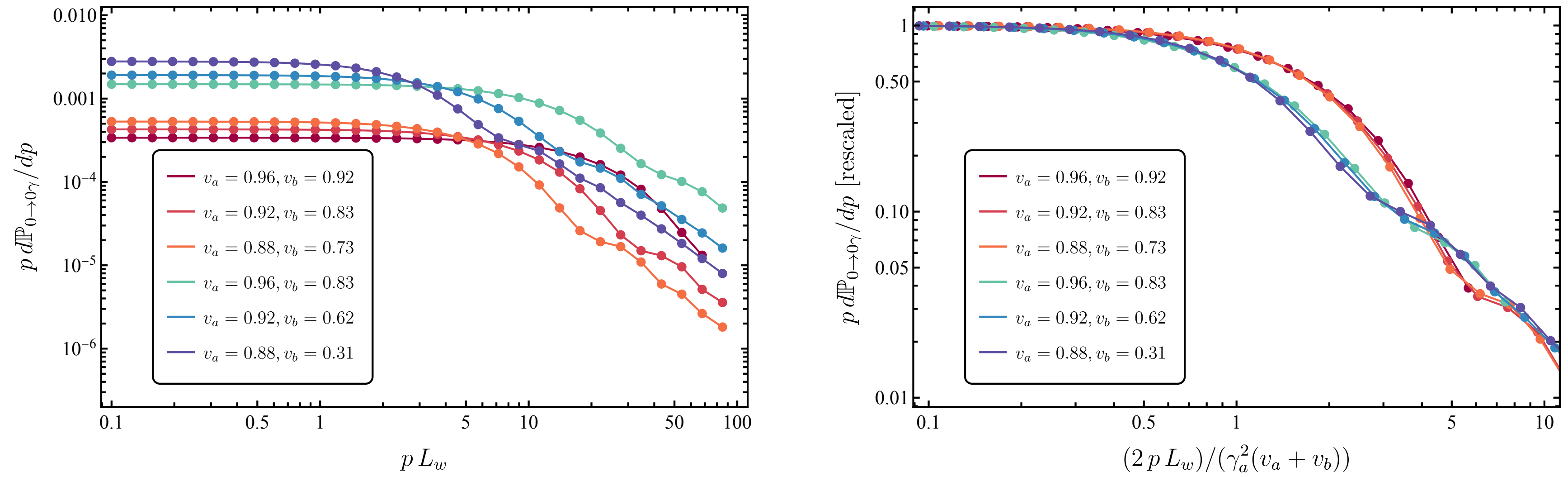
Nonzero wall thickness as a UV cutoff

Until this point we have been neglecting the thickness of the bubble wall: L_w

However, the inverse wall thickness enters as a UV cutoff, which is lower than $\gamma_w T$.

$$E_{c,\mathrm{UV}} \sim \gamma_w^2 L_w^{-1} \ll \gamma_w T$$

(same as classical bremsstrahlung)



This leads to a different γ_w scaling for the p_b choice with UV sensitivity.

Summary
& conclusion

Summary & conclusion

formalism	channel	how to choose $\boldsymbol{p}_b$	$\langle \Delta p_z \rangle$	UV cutoff p_{UV}	P_{therm}
C	$a \to b$	$\circ$	$\frac{m_b^2 - m_a^2}{2E_a}$	$\circ$	$\propto \gamma_w^0$
QPS	$a \to bc$	$\circ$	$\frac{q^2 e^2}{4\pi^2}\left(\log\frac{m_b}{m_a} - \frac{m_b^2 - m_a^2}{m_b^2 + m_a^2}\right)\frac{m_b^2 + m_a^2}{E_a}\log\frac{p_{\text{UV}}}{p_{\text{IR}}}$	$\circ$	$\propto \gamma_w^0$
SCR	$a \to bc$	1-to-1 kinematics	$\frac{q^2 e^2}{4\pi^2}\left(\log\frac{m_b}{m_a} - \frac{m_b^2 - m_a^2}{m_b^2 + m_a^2}\right)\frac{m_b^2 + m_a^2}{E_a}\log\frac{p_{\text{UV}}}{p_{\text{IR}}}$	$\circ$	$\propto \gamma_w^0$
	$a \to bc$	1-to-2 kinematics	$\frac{q^2 e^2}{2\pi^2}\, p_{\text{UV}}$	E_a	$\propto \gamma_w^2$

For models with massive radiator / massless radiation, both QPS and SCR formalisms yield:

$$e^- \to e^- + \gamma \quad \Rightarrow \quad P_{\text{therm}} \propto \gamma_w^0 \quad \Rightarrow \quad \text{runaway possible}$$

Take care to choose p_b when using the SCR formalism. Some choices gives $P_{\text{therm}} \propto \gamma_w^2$

Possibly interesting directions for future studies:
- Revisit massless radiator / massive radiation using the SCR formalism.
- Consider models with no mass change, but coupling change instead.

backup slides

SCR formalism

[AL & Turner (2024)]

How to choose p_b? Suppose that rather than taking 1-to-1 kinematics or 1-to-2 kinematics, you just treat p_b and p_a as separate free parameters. But suppose that they're colinear for simplicity.

differential emission probability

$$\frac{d\mathbb{P}_{0\to0\gamma}}{d\Omega} = \frac{q^2e^2}{16\pi^3}\mathbb{P}_{0\to0}\frac{dp}{p}\frac{(\sin^2\theta)\,(v_a - v_b)^2}{(1-v_a\,\cos\theta)^2(1-v_b\,\cos\theta)^2}$$

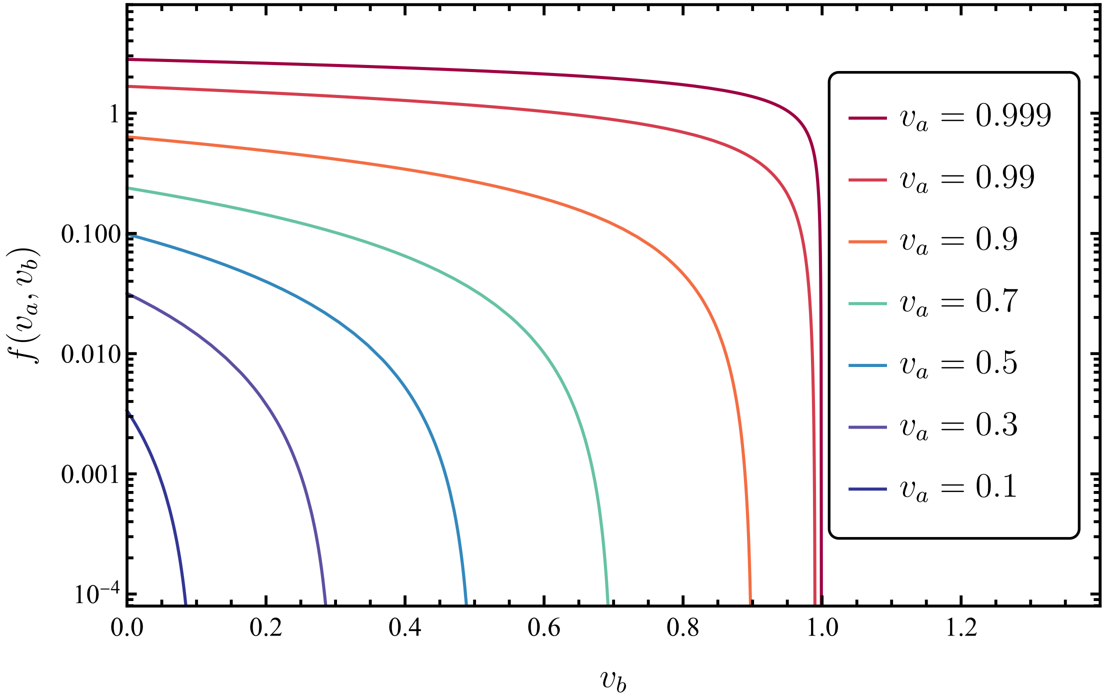
after the angular integral

$$p\frac{d\mathbb{P}_{0\to0\gamma}}{dp} = \left(\frac{q^2e^2}{2\pi^2}\mathbb{P}_{0\to0}\right) f(v_a, v_b)$$

average momentum transfer (v_a=1)

$$\langle\Delta p_z\rangle \approx \left(\frac{q^2e^2}{4\pi^2}\mathbb{P}_{0\to0}\right)\left(\frac{1-v_b}{v_b^2}\right)\left(1-\frac{1-v_b^2}{2v_b}\log\frac{1+v_b}{1-v_b}\right)p_{\mathrm{UV}}$$

$$f(v_a, v_b) = \frac{1}{2}\left(\frac{1-v_av_b}{v_a-v_b}\right)\log\frac{(1+v_a)(1-v_b)}{(1-v_a)(1+v_b)} - 1$$



massless radiator / massive radiation

[Bodeker & Moore (2017)]
[appendix of AL & Turner (2024)]

$$\mathrm{d}\mathbb{P}_{a\to bc} = \frac{1}{2E_a}\frac{\mathrm{d}^3\vec{p}_{b,\mathrm{s}}}{(2\pi)^3}\frac{1}{2E_b}\frac{\mathrm{d}^3\vec{p}_{c,\mathrm{s}}}{(2\pi)^3}\frac{1}{2E_c}(2\pi)^3\frac{p_{a,z,\mathrm{s}}}{E_a}\,\delta(\vec{p}_{a,\perp}-\vec{p}_{b,\perp}-\vec{p}_{c,\perp})\,\delta(E_a-E_b-E_c)\,|\mathcal{M}_{a\to bc}|^2$$

$$\mathrm{d}\mathbb{P}_{a\to bc} = \frac{\mathrm{d}^3\vec{p}_{c,\mathrm{s}}}{(2\pi)^3}\,\frac{1}{8E_aE_bE_c}\,|\mathcal{M}_{a\to bc}|^2\Big|_{\vec{p}_{b,\perp}=\vec{p}_{a,\perp}-\vec{p}_{c,\perp},E_b=E_a-E_c}$$

$$\mathrm{d}\mathbb{P}_{a\to bc} = \mathrm{d}k_\perp\,\mathrm{d}x\,\frac{k_\perp}{32\pi^2(1-x)E_a\sqrt{x^2E_a^2-k_\perp^2-m_{c,\mathrm{s}}^2}}\,|\mathcal{M}_{a\to bc}|^2\Big|_{\vec{p}_{b,\perp}=\vec{p}_{a,\perp}-\vec{p}_{c,\perp},E_b=E_a-E_c}$$

where: $x = E_c/E_a$ and $\vec{p}_{a,\perp} = 0$ and $k_\perp = |\vec{p}_{c,\perp}|$

$$|\mathcal{M}^{(0)}_{a\to bc}|^2 \approx 16g^2C_2[R]\frac{k_\perp^2\left(m_{c,\mathrm{h}}^2-m_{c,\mathrm{s}}^2\right)^2(1-x)^4}{\left(k_\perp^2+(1-x)^2m_{c,\mathrm{h}}^2\right)^2\left(k_\perp^2+(1-x)^2m_{c,\mathrm{s}}^2\right)^2}E_a^2+O(E_a^0)$$

$$x\frac{\mathrm{d}\mathbb{P}_{a\to bc}}{\mathrm{d}x} \approx \frac{g^2C_2[R]}{2\pi^2}\left(\frac{m_{c,\mathrm{h}}^2+m_{c,\mathrm{s}}^2}{m_{c,\mathrm{h}}^2-m_{c,\mathrm{s}}^2}\log\frac{m_{c,\mathrm{h}}}{m_{c,\mathrm{s}}}-1\right)(1-x)+O(E_a^{-2})$$

$$\langle\Delta p_z\rangle \approx \frac{g^2C_2[R]}{4\pi^2}\left(\log\frac{m_{c,\mathrm{h}}}{m_{c,\mathrm{s}}}-\frac{m_{c,\mathrm{h}}^2-m_{c,\mathrm{s}}^2}{2m_{c,\mathrm{h}}^2}\right)\frac{m_{c,\mathrm{h}}^2}{E_{c,\mathrm{IR}}}+O(E_a^{-3})$$

massive radiator / massless radiation

[Bodeker & Moore (2017)]
[appendix of AL & Turner (2024)]

$$\mathrm{d}\mathbb{P}_{a\to bc} = \frac{1}{2E_a}\frac{\mathrm{d}^3\vec{p}_{b,\mathrm{s}}}{(2\pi)^3}\frac{1}{2E_b}\frac{\mathrm{d}^3\vec{p}_{c,\mathrm{s}}}{(2\pi)^3}\frac{1}{2E_c}(2\pi)^3\frac{p_{a,z,\mathrm{s}}}{E_a}\,\delta(\vec{p}_{a,\perp}-\vec{p}_{b,\perp}-\vec{p}_{c,\perp})\,\delta(E_a-E_b-E_c)\,|\mathcal{M}_{a\to bc}|^2$$

$$\mathrm{d}\mathbb{P}_{a\to bc} = \frac{\mathrm{d}^3\vec{p}_{c,\mathrm{s}}}{(2\pi)^3}\frac{1}{8E_aE_bE_c}|\mathcal{M}_{a\to bc}|^2\Big|_{\vec{p}_{b,\perp}=\vec{p}_{a,\perp}-\vec{p}_{c,\perp},E_b=E_a-E_c}$$

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$$x\frac{\mathrm{d}\mathbb{P}_{a\to bc}}{\mathrm{d}x} \approx \frac{g^2C_2[R]}{2\pi^2}\left(\frac{m_{b,\mathrm{h}}^2+m_{a,\mathrm{s}}^2}{m_{b,\mathrm{h}}^2-m_{a,\mathrm{s}}^2}\log\frac{m_{b,\mathrm{h}}}{m_{a,\mathrm{s}}}-1\right)(1-x)+O(E_a^{-2})$$

$$\langle\Delta p_z\rangle \approx \frac{g^2C_2[R]}{4\pi^2}\left(\log\frac{m_{b,\mathrm{h}}}{m_{a,\mathrm{s}}}-\frac{m_{b,\mathrm{h}}^2-m_{a,\mathrm{s}}^2}{m_{b,\mathrm{h}}^2+m_{a,\mathrm{s}}^2}\right)\frac{m_{b,\mathrm{h}}^2+m_{a,\mathrm{s}}^2}{E_a}\log\frac{E_{c,\mathrm{UV}}}{E_{c,\mathrm{IR}}}+O(E_a^{-3})$$