Many-body Localization: Quantum coherence, single-particle excitations and nature of the transition

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Indo-French Discussion Meeting on classical and quantum dynamics of out of equilibrium many-body systems, ICTS (16-20 Dec,2024)

Plan of the talk

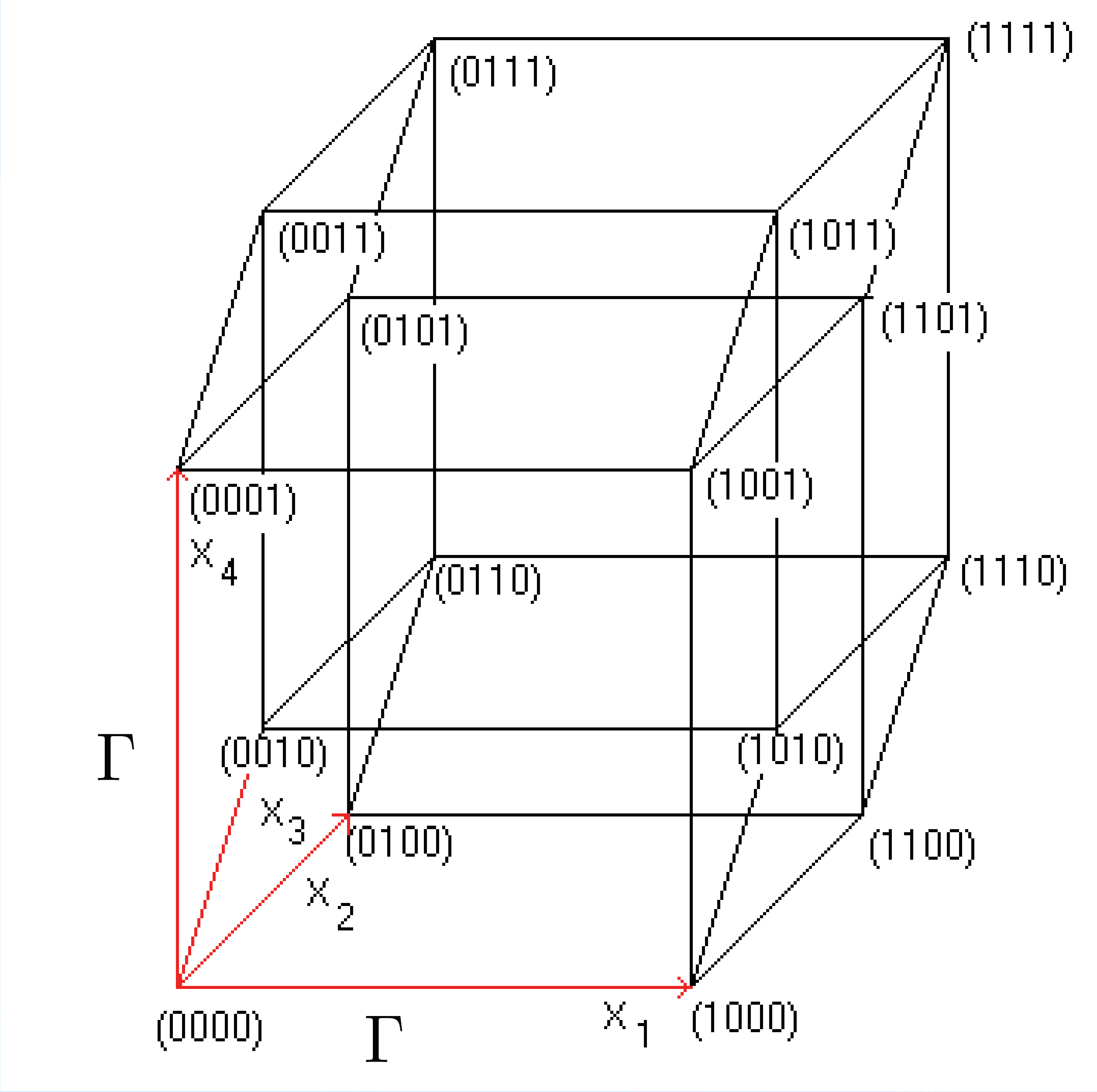
- Exact relations between coherence and measure of localization: application to MBL and quantum devices.
  Reference: A. Garg and A.K. Pati, arXiv:2409.10449.

- Single-particle excitations across many-body localization transition and nature of MBL transition
  References:
  A. Jana, V.R.Chandra and A. Garg, PRB109, 214209 (2024)
  Y. Prasad and A. Garg, PRB 109 (9), 094204 (2024).
  A. Jana, V. R. Chandra. A. Garg PRB(Letter) 104, L140201(2021)

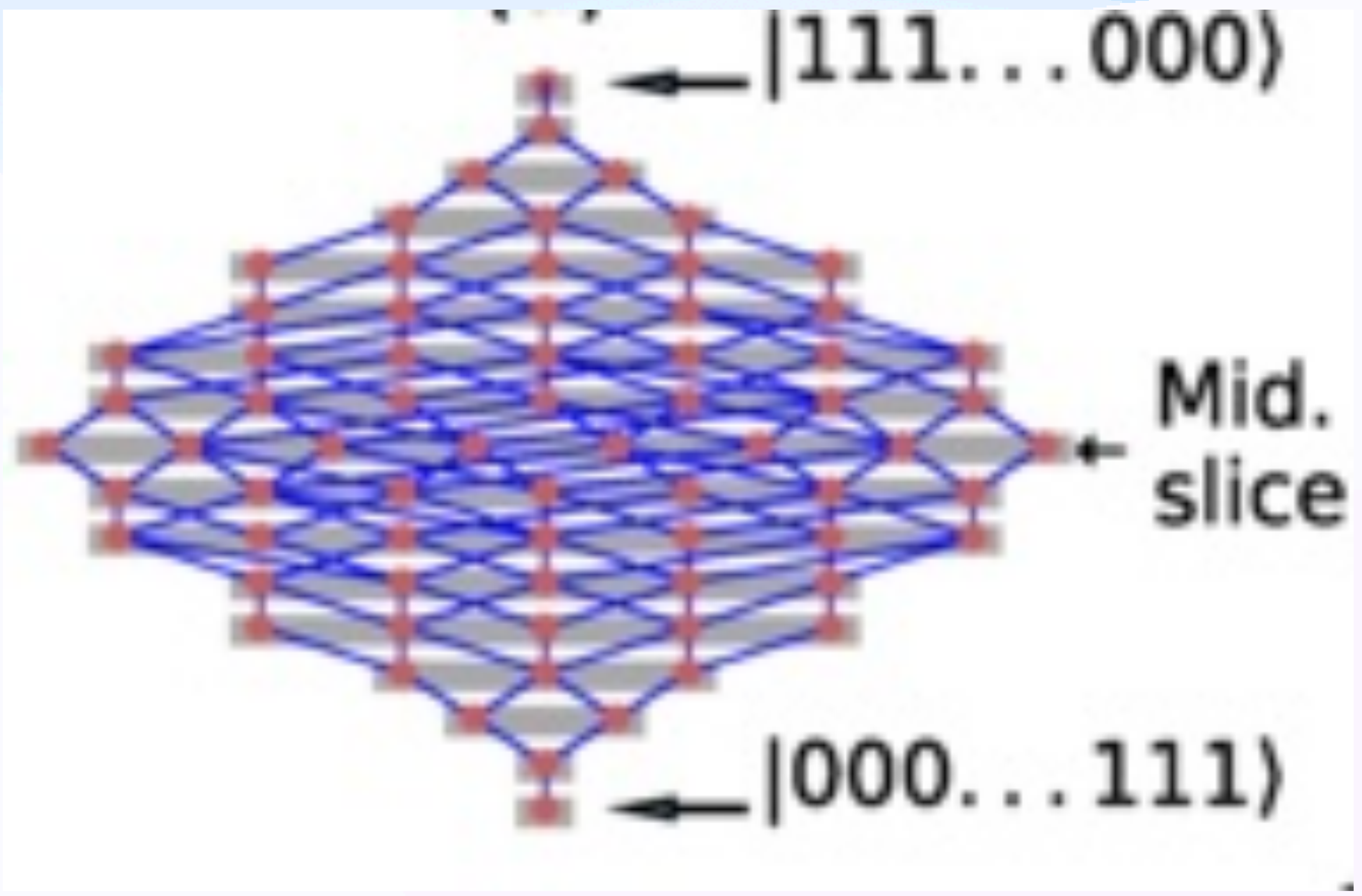
Anderson Localization in the presence of interactions

$$H = -t\sum_{ij} C_i^\dagger C_j + h.c. + \sum_i h(i)n(i) + V\sum_{ij} n_i n_j$$

$$\equiv \sum_l \epsilon_l |l\rangle\langle l| + \sum_{lm} T_{lm} |l\rangle\langle m| \qquad \epsilon_l = \sum_{i=1}^{L} h(i)n_l(i) + V\sum_{ij} n_l(i)n_l(j)$$

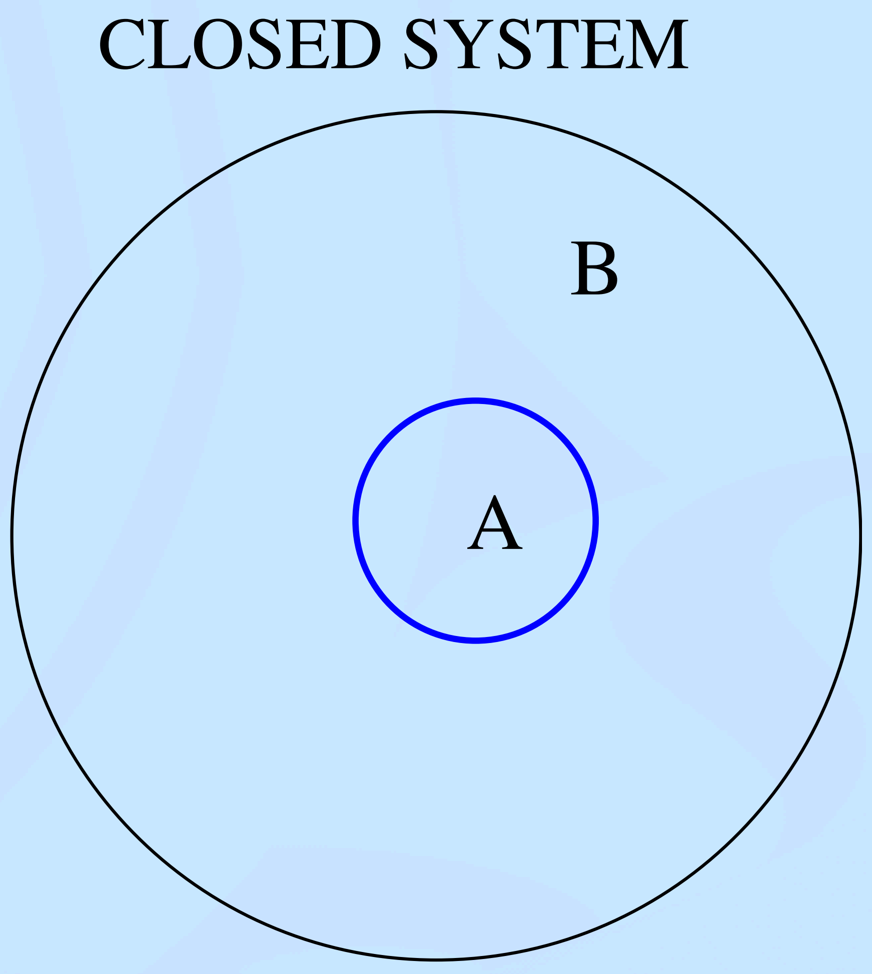


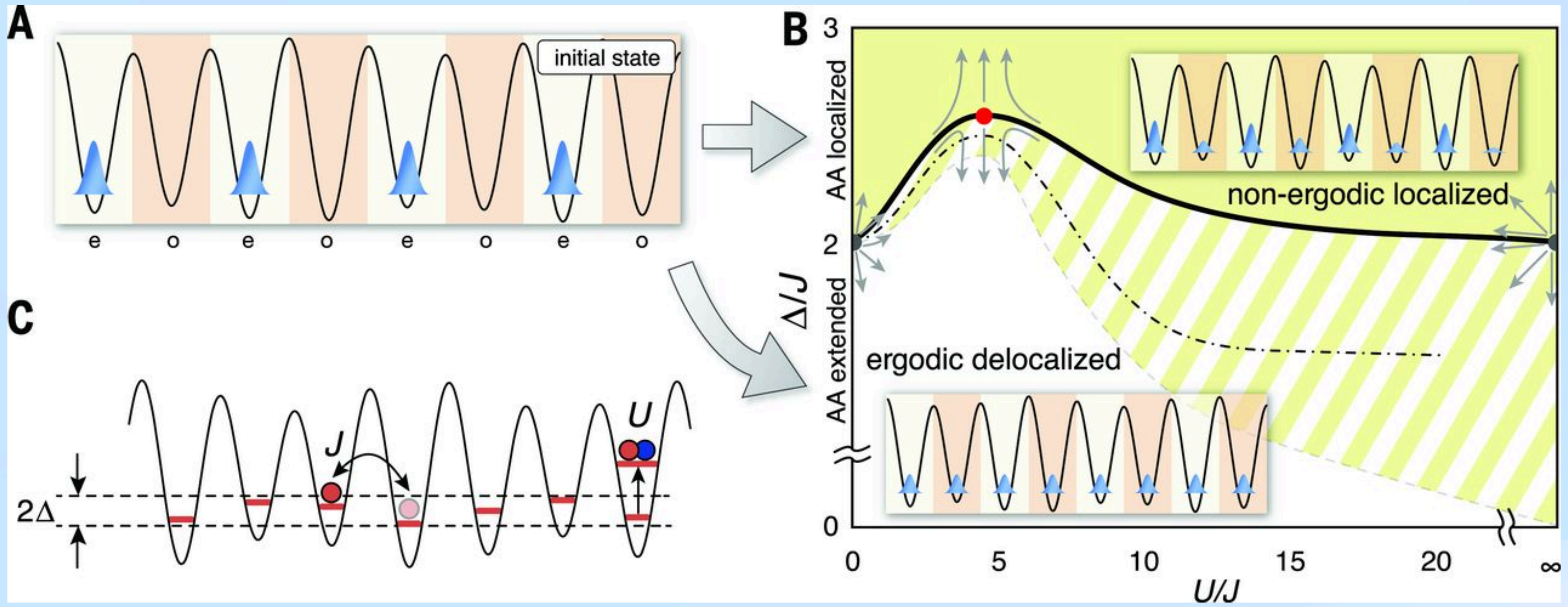
- Extended state has finite fraction of basis states
- MBL phase: system explores only an exponentially small fraction of basis states⇒ **Ergodicity Breaking**

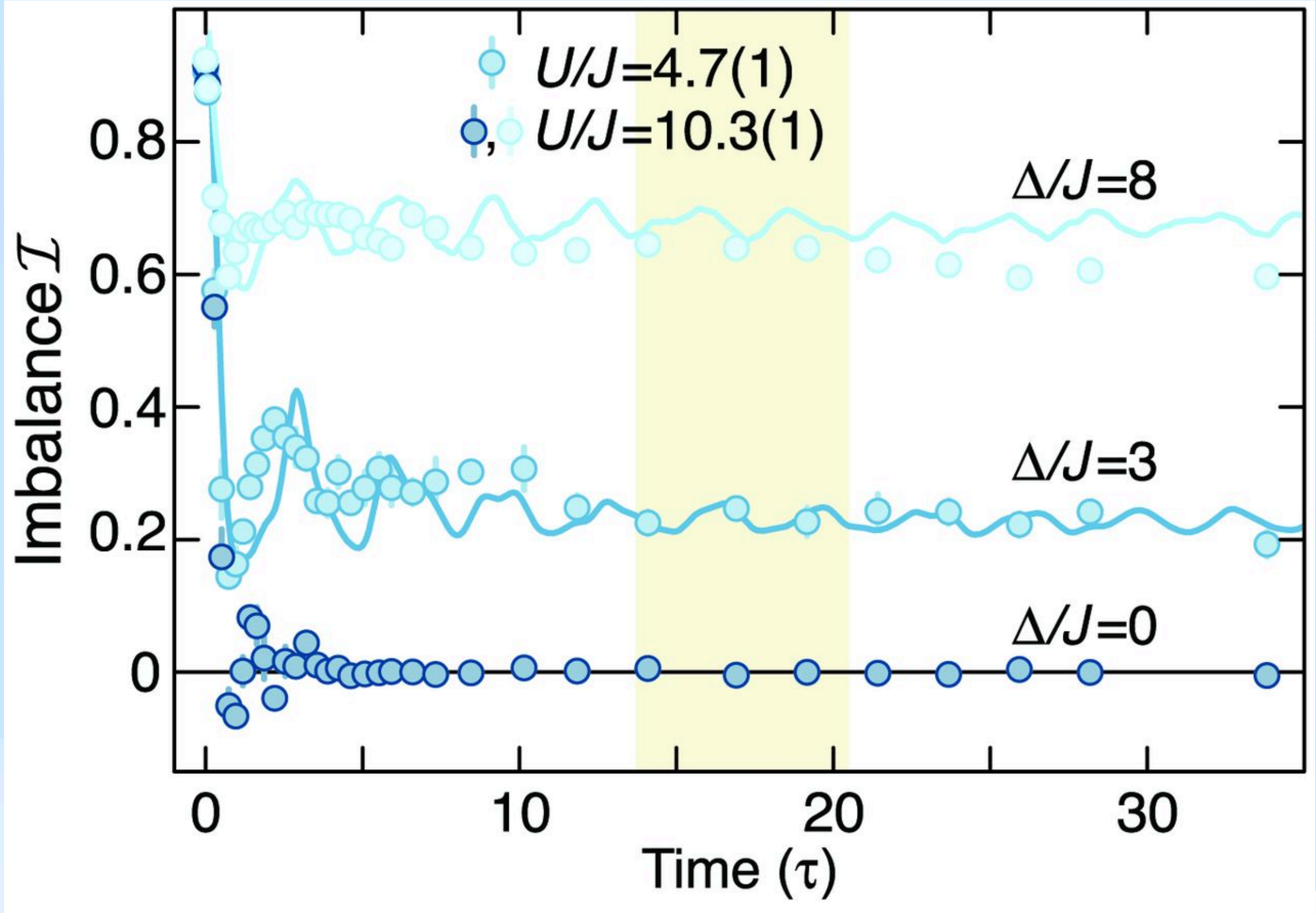


Many-body Localization

Lack of thermalization and strong memory of initial states



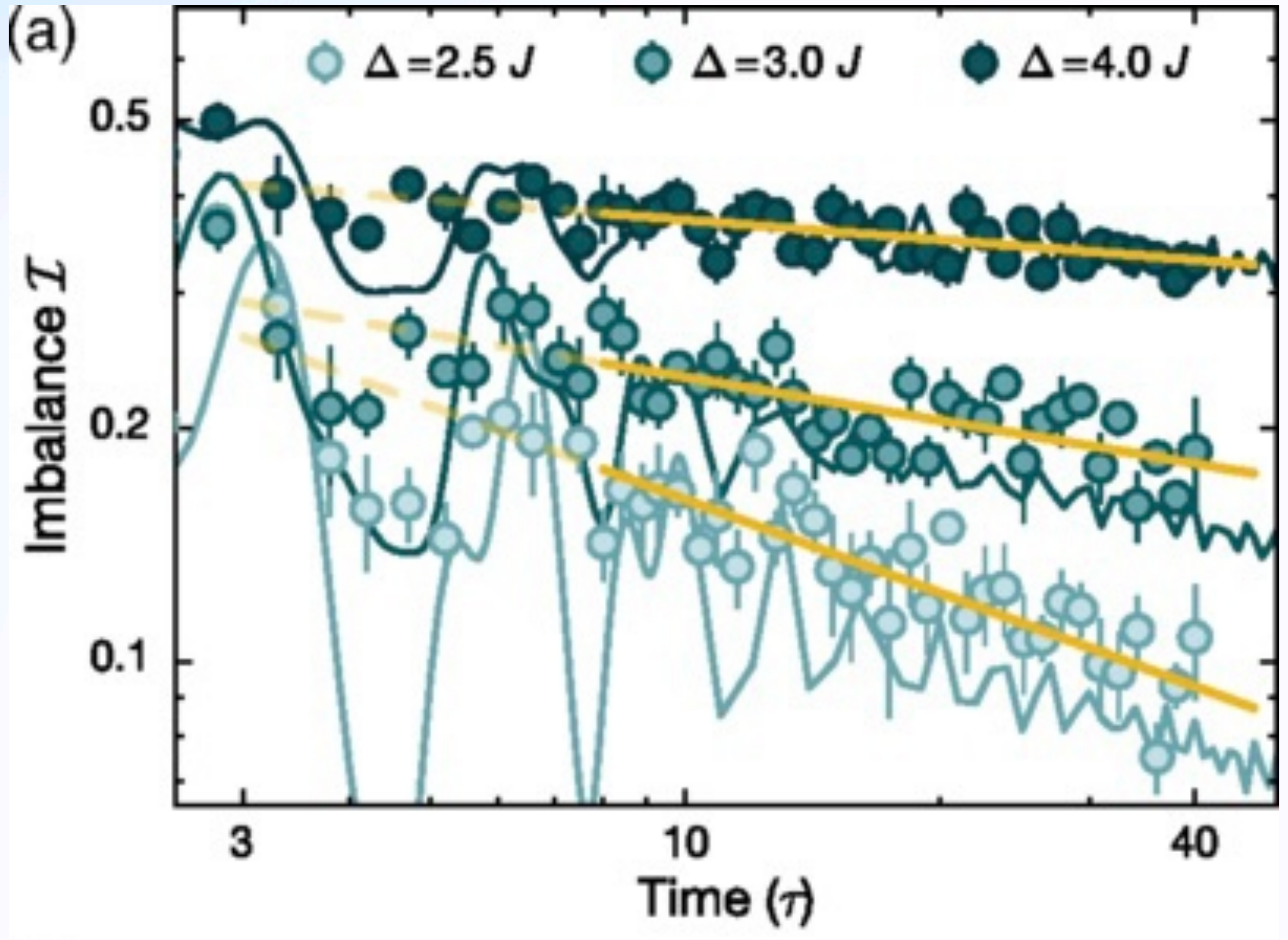


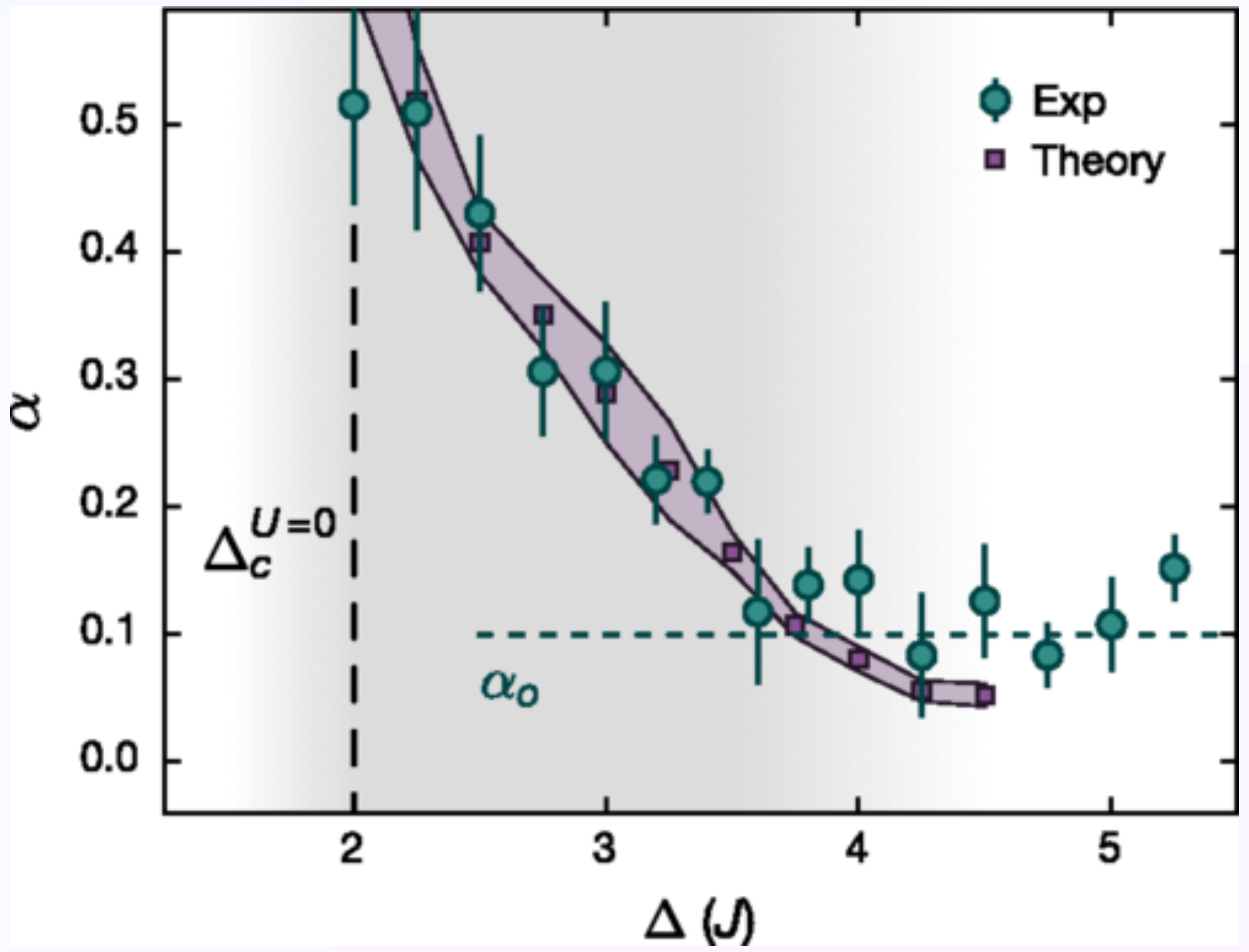


- Prepare system in CDW initial state and measure its relaxation

- $|\Psi(t)\rangle = \exp(-iHt)|\Psi(0)\rangle$

- $$I(t) = \frac{N_e - N_o}{N_e + N_o}$$

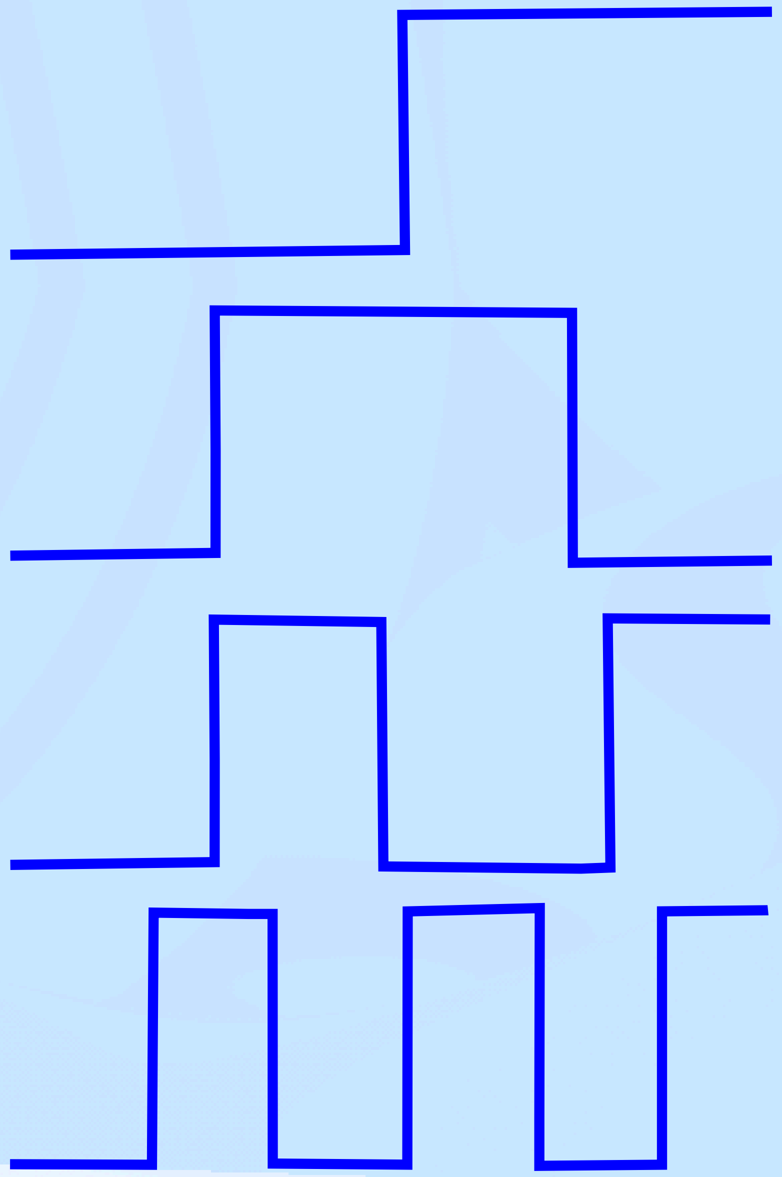




Schreiber et.al Science (2015)

Kink Dependent Density Imbalance

- Y.Prasad and A. Garg PRB105, 214202 (2022).

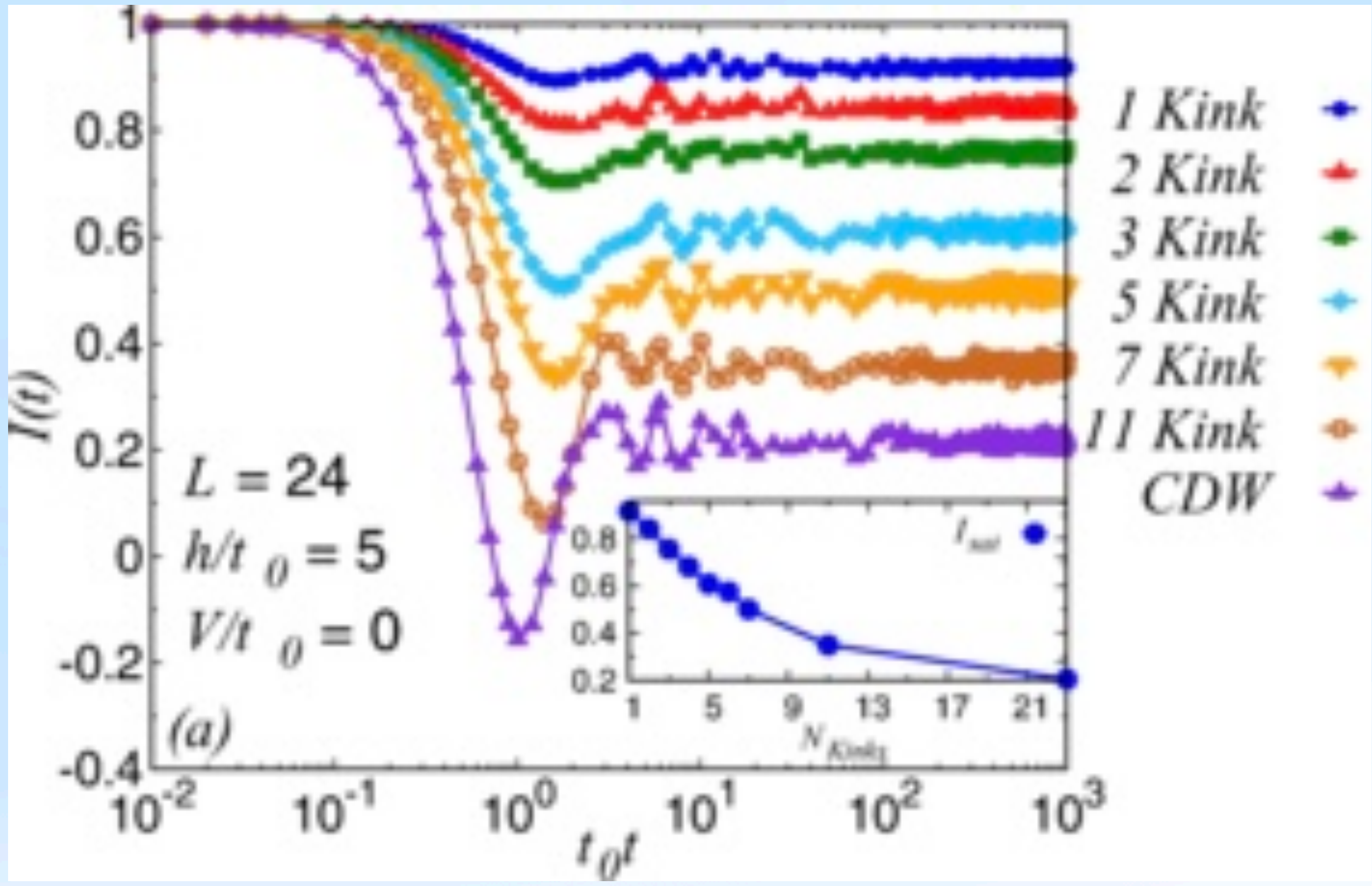


$|\psi_0\rangle = |000000000000111111111111\rangle$

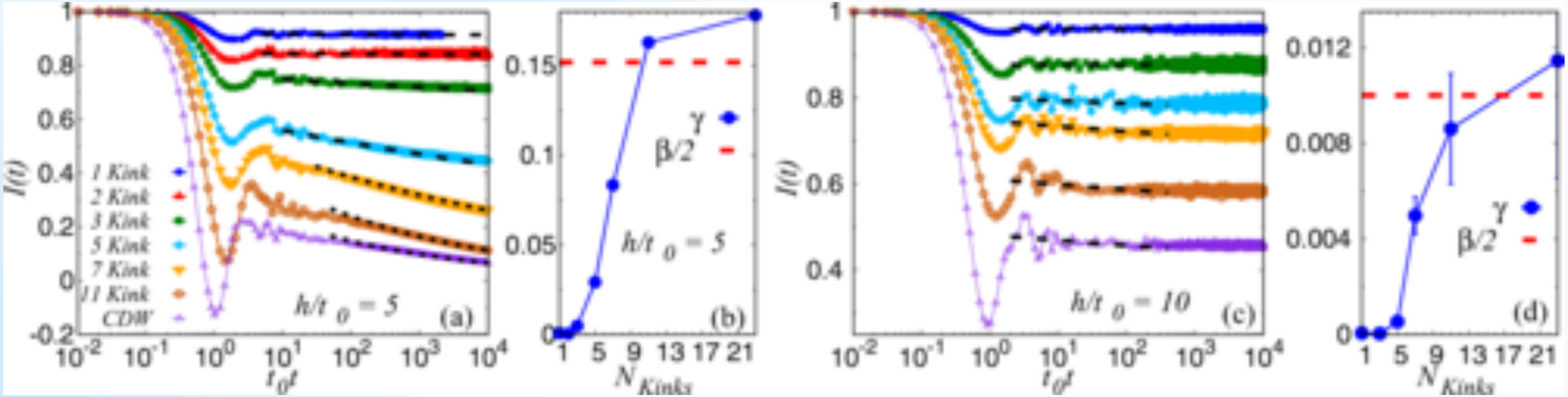
$|\psi_0\rangle = |000000111111111111000000\rangle$

$|\psi_0\rangle = |000000111111000000111111\rangle$

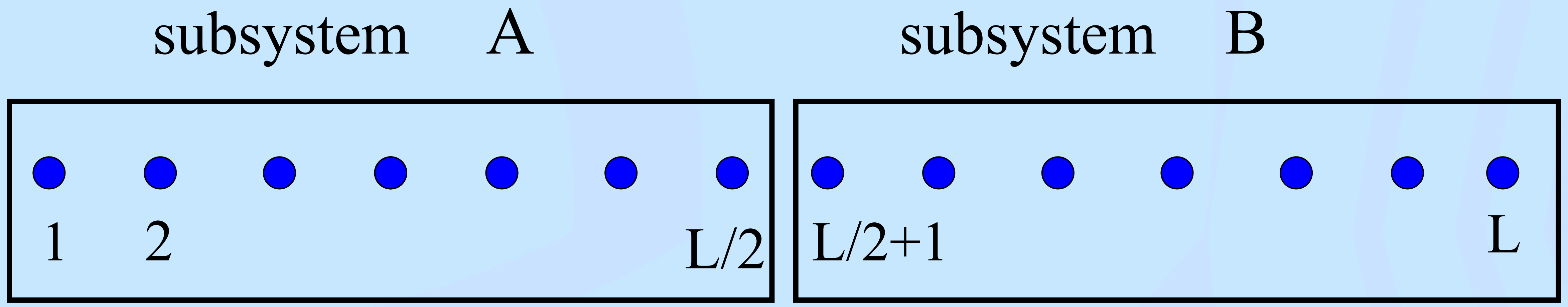
$|\psi_0\rangle = |000011110000111100001111\rangle$



Non-interacting Case

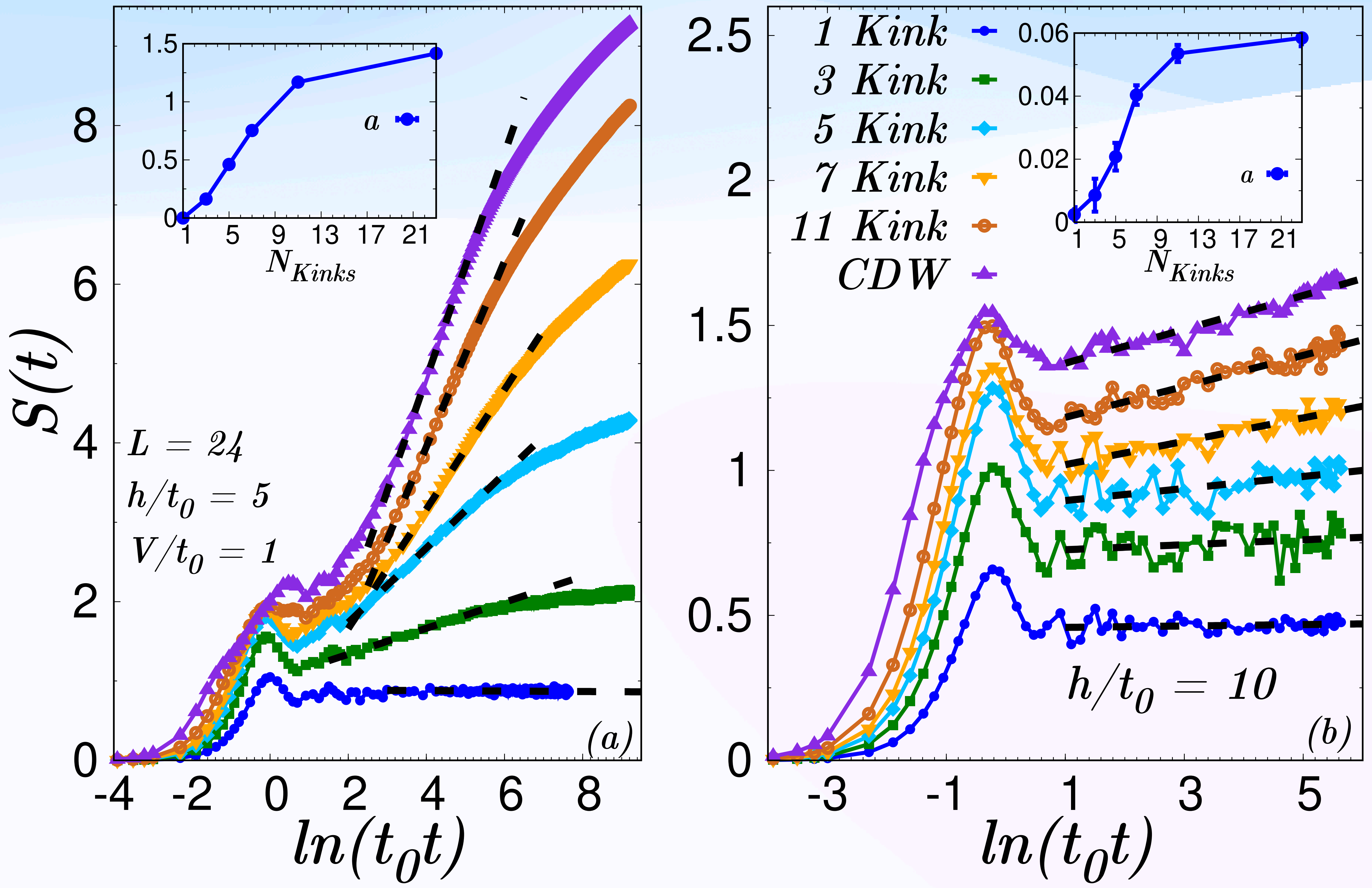


Entanglement entropy

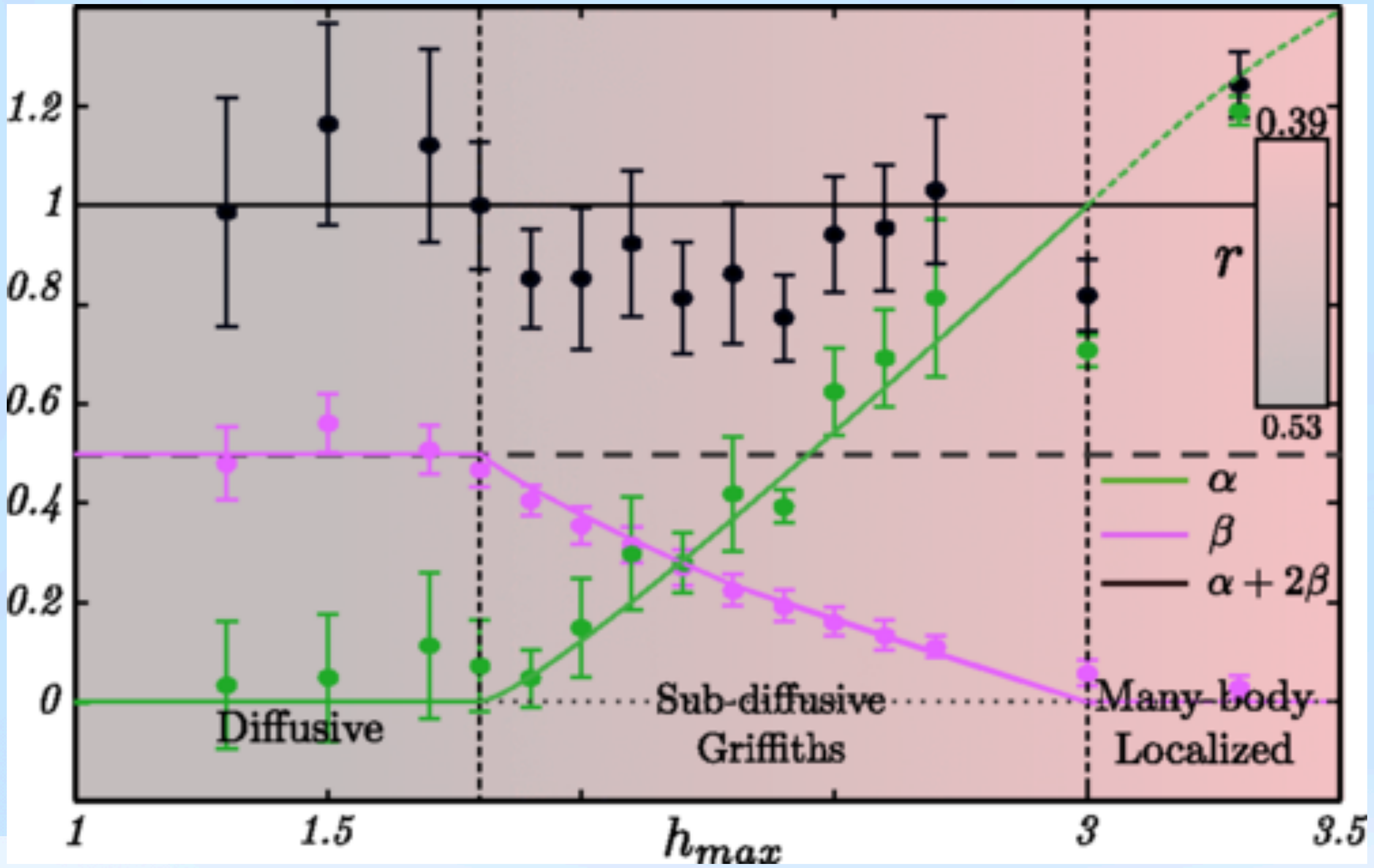


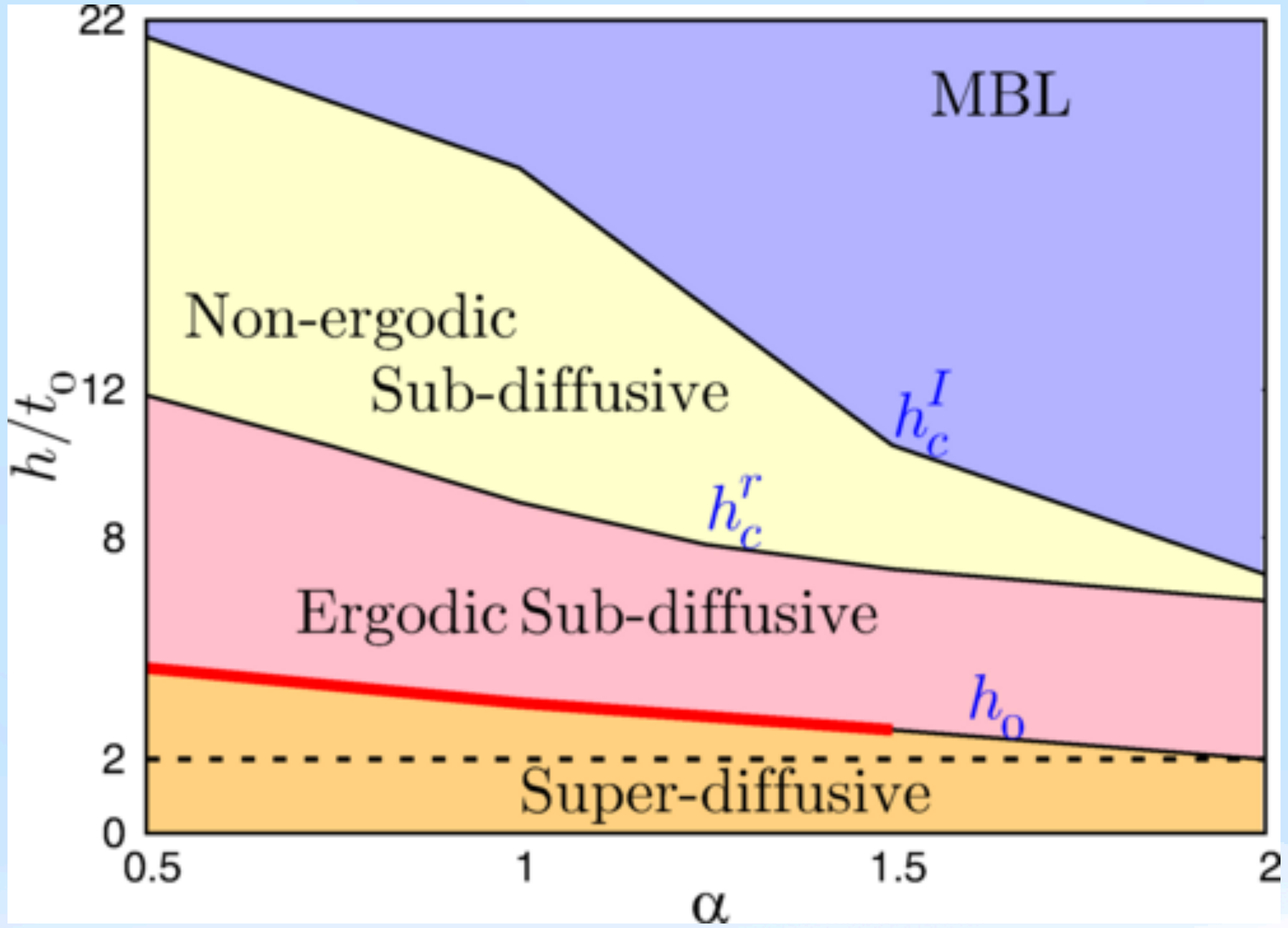
- $\rho_{total}(E_n) = |\Psi_n\rangle\langle\Psi_n|, \quad \rho_A = Tr_B \rho_{total}$
- $Renyi\ entropy\ R(E_n) = -\log[Tr_A \rho_A^2(E_n)]$
- Since ρ_A is not thermal in the MBL phase, $R(E) \sim L^{d-1}$
- In the ergodic delocalized phase $R(E) \sim L^d$

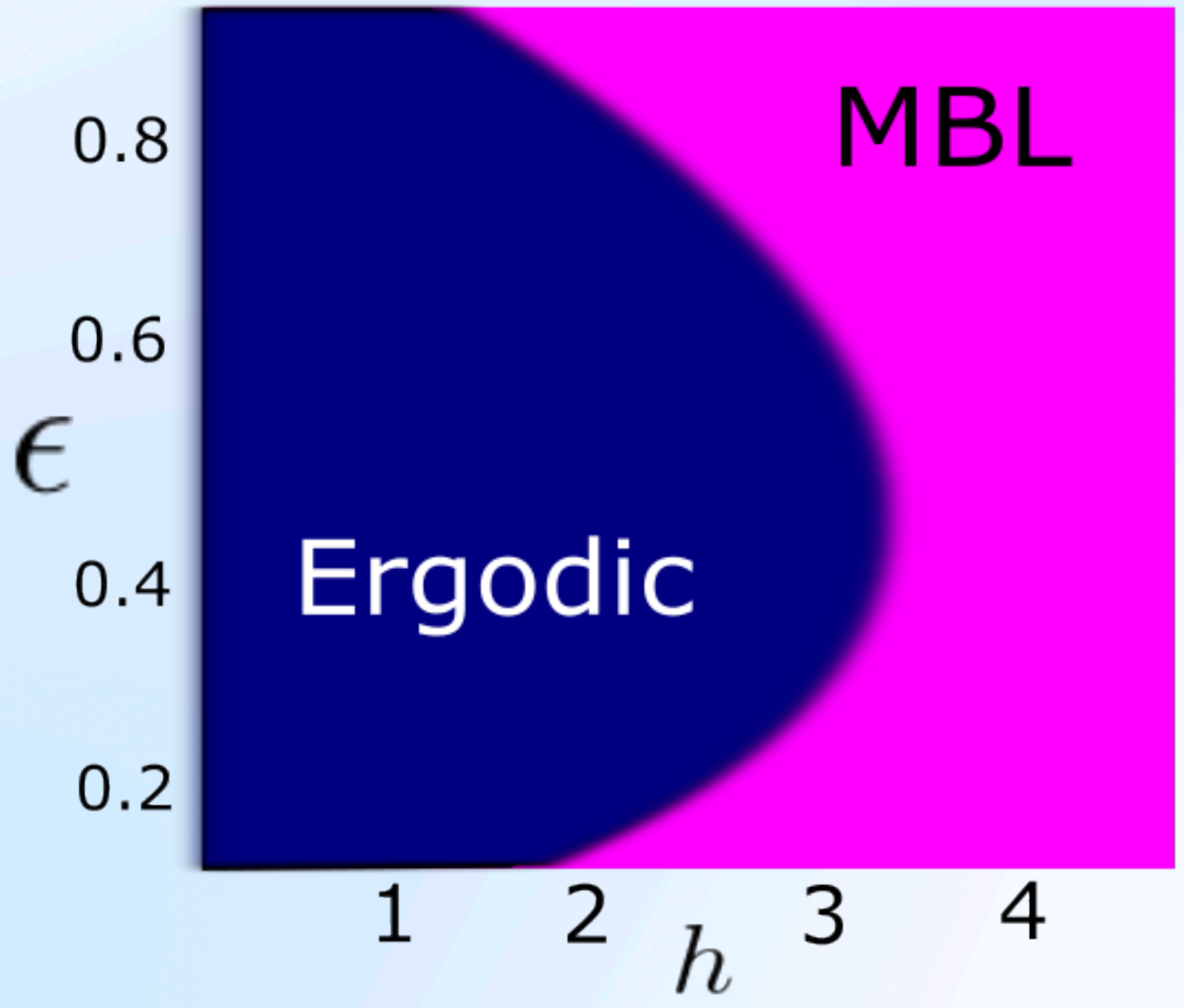
  - Y.Prasad and A. Garg PRB105, 214202 (2022).



Phase Diagram







Random Long-range interactions

$$H = -t\sum_i C_i^\dagger C_{i+1} + h.c. + \sum_i h(i)n(i) + \sum_{ij}\frac{V_{ij}}{r_{ij}^\alpha}$$

- h(i)=$h\cos(2\pi\beta\sqrt{i}+\phi)$; ϕ a random offset
- $V_{ij} \in [-V:V]$, we set V=t.

- Y.Prasad and A.Garg PRB103, 064203 (2021).

Single-particle excitations across MBL transition

$$H = -t\sum_i C_i^\dagger C_{i+1} + h.c. + V\sum_i n_i n_{i+1} + \sum_i h(i)n(i)$$

$$h(i) \in [-W/t, W/t] \quad and \quad h(i) = h\cos(2\pi\beta i + \phi) \quad with \beta = \frac{\sqrt{5}-1}{2}, \quad \phi \ an \quad offset$$

Green's function in nth eigenstate

$$G_n(i,j,t) = -\imath\Theta(t)\langle\Psi_n|\{C_i(t), C_j^\dagger(0)\}|\Psi_n\rangle$$

Generalised Dyson equation

$$\mathbf{\Sigma}(n,\omega) = \mathbf{G}_o^{-1}(\omega) - \mathbf{G}^{-1}(n,\omega)$$

Local Density of States

$$\rho_n(i,\omega) = -\frac{1}{\pi}Im[G_n(i,i,\omega)]$$

Scattering rate

$$\Gamma_n(i,\omega) = -\frac{1}{\pi}Im[\Sigma_n(i,i,\omega)]$$

LDOS across MBL transition

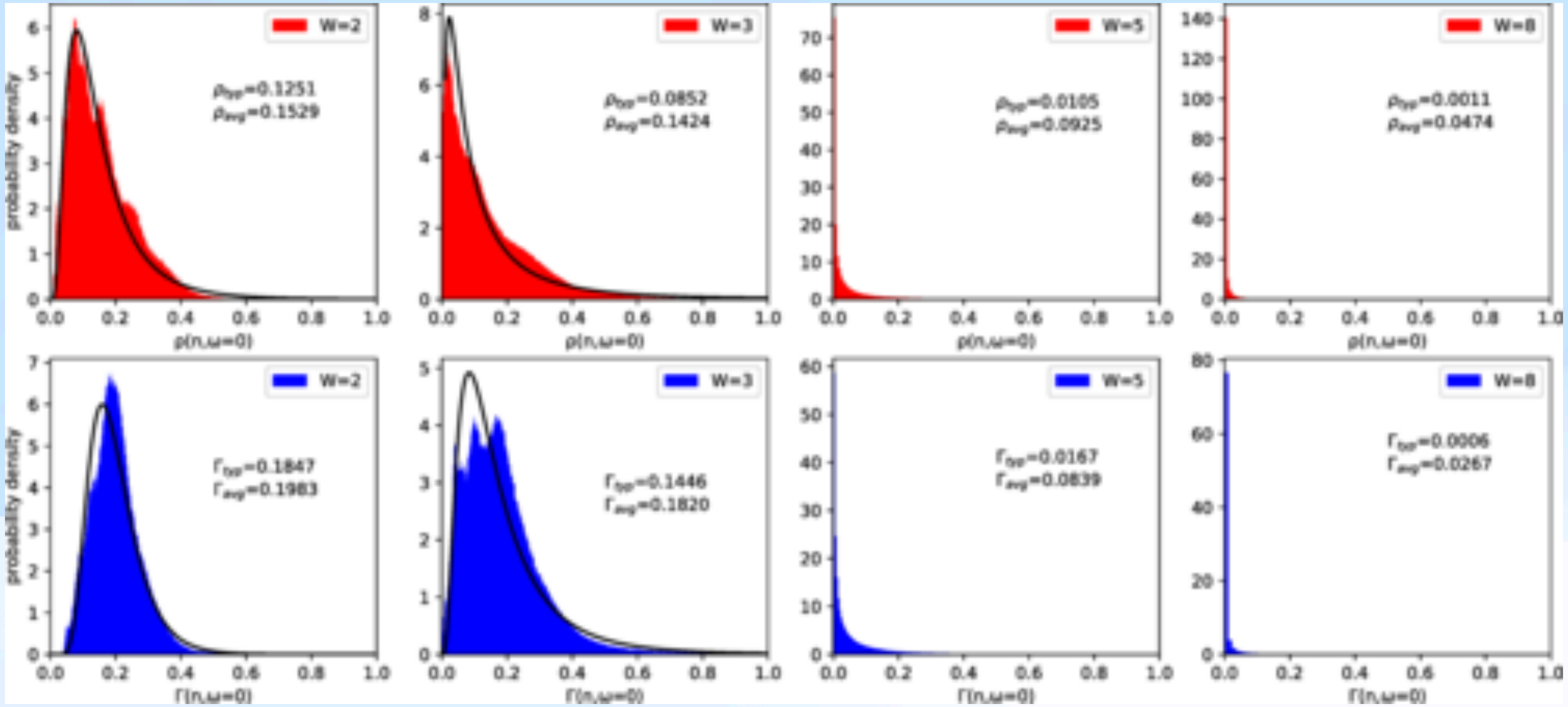
$$\rho_i(n,\omega) = \sum_m |\langle\Psi_m|c_i^\dagger|\Psi_n\rangle|^2\delta(\omega - E_m + E_n)$$
$$+|\langle\Psi_m|c_i|\Psi_n\rangle|^2\delta(\omega + E_m - E_n)$$

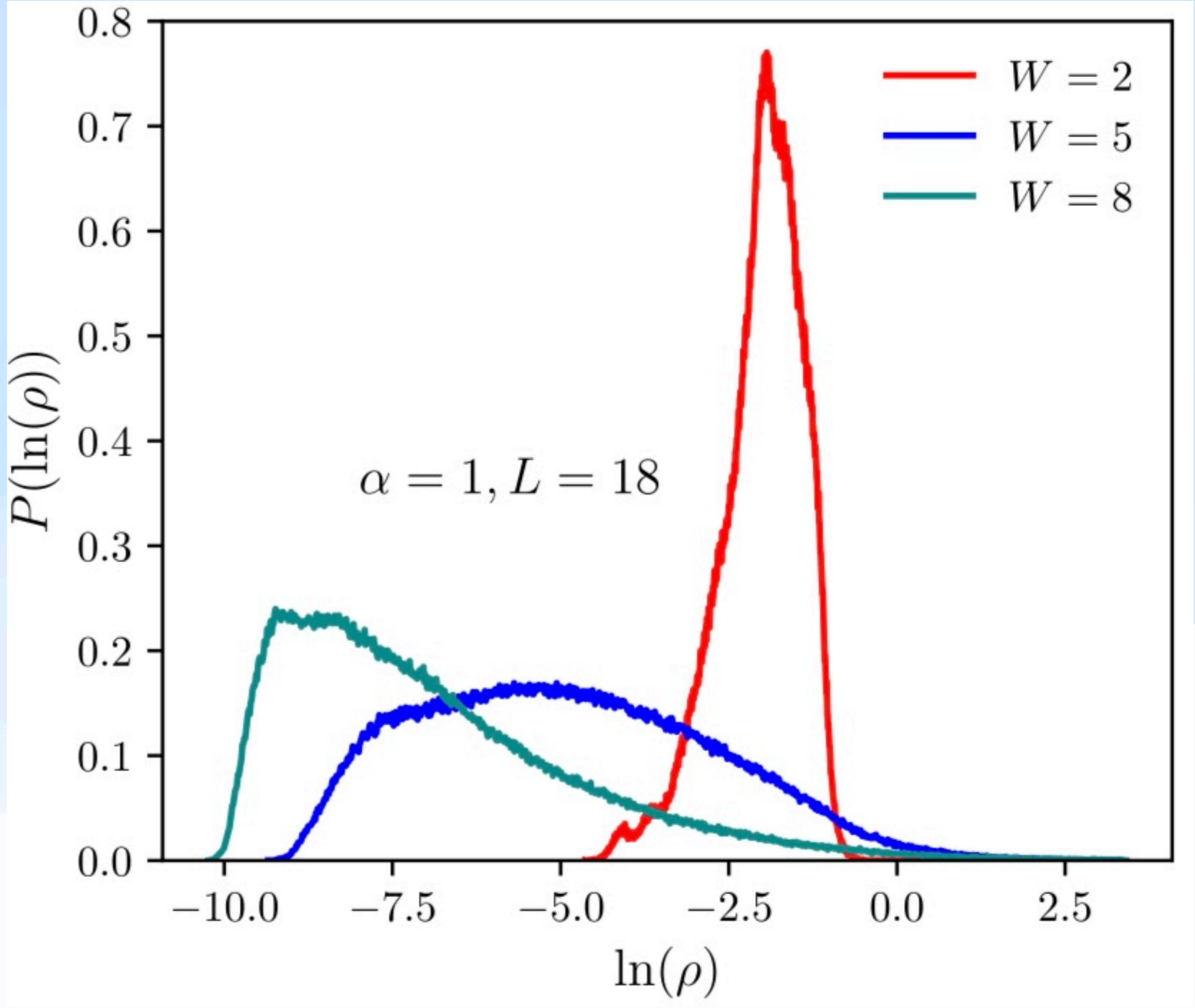
Create a particle-hole pair on top of $|\Psi_n\rangle$. Excited state $|\Psi_{ex,n}\rangle = C_i^\dagger|\Psi_n\rangle = \sum_m a_m|\Psi_m\rangle$

- If $|\Psi_n\rangle$ is localized, number of eigenstates contributing to $|\Psi_{ex,n}\rangle$ is of measure zero $\Rightarrow$ excitation can not propagate over all eigenstates allowed by the energy conservation. Hence $\rho_{typ}(\omega)$ is vanishingly small.

- If $|\Psi_n\rangle$ is extended, $|\Psi_{ex,n}\rangle$ will get contribution from a significant fraction of many-body eigenstates making $\rho_{typ}(\omega)$ finite in the delocalised phase.

Probability Distributions

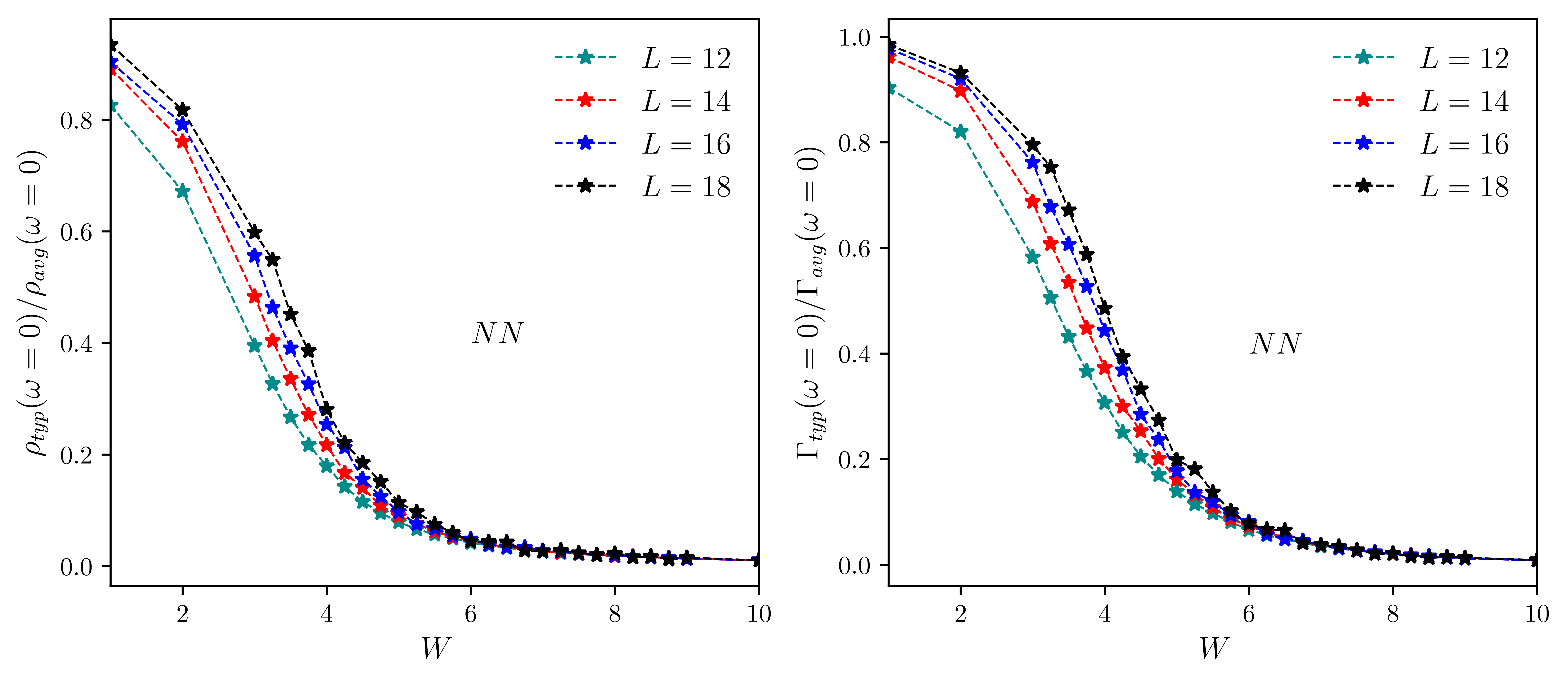




- Distribution is log-normal for weak to intermediate disorder, with larger weight for smaller values as W increases.

- Excitations become more localized as disorder increases $\rightarrow \rho_{typ}(\omega)$ decreases. Excitations are longer lived for strong disorder $\rightarrow \Gamma_{typ}(\omega) \rightarrow 0$in the MBL phase.

Single particle excitations



- For small W, $\rho_{typ}(\omega = 0) \sim \rho_{avg}(\omega = 0)$ and $\Gamma_{typ}(\omega = 0) \sim \Gamma_{avg}(\omega = 0)$
- For strong disorder. $\rho_{typ} \sim 0 \ll \rho_{avg}, \ \Gamma_{typ} \sim 0 \ll \Gamma_{avg}$

Finite Size Scaling of LDOS and Scattering Rates

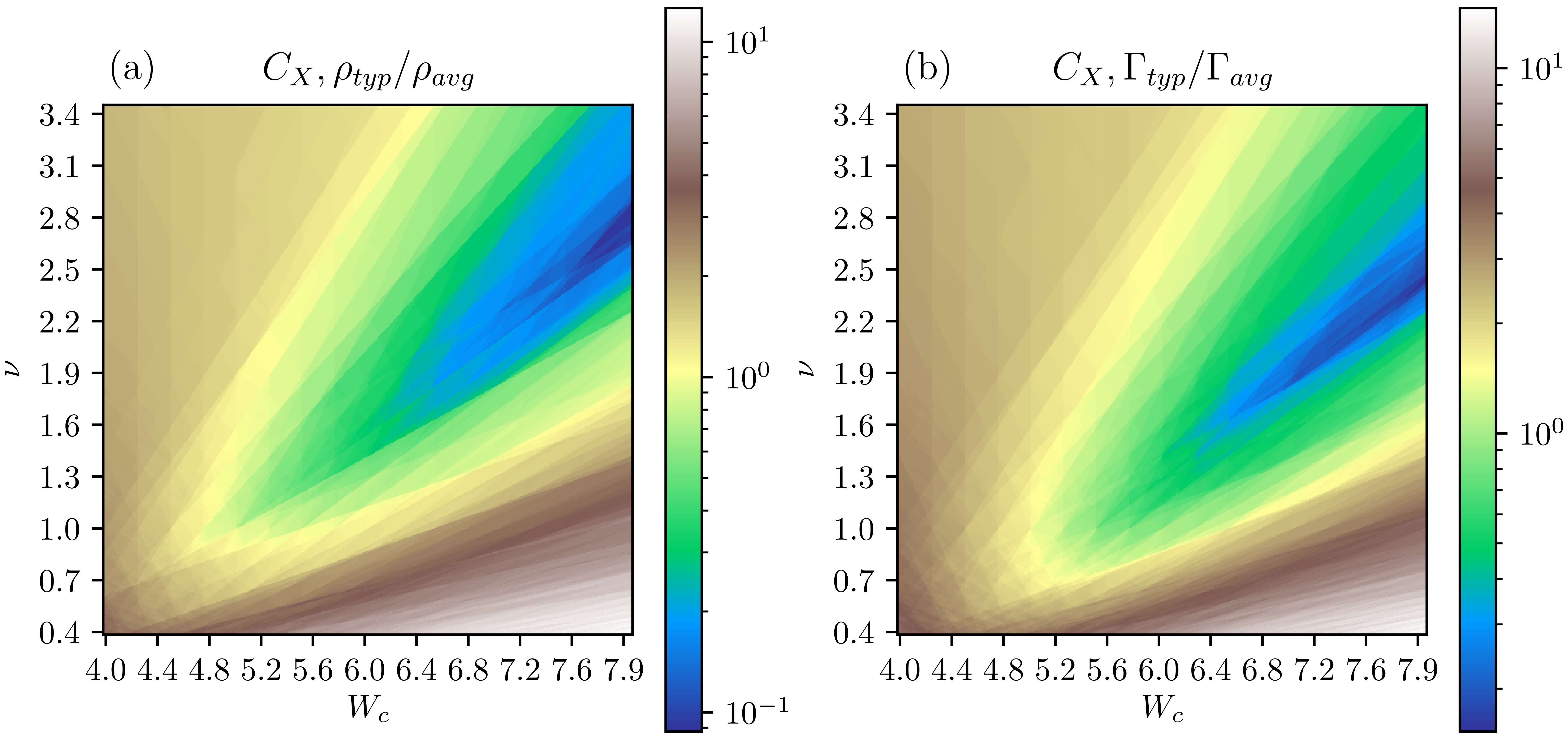
Assuming $\xi \sim |W - W_c|^{-\nu}$

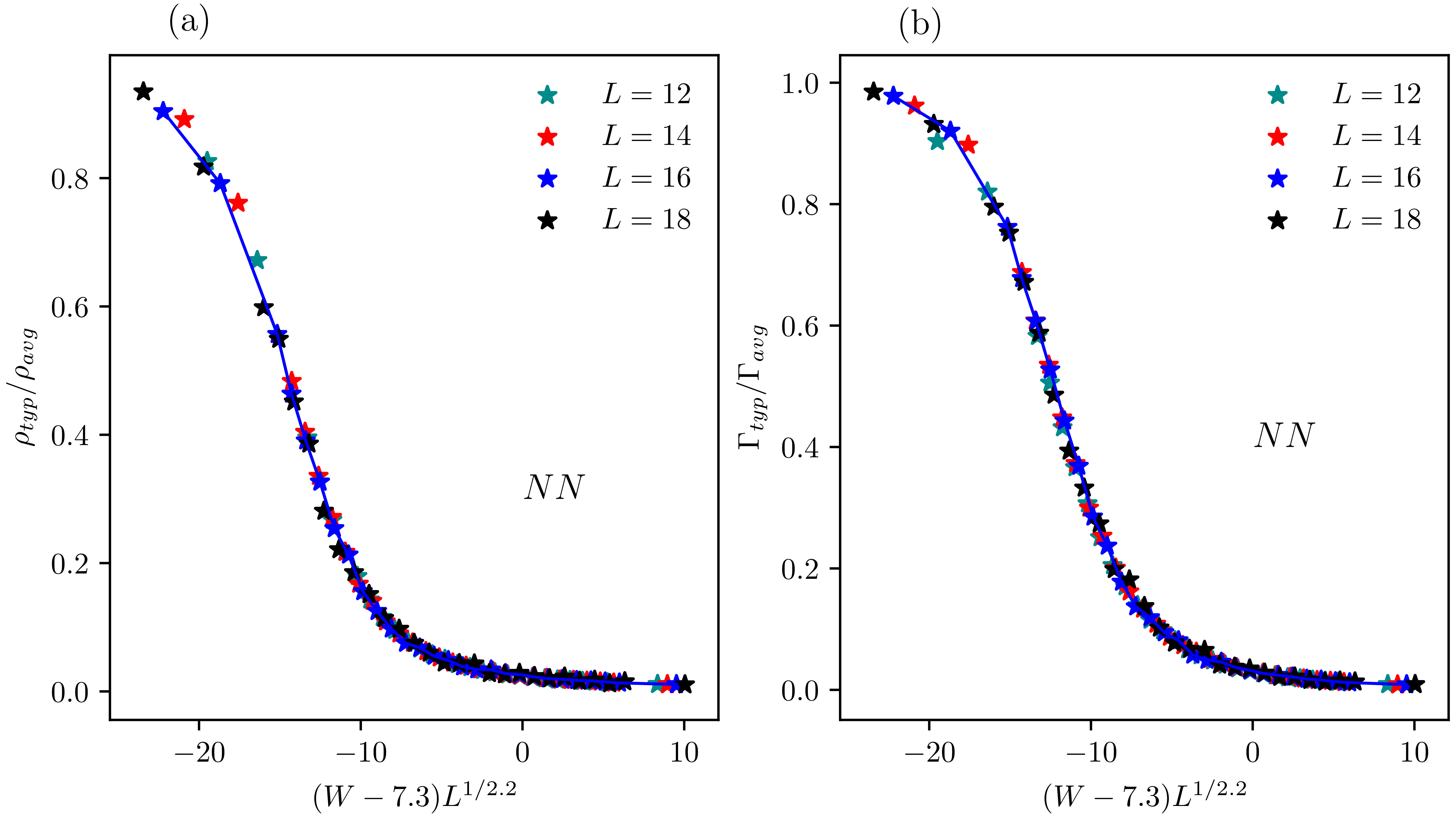
$$X[\delta, L] \sim \tilde{X}(\delta L^{1/\nu})$$

$$\nu > 2/d$$

Satisfies CCFS bound

A.Jana, VRC, AG, PRB(2024)





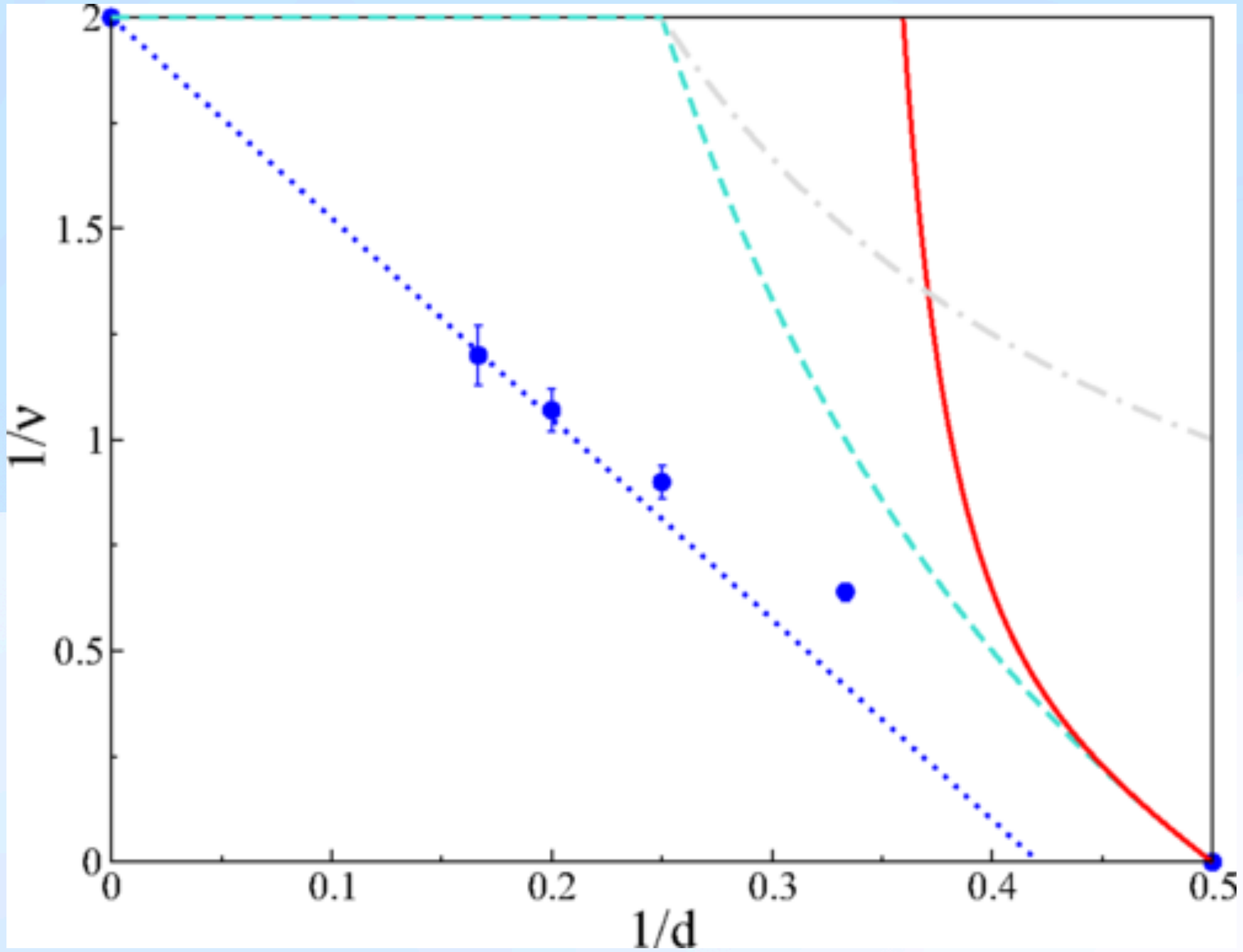
- $$C_X = \frac{\sum_{j=1}^{N_{total}-1} |X_{j+1} - X_j|}{max\{X_j\} - min\{X_j\}} - 1$$
- N_{total} is total number of X_i for various sizes L and disorder W, $\{X_i\}s$ are arranged in order of increasing $(W - W_c)L^{1/\nu}$
- $W_c \sim 7.96t, \nu \sim 2.7$ for LDOS
- $W_c \sim 7.8t, \nu \sim 2.3,\ for\ \Gamma$

Transition in systems with random disorder

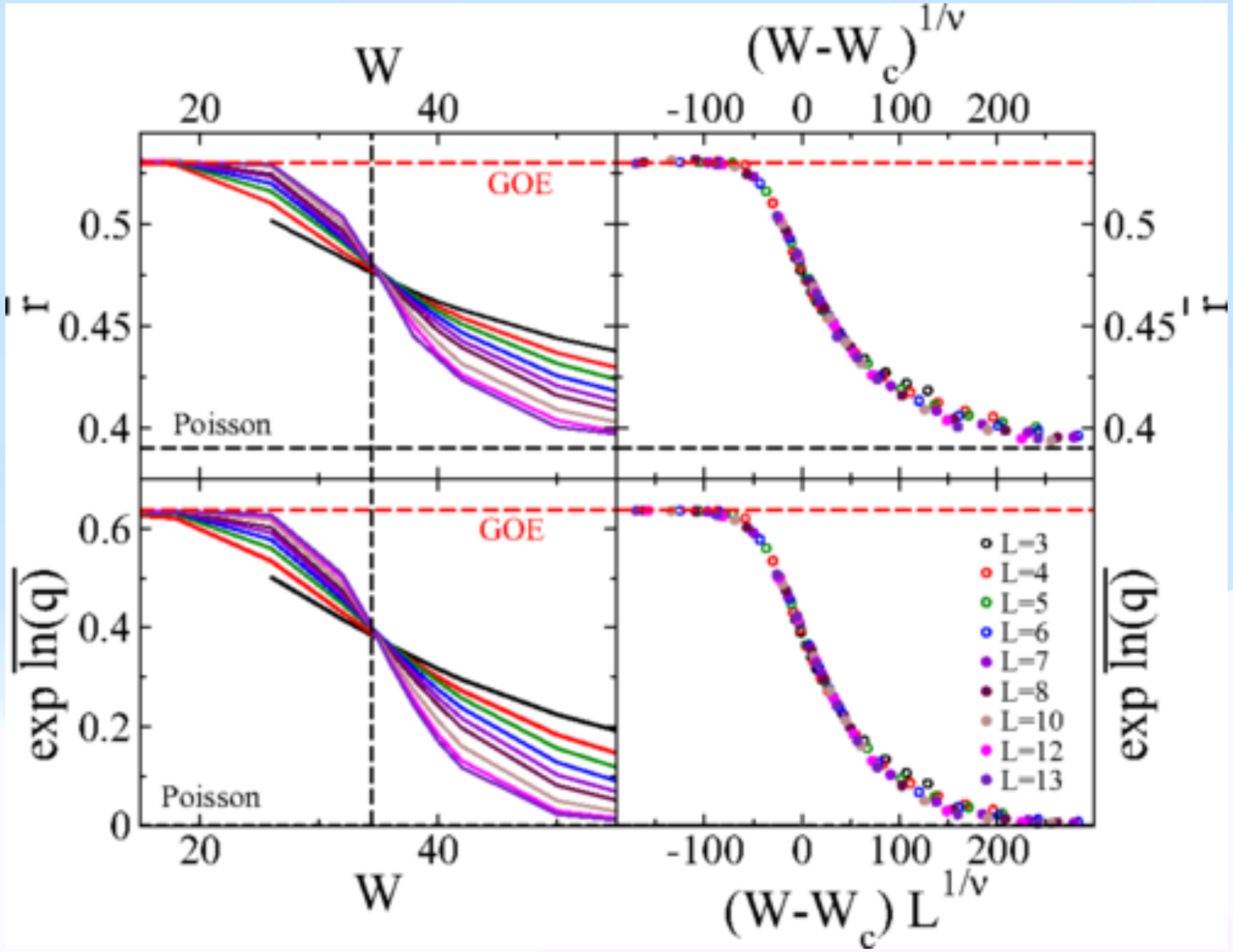
Chayes-Chayes-Fisher-Spencer (PRL 57, 2999 (1986))

- For all systems with quenched random disorder, that undergo continuous transition with a power-law divergence of the correlation length at the critical point
- $\xi \sim |W - W_c|^{-\nu}, \nu \geq 2/d$ irrespective of whether the clean system has any transition or not.
- This results applies not only to disordered magnets but also to systems showing percolation and Anderson Localization. Should also apply to Many -body localization.
- This bound was obtained mainly based on probability distribution of order parameter.

CCFS bound and Anderson Localization



- PRB 95 , 094204 (2017)
- **What about the MBL transition? Does it satisfy CCFS criterion?**

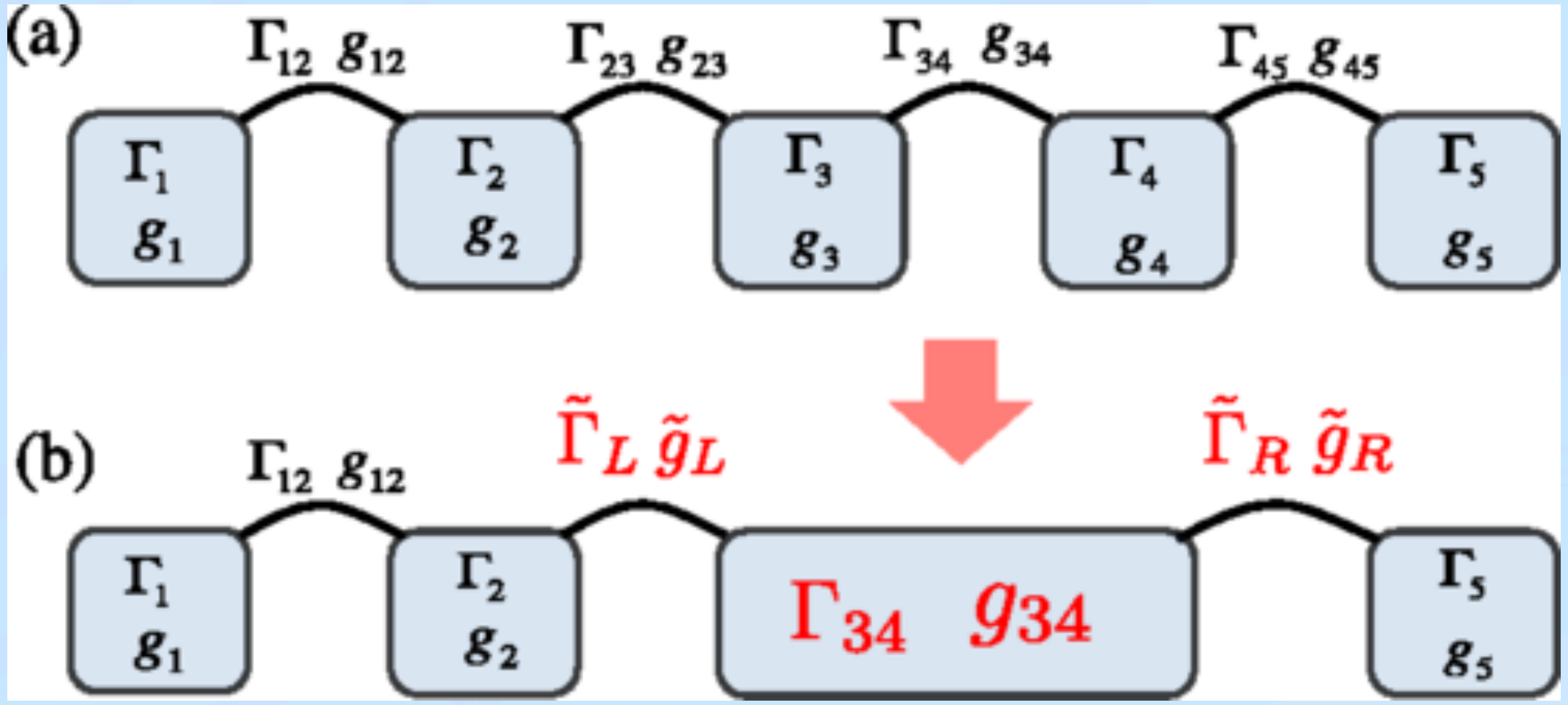


4d Anderson model

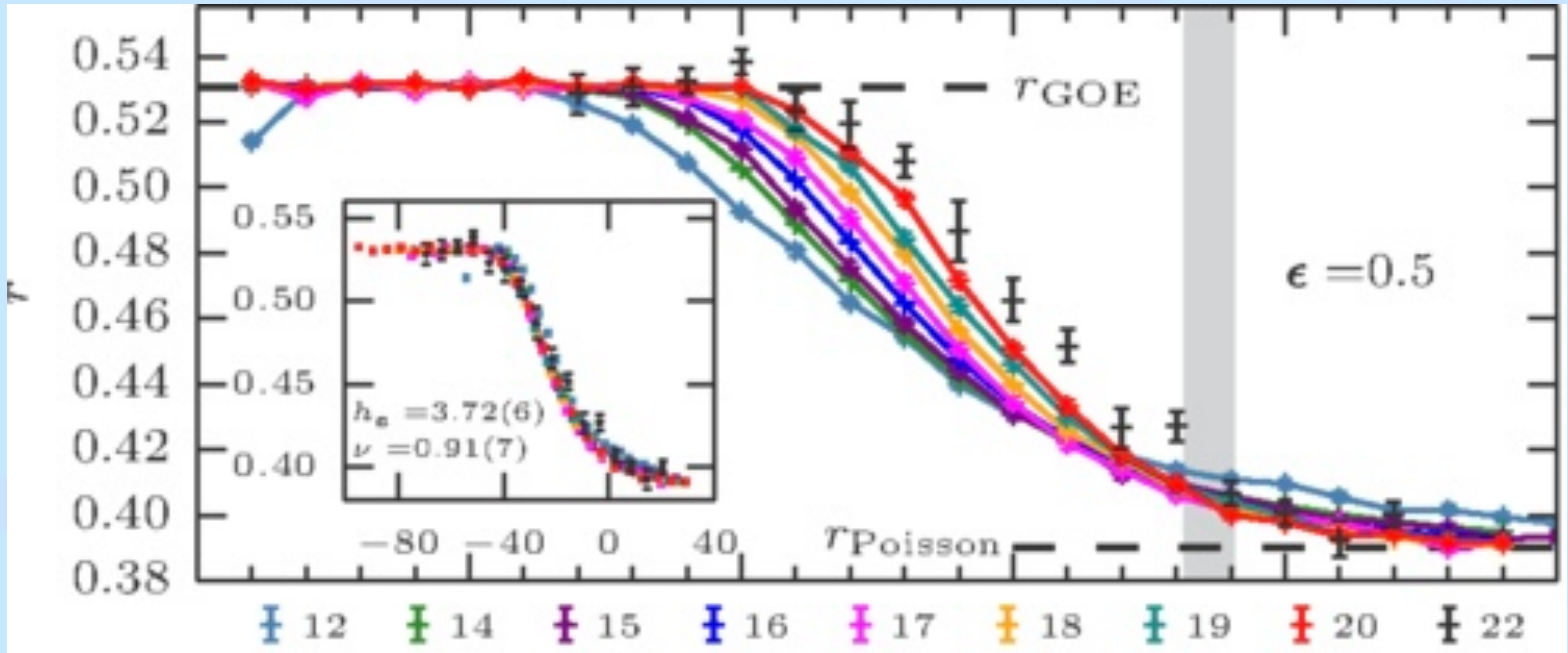
$$\nu = 1.11 \geq 2/d$$

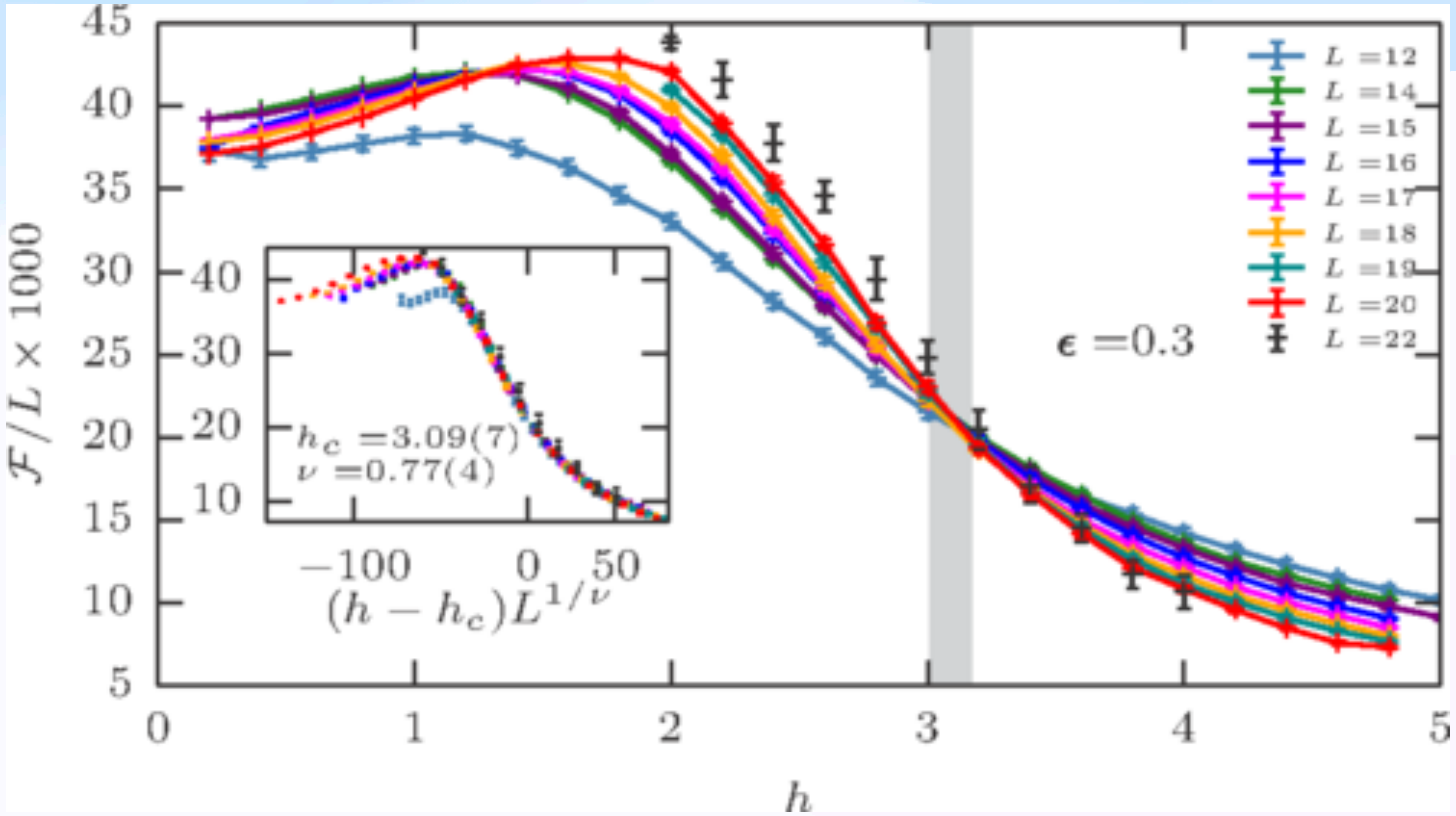
Tarquinius et. al. PRB 95, 094204 (2017)
3d model: Shklovskii PRB (1993),Kramer PRL (1997), Slevin PRL (1999)…

Real Space RG approach



- Δ_i: mean level spacing in the block
- Γ_i : rate of entanglement spread across the block, $g_i = \Gamma_i / \Delta_i$.
- $\nu = 3.1$



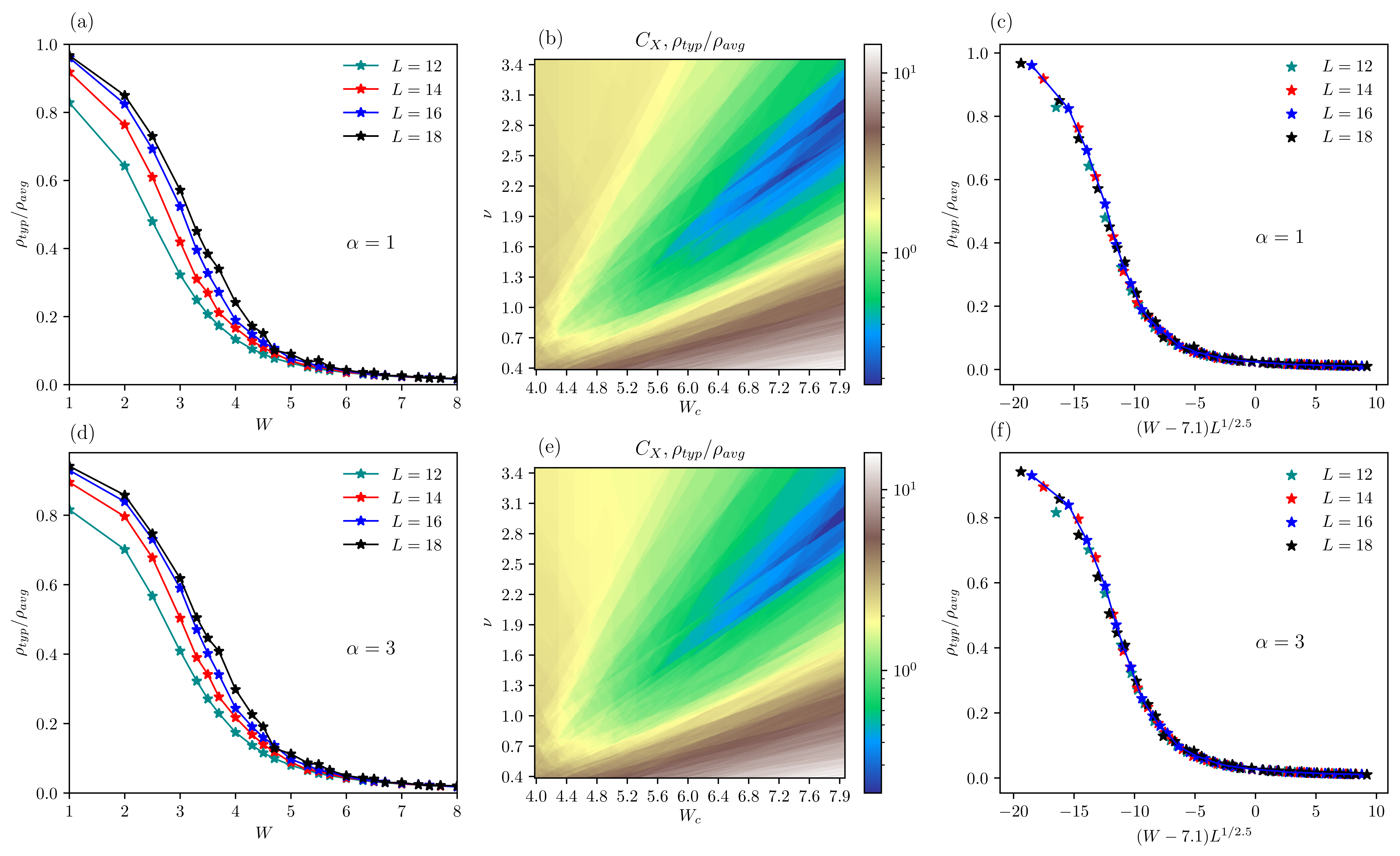


Vosk et.al PRX (2015), Potter et.al. PRX (2015), Zhang PRB (2016) ,Dumitrescu et.al. PRL (2017)…

Kjall et.al. (2014), Luitz et.al.(2015),Khemani et.al.(2017),Sierant et.al (2021)

Single Particle Excitations in long-range interacting MBL system

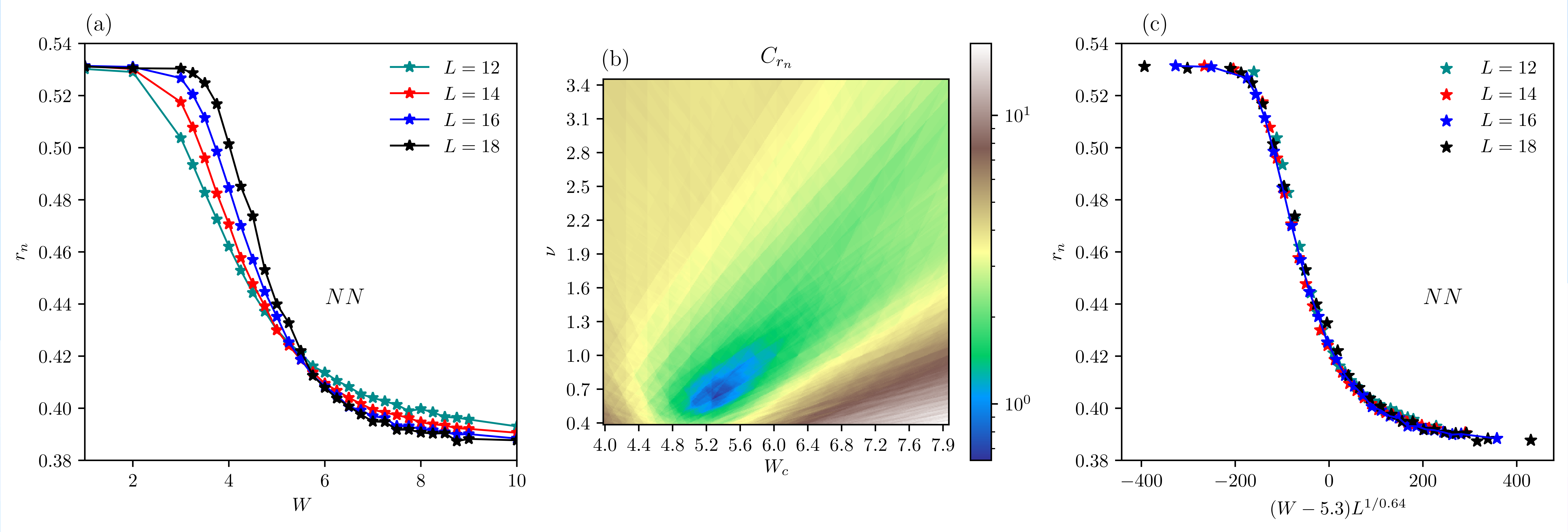
- Typical values of LDOS for $\alpha = 1{,}3$
- The cost function for ratio of typical to average values shows minima around $W_c \sim 7.66,\ \alpha = 1$ $\nu \sim 2.62$
- $\alpha = 3 : W_c \sim 7.89$
- $\nu \sim 2.88$



Are MBL systems Hyperuniform?

Finite Size Scaling of Level Spacing Ratio

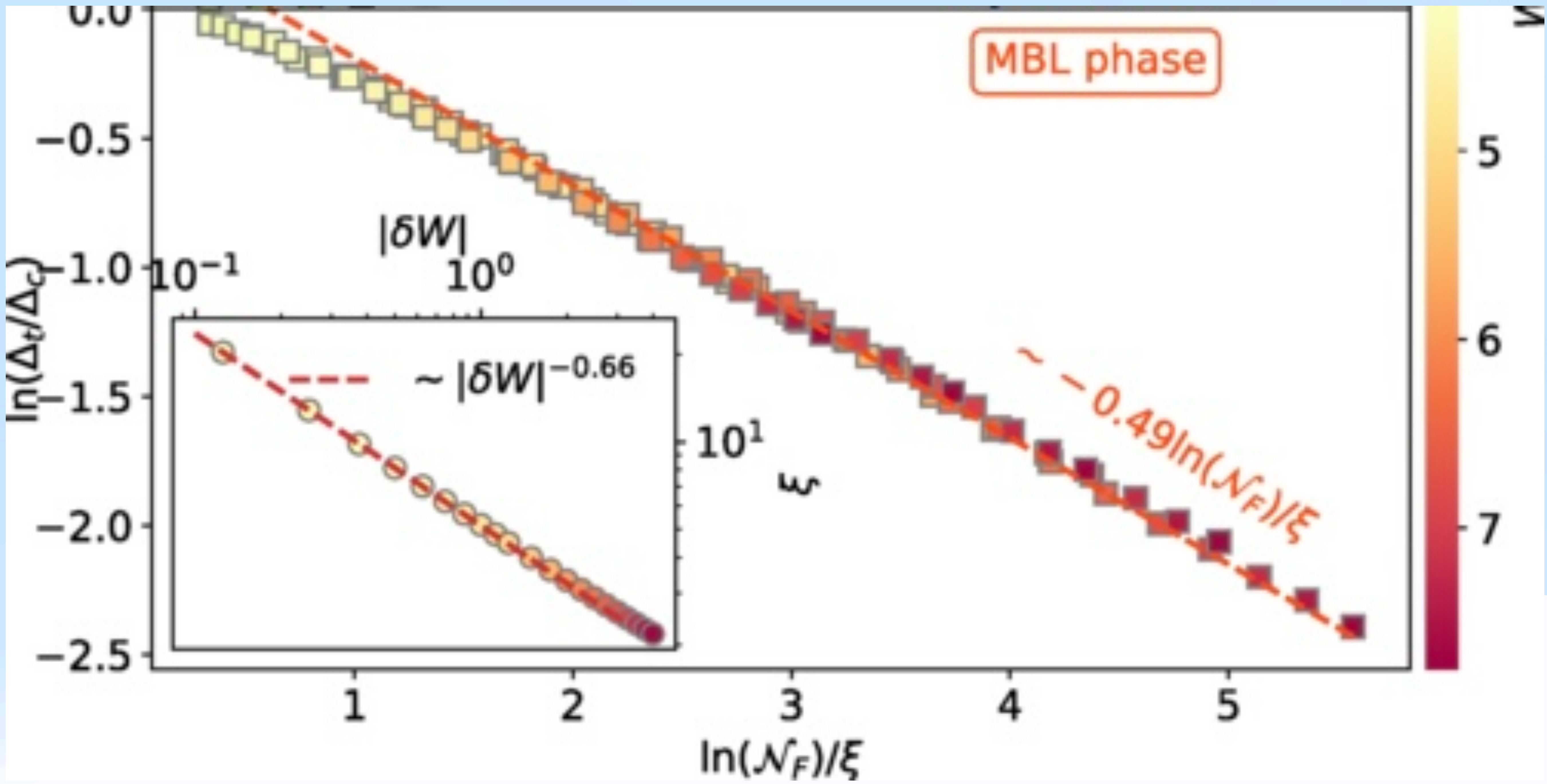
A.Jana, VRC, AG, PRB(2024)



- Cost function has minm around $W_c = 5.3$and $\nu = 0.6 \pm 0.1$
- Best minima at $\nu = 0.64$
- For Green's function quantities $W_c \sim 7.9$ and $\nu \sim 2.76$
- For all ranges of interactions $\nu \geq 2$

Critical exponent for Level Spacing Ratio

- $\nu_{LSR} \sim 0.64 \sim \nu_{FS}$



J. Sutradhar et.al. PRB (2022)

- $$H_{eff} = \sum_{l,m} T_{l,m} |l\rangle\langle m| + \sum_l \epsilon_l |l\rangle\langle l|$$

ϵ_l is correlated.

- ν from LSR is also close to critical exponent obtained Anderson model on RRG.

Eigen values of H_{eff} in strong disorder limit: $E_n \sim \epsilon_n + O(t)$

LSR follows dimensionality of H_{eff} in Fock space and hence $d_{FS} \sim L$. CCFS criterion may not hold as it is for $d < 4$ or $\nu \geq 2/L$.

MBL Transition in Quasiperiodic Systems

CCFS criterion does not hold for systems with quasiperiodic potential.

EUROPHYSICS LETTERS 10 November 1993

Europhys. Lett., **24** (5), pp. 359-364 (1993)

A Classification of Critical Phenomena on Quasi-Crystals and Other Aperiodic Structures.

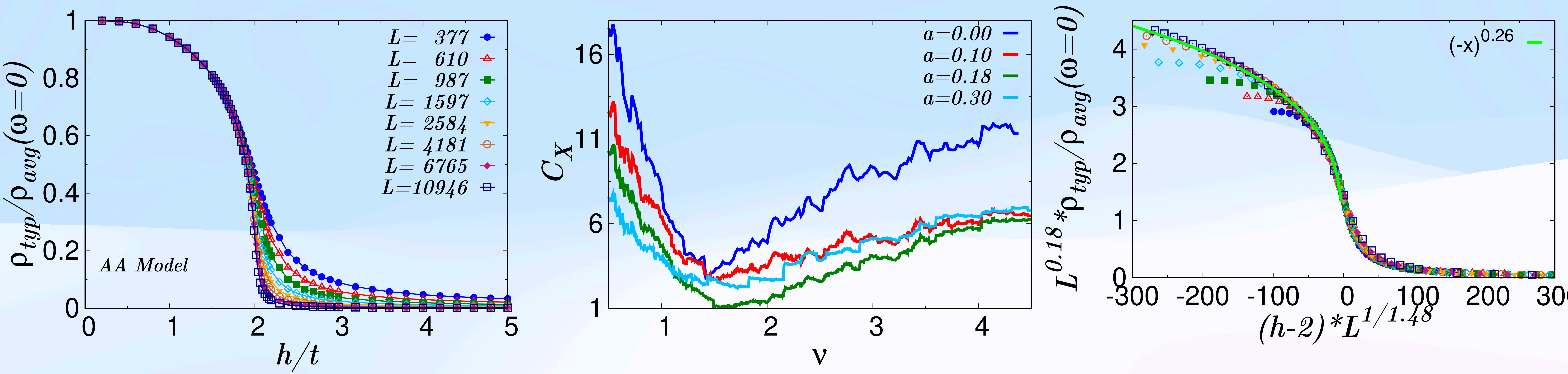
J. M. Luck (*)

Service de Physique Théorique, Centre d'Etudes de Saclay
91191 Gif-sur-Yvette Cedex, France

A continuous transition is stable w.r.t quasiperiodicity if $\nu \geq 1/d$.
Not applicable to MBL transitions driven by quasiperiodicity itself.

Local DOS for Aubry-Andre model

Non-interacting case



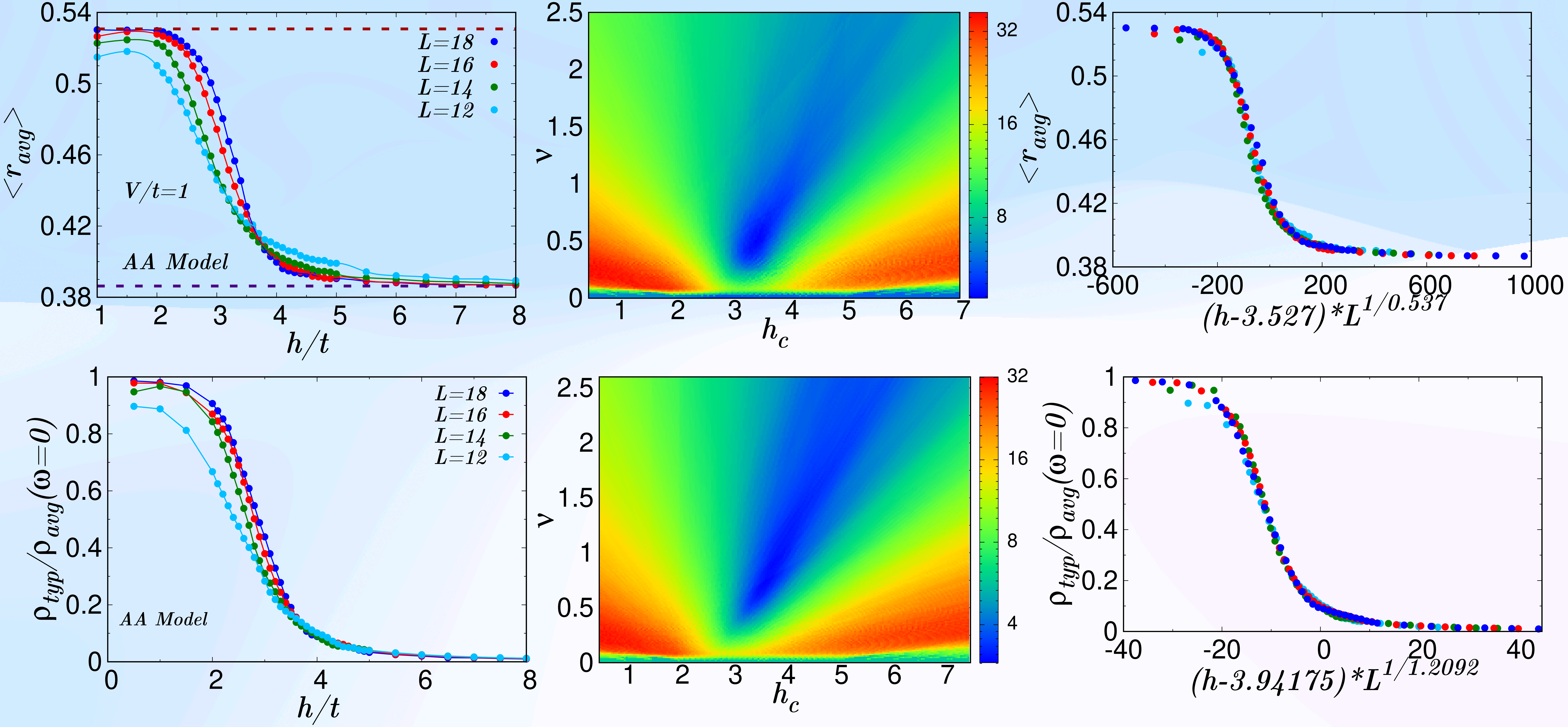
- $\nu = 1.48 \geq 1/d$

Y. Prasad, AG, PRB(2024)

Interacting AA model

Y. Prasad, AG, PRB(2024)

- $\nu_{lsr} = 0.537, \nu_{ldos} = 1.21$



Comparison of random and deterministic models

- $\nu_{lsr} \ll 1$ for all the models supporting our argument that LSR follows dimensionality of effective Anderson model on Fock space and at most $\nu_{lsr} \geq 2/L$.
- **Finite size scaling of LDOS shows**
  $\nu \geq 2/d \;\; for \; Random \; Disorder$
- $\nu \geq 1/d \; for \; quasiperiodic \; systems$
- W_c for LSR and LDOS not very different for AA model, while it is significantly different for random disorder model. No Griffiths phase in AA model due to absence of rare regions.

Can combined effect of disorder and interactions preserve coherence ?

AG, A.K. Pati , arXiv: 2409.10449

Coherence: Basic Definition

- $C_1 = \sum_{\alpha \neq \beta} |\rho_{\alpha \neq \beta}|$
- $C_2 = \sum_{\alpha \neq \beta} |\rho_{\alpha\beta}|^2$
- $C_{rel} = -\sum_{\alpha} \rho_{\alpha\alpha} \log(\rho_{\alpha\alpha}) - S(\rho)$

$$S(\rho) = -Tr[\rho \log \rho]$$

- $C(\rho) \geq 0$
- Under incoherence operations $C(\rho)$does not increase
- $C(\rho)$ is a convex function of quantum states

$$C\left(\sum_k p_k \rho_k\right) \leq \sum_k p_k C(\rho_k)$$

Exact relations between coherence and measure of localization

Pure States: $\rho = |\Psi\rangle\langle\Psi|$, $|\Psi\rangle = \sum_{\alpha=1}^{N_f} \Psi_\alpha |\alpha\rangle$

- $C_1 = \sum_{\alpha\neq\beta} |\rho_{\alpha\beta}|$

  $1 \leq C_1 + IPR \leq N_f$
- $1 \leq C_{rel} + IPR \leq log(N_f)$
- $IPR = \sum_{\alpha} |\Psi_\alpha|^4$

- For conventionally extended state,
  $IPR \sim (1/N_f), C_1 \sim N_f - 1$

$$C_{rel} \sim log(N_f)$$

For localized state, $IPR \sim O(1)$

$C_1 \sim 0, C_{rel} \sim 0$

Trade-off relations for mixed states

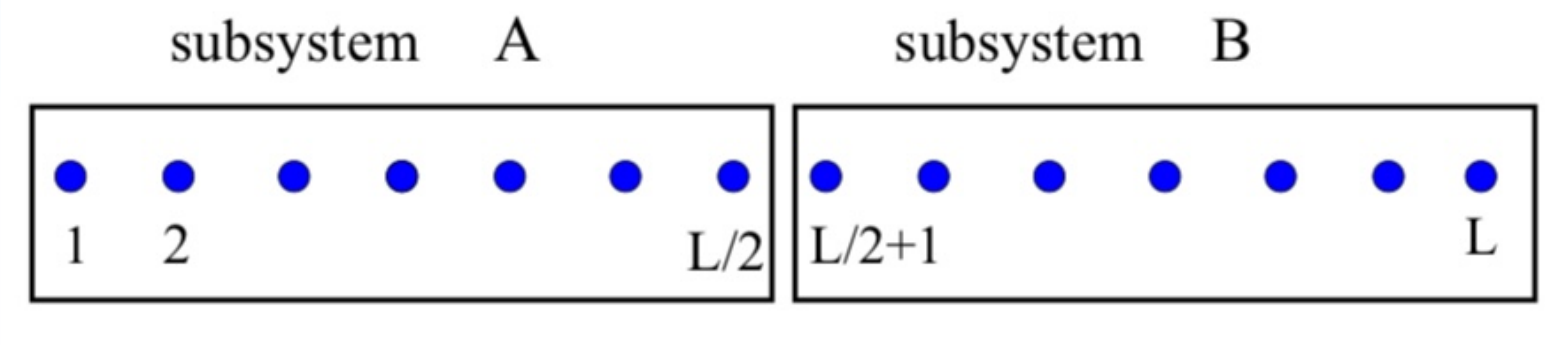
- $C_2 + IPR \leq 1$
- $C_{rel}(\rho) + IPR(\rho) + M(\rho) \leq d, \quad M(\rho) = 1 - Tr(\rho^2)$
- $C_{rel}(\rho) + d_n IPR(\rho) \geq 1$

$$|\Psi\rangle = \sum_{\alpha=1}^{2^L} \Psi(\alpha)|\alpha\rangle, \rho = |\Psi\rangle\langle\Psi|$$

Reduced Density Matrix for subsystem A

$$\rho^A = Tr_B \rho, C_2^A = \sum_{\alpha \neq \beta} |\rho^A_{\alpha\beta}|^2$$

$$IPR^A = \sum_{\alpha} |\rho^A_{\alpha\alpha}|^2$$



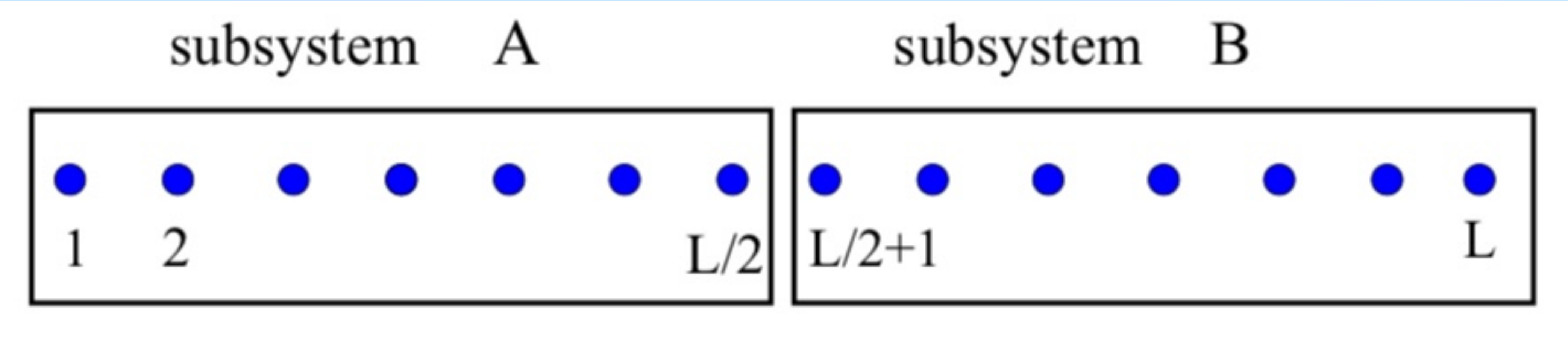
For extended states,

$$C_2^A \sim \left(\frac{1}{2^L}\right)^2 (2^{L/2}(2^{L/2} - 1)) \sim 0$$

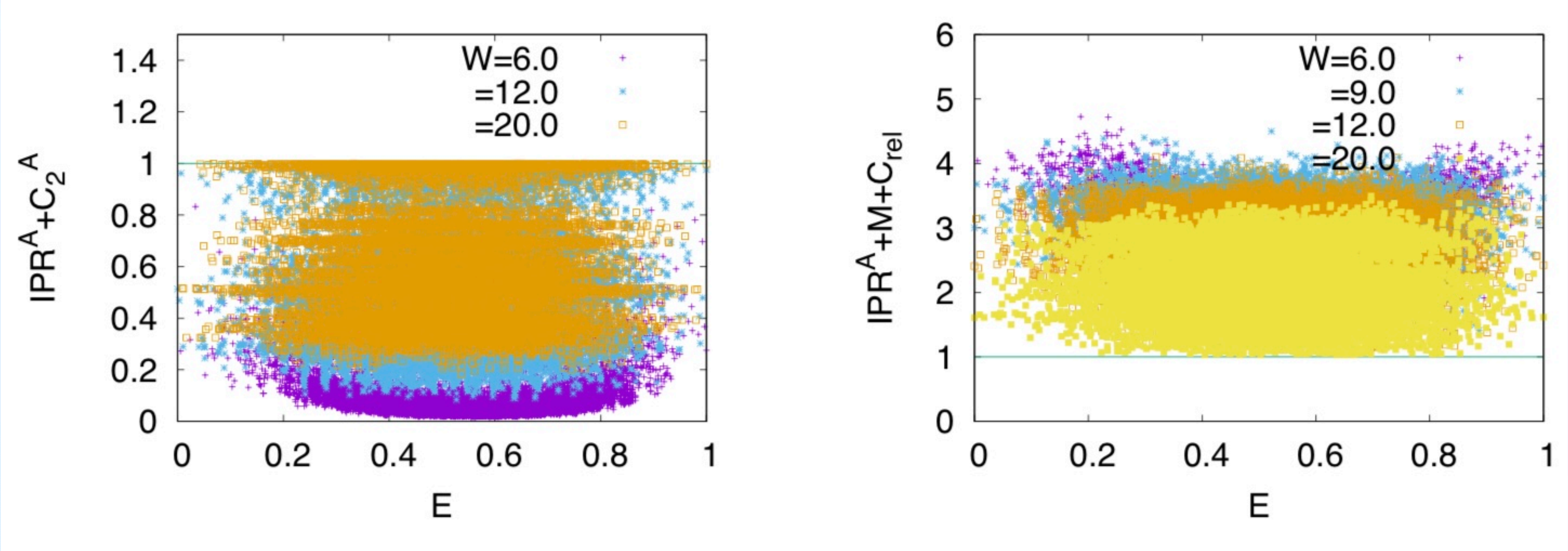
Localized states, $C_2^A \sim O(1)$

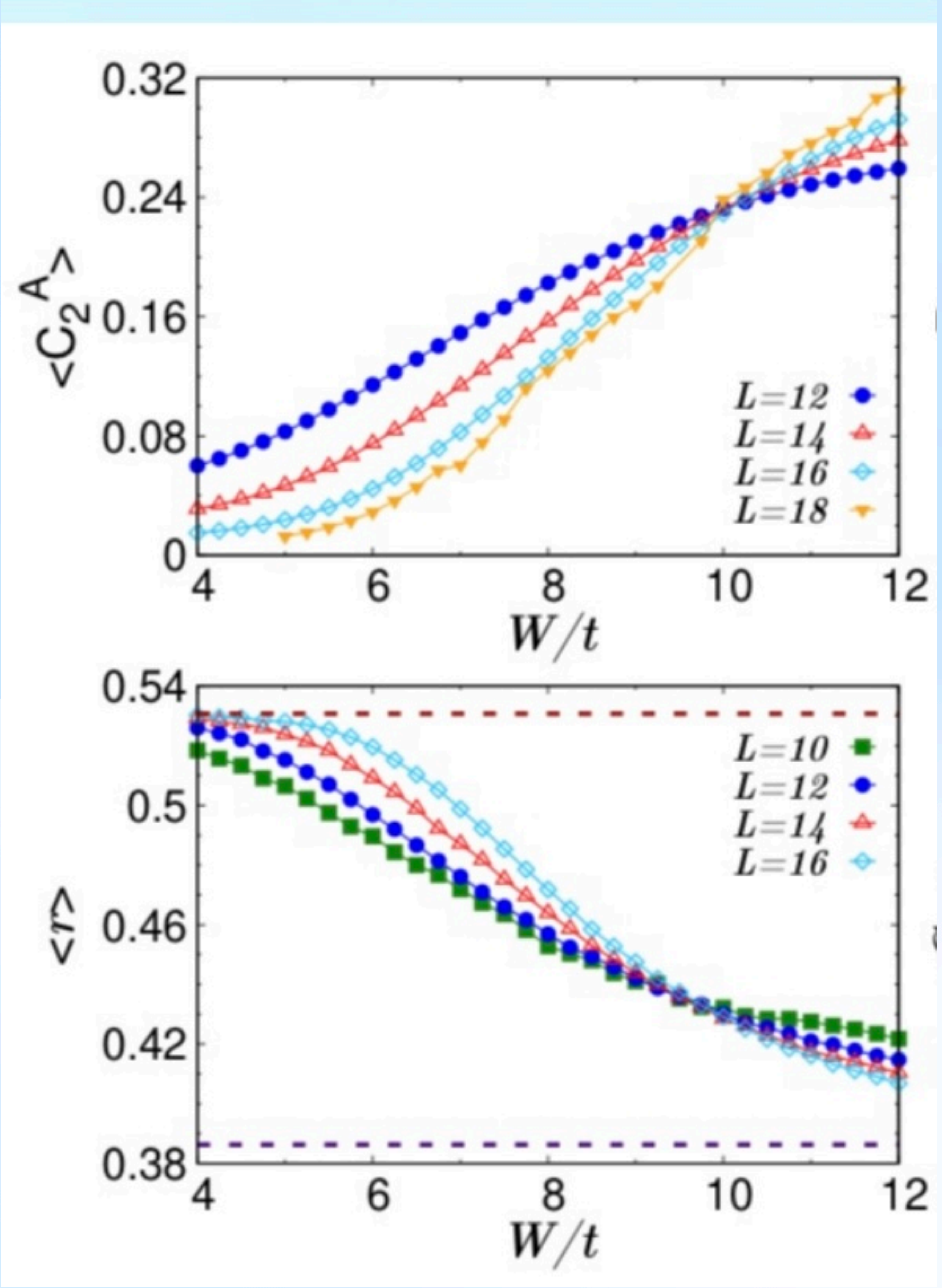
$$H = -t\sum_i [c_i^\dagger c_{i+1} + h.c.] + \sum_i h_i n_i + \sum_i V n_i n_{i+1} + V_2 n_i n_{i+2}$$

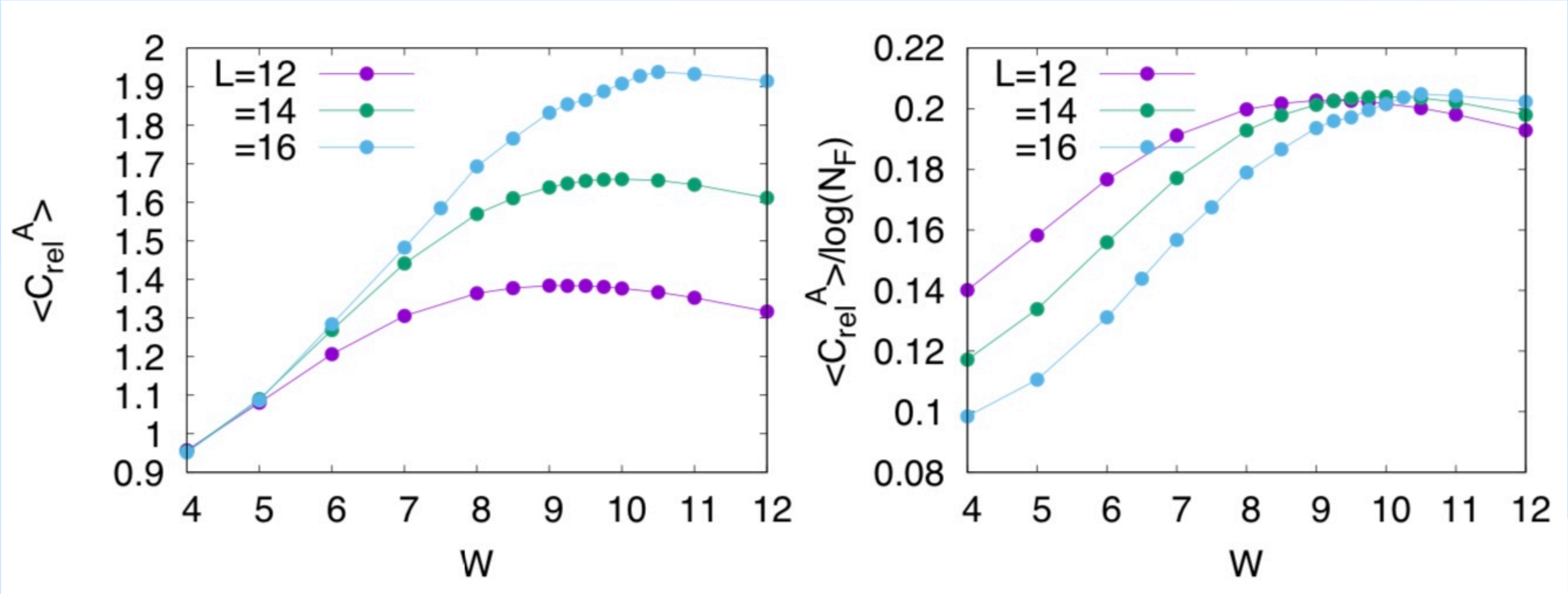
$h_i \in [-W, W], V_1 = 1, V_2 = 1/2$



- $C_2 + IPR \leq 1$
- $C_{rel}(\rho) + IPR(\rho) + M(\rho) \leq d, \quad M(\rho) = 1 - Tr(\rho^2)$
- $C_{rel}(\rho) + d_n IPR(\rho) \geq 1$





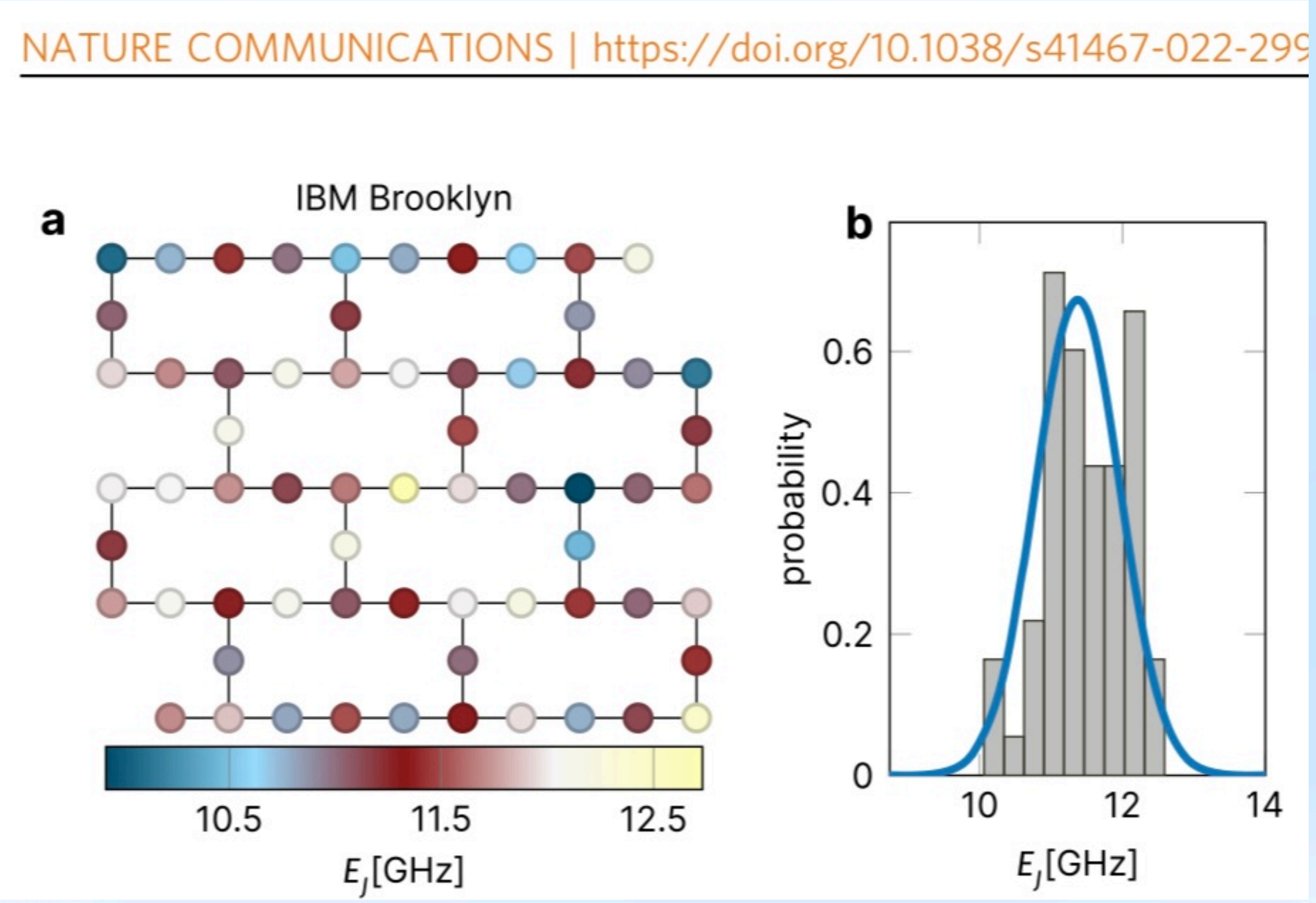


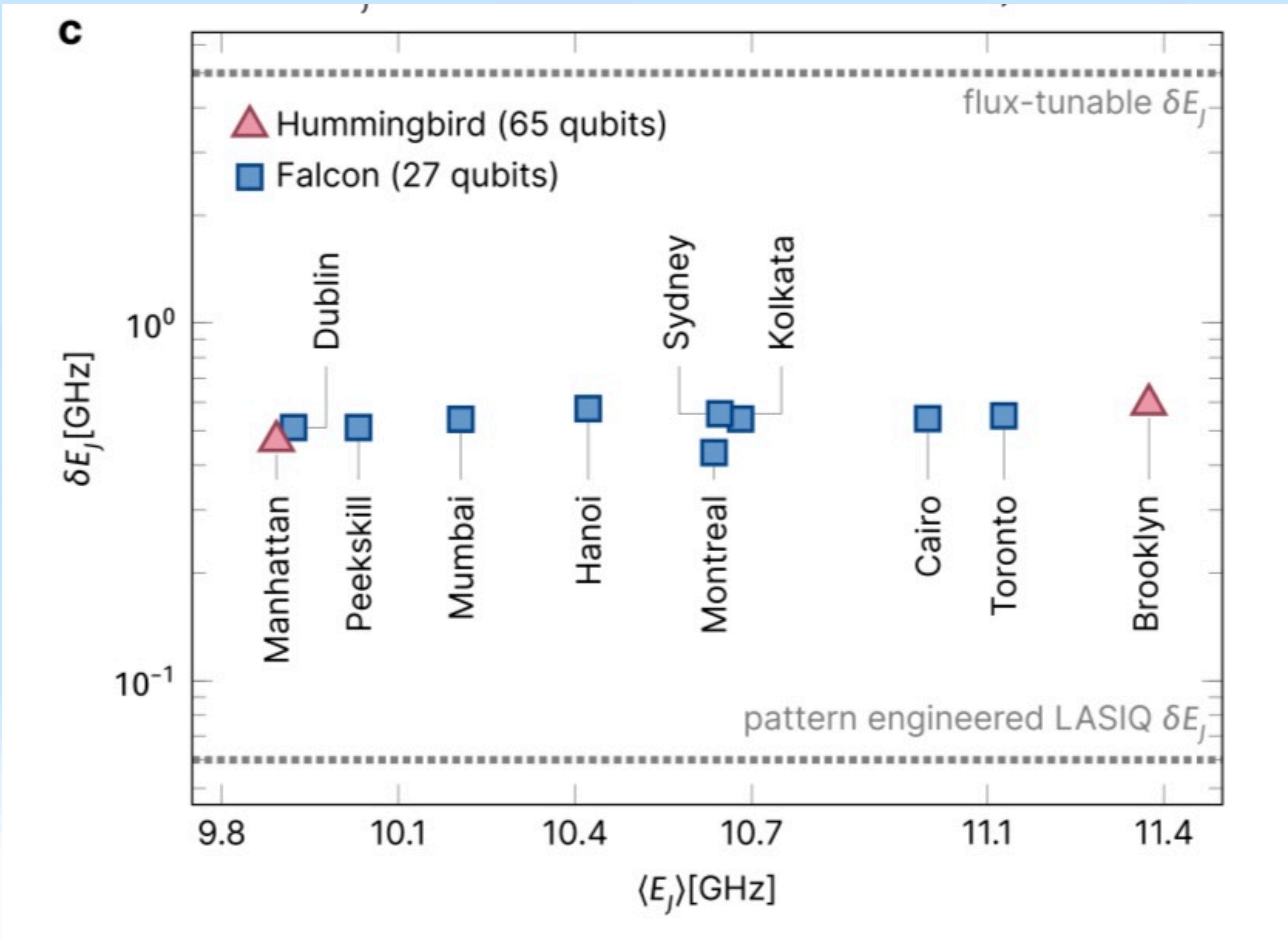
Coherence for subsystem A gets enhanced due to combined effect of disorder and interactions!

$W_c \sim 9.75t$

AG, A.K. Pati , arXiv: 2409.10449

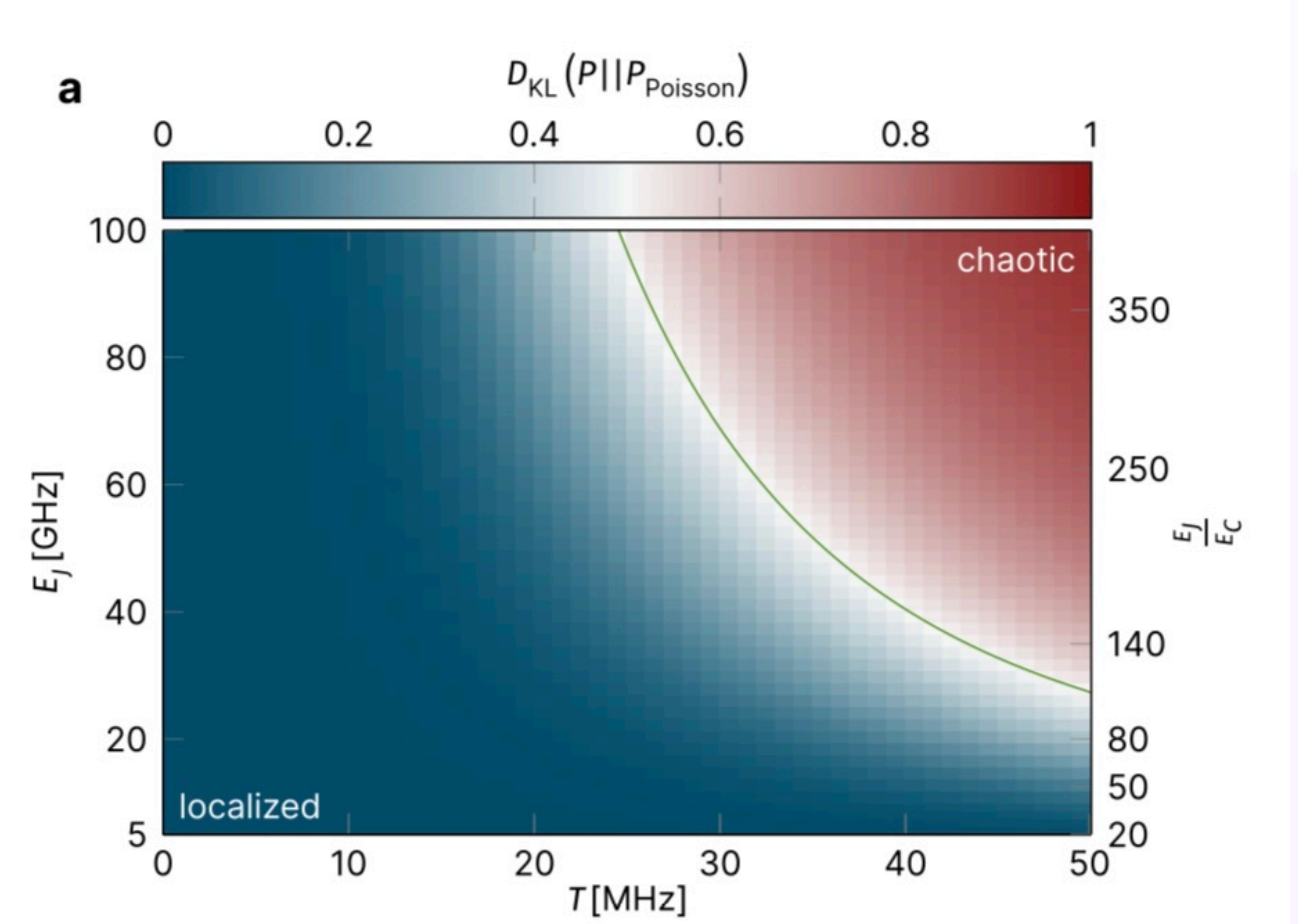
Application for Superconducting Qubit Arrays





$$H = 4E_C \sum_i n_i^2 - \sum_i E_{J_i} \cos\phi_i + T \sum_{\langle i,j \rangle} n_i n_j .$$

- Our work shows that disorder in Josephson energies may help in enhancing coherence of a subsystem of SC qubit arrays.



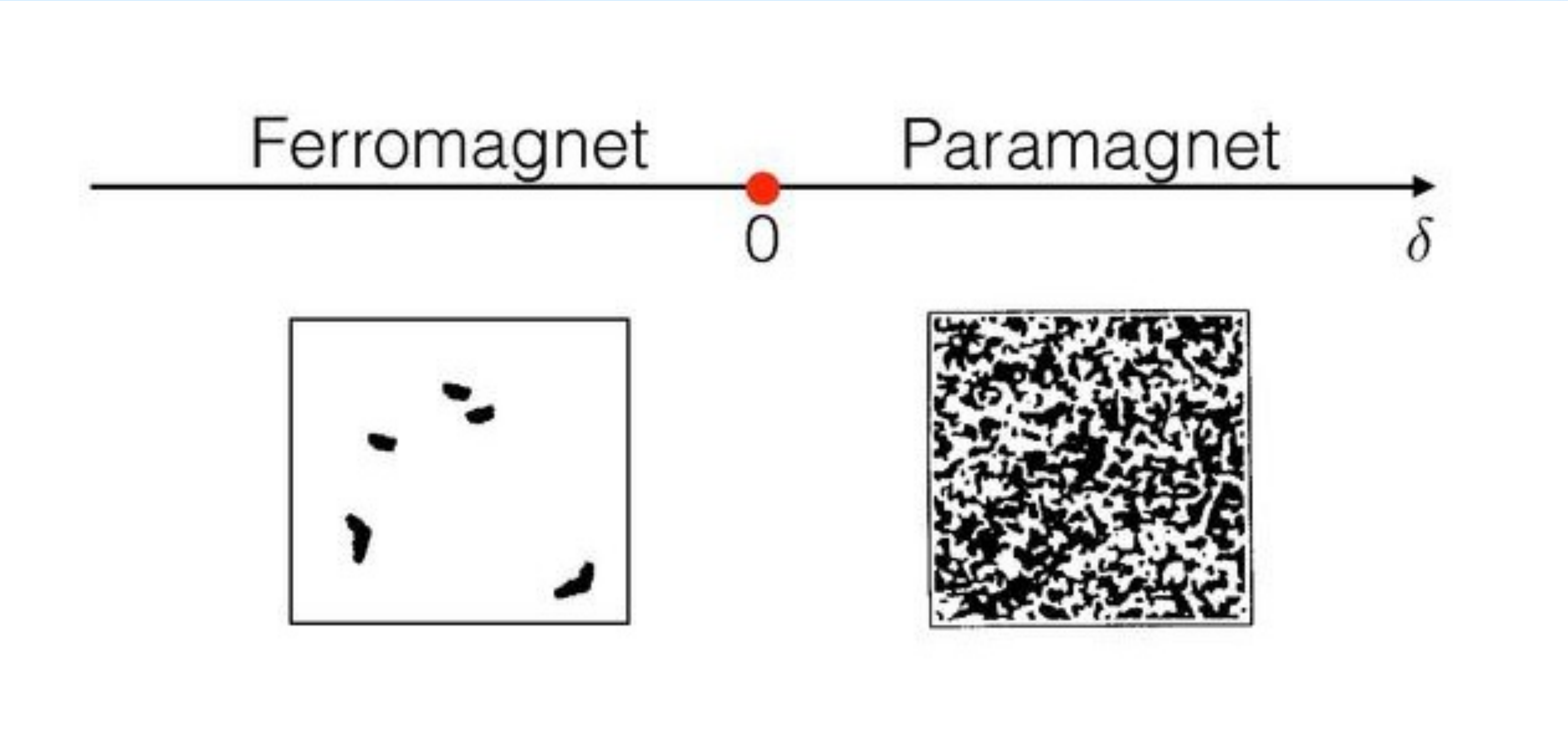
- C. Berke, Nature Communications (2022)

Summary

- We proved exact trade-off relations between various norms of quantum coherence and measure of localization.
- Many-body localization might preserve coherence for certain quantum states.
- Coherence of a subset of SC qubit arrays with inhomogeneous Josephson energies might have enhanced coherence due to combined effect of disorder and interactions.
- Finite-size scaling of single-particle excitations and scattering rates shows that MBL systems with random and quasiperiodic potential belong to different universality class.

Harris Criterion

- a continuous transition in Ising model as a function of temperature with correlation length diverging at the transition point
- $\xi \sim \delta^{-\nu}$



- Add disorder , δ is not constant now
- In a box of size ξ, $\delta = \bar{\delta} + C\sqrt{\frac{1}{\xi^d}}$
- r.m.s. < mean for a stable transition $\Rightarrow \nu \geq 2/d$
- **Harris, J. Phys. C (1974)**

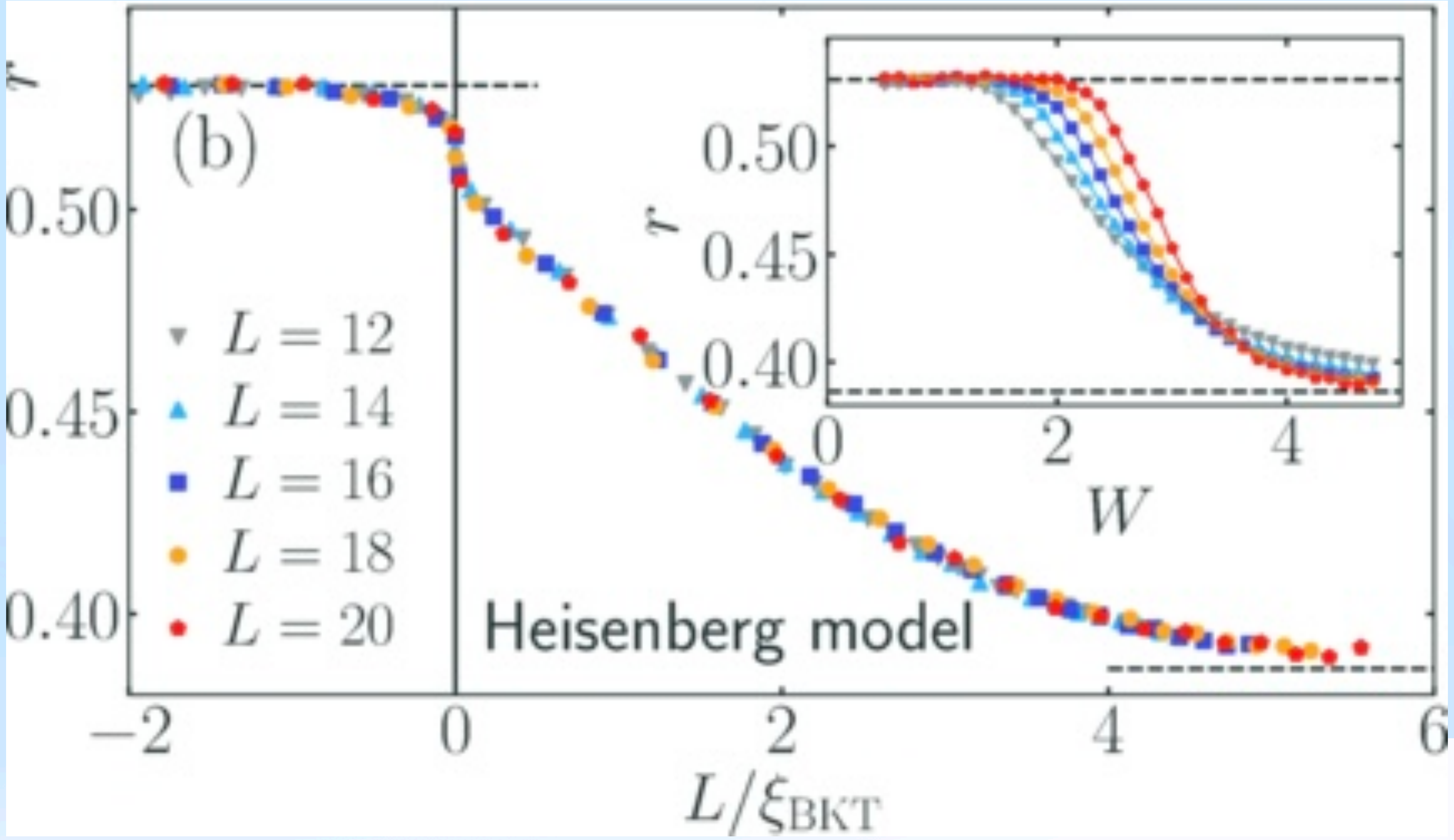
- CCFS criterion is much stronger and is applicable to transitions driven by disorder itself e.g. Anderson Localization Transition.

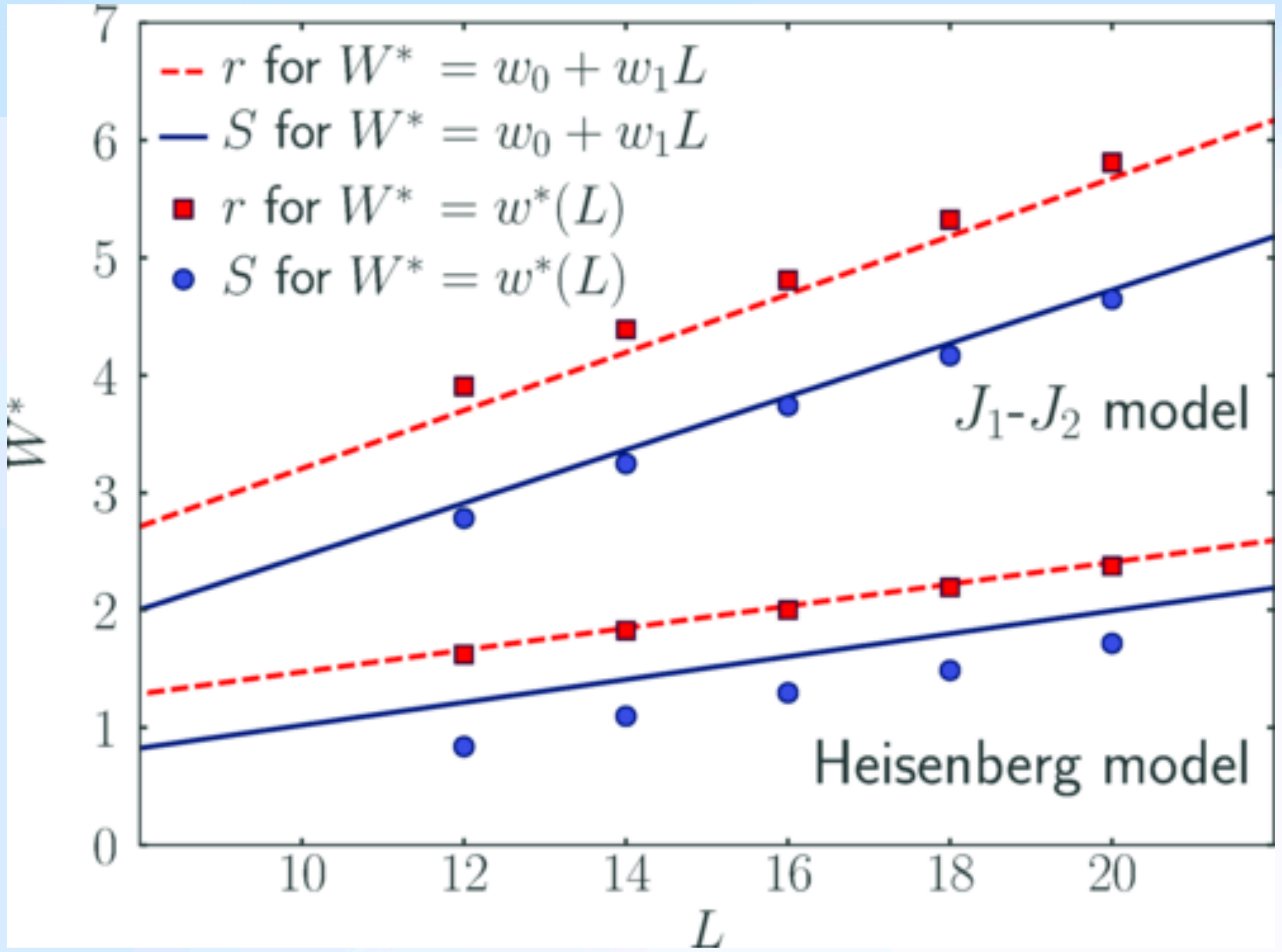
CCFS violation in MBL systems

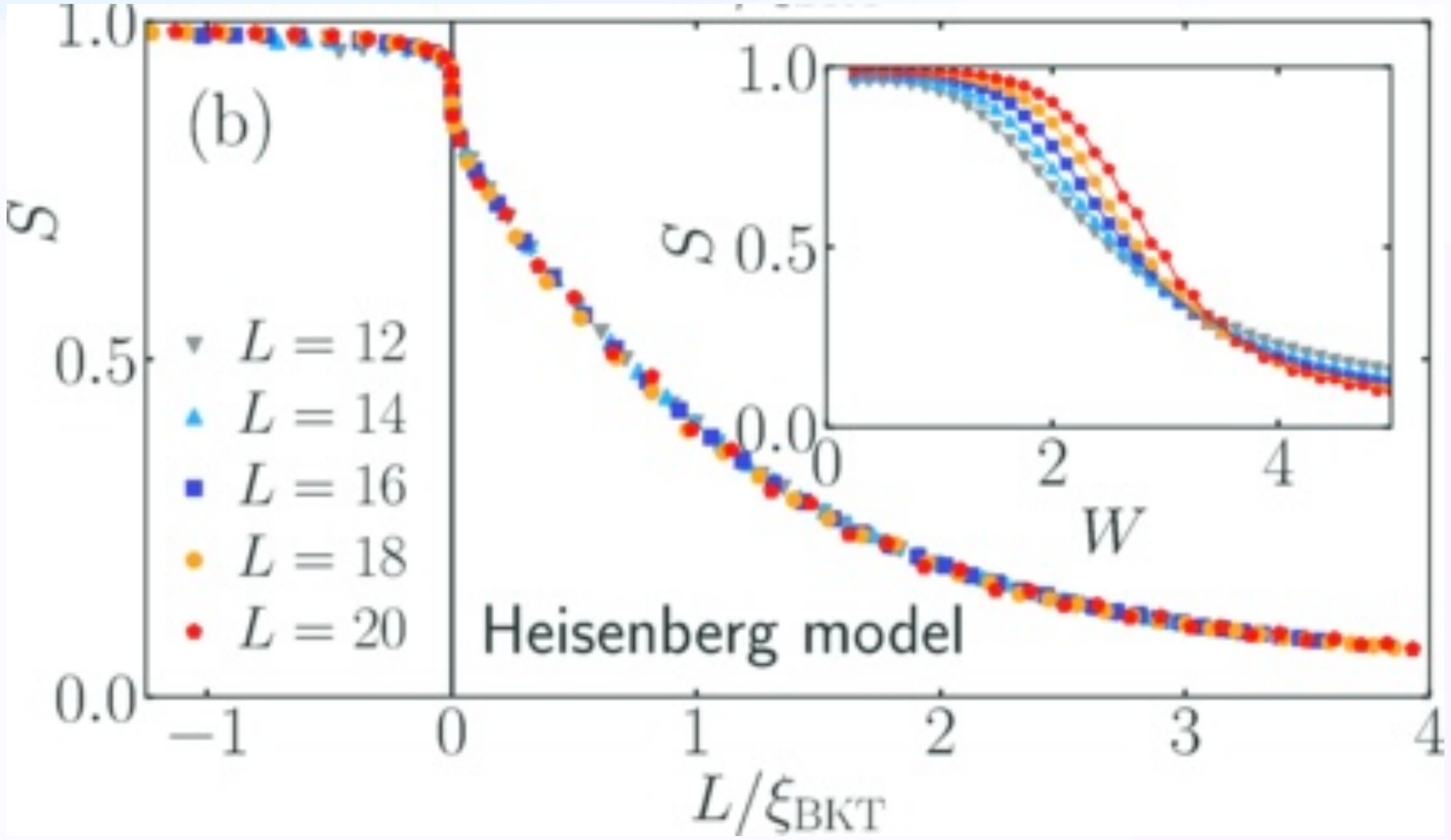
- Numerics can not handle large enough sizes to see correct trend which is seen in RG calculations which show $\nu \sim 3$.
- MBL transition is not continuous in nature…??
- This prompted modified RG calculations which predicted Kosterlitz-Thouless (KT) type transition (Goremykina (2019), Dumitrescu (2019), Morningstar(2019))
- $$\xi_{KT} = \exp \frac{b_{\pm}}{\sqrt{|W - W_c|}}$$

Scaling with KT ansatz

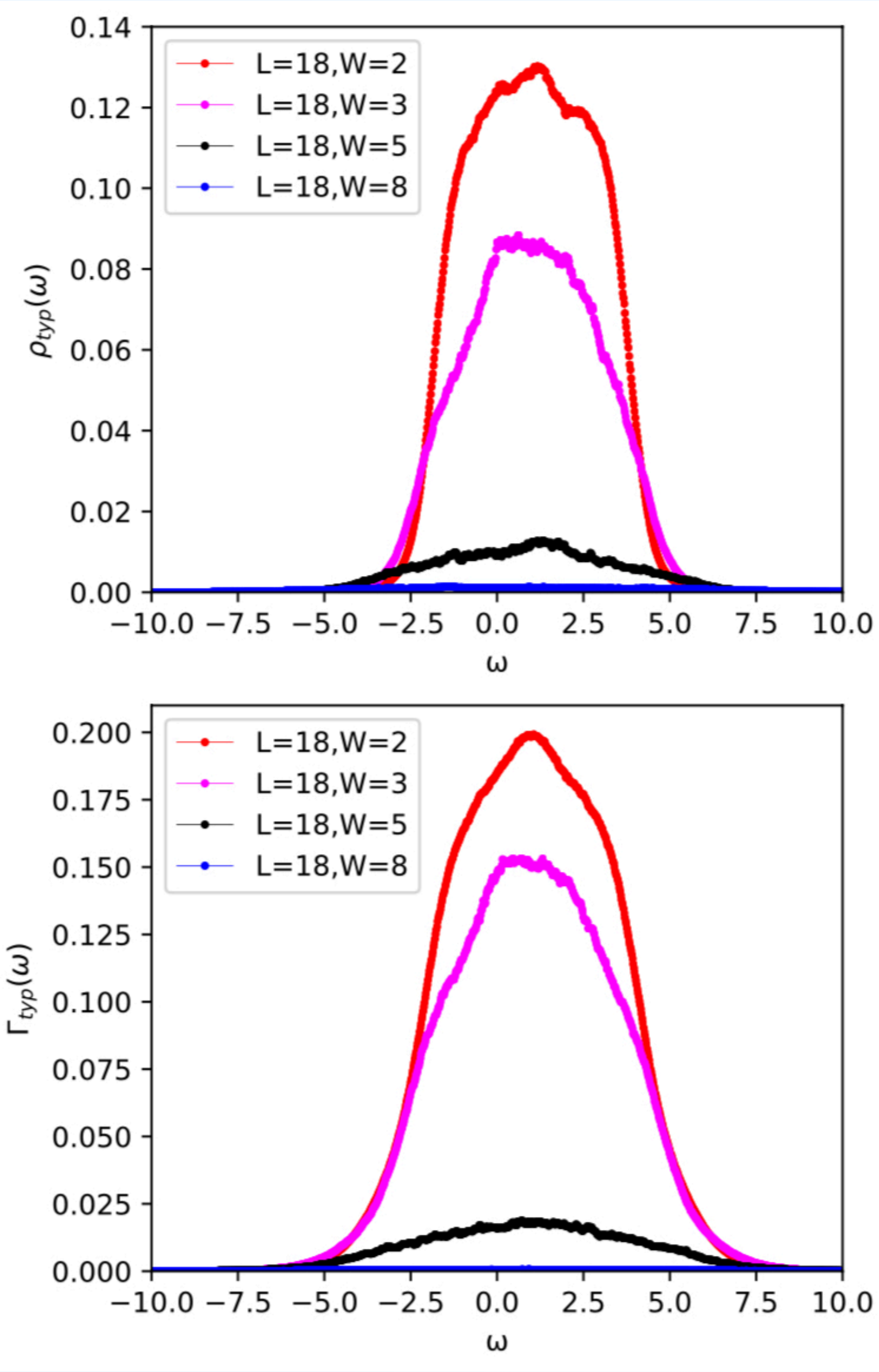
- $W^\star = W_0 + W_1 L$ gives best collapse
- Suntajs PRB (2020)

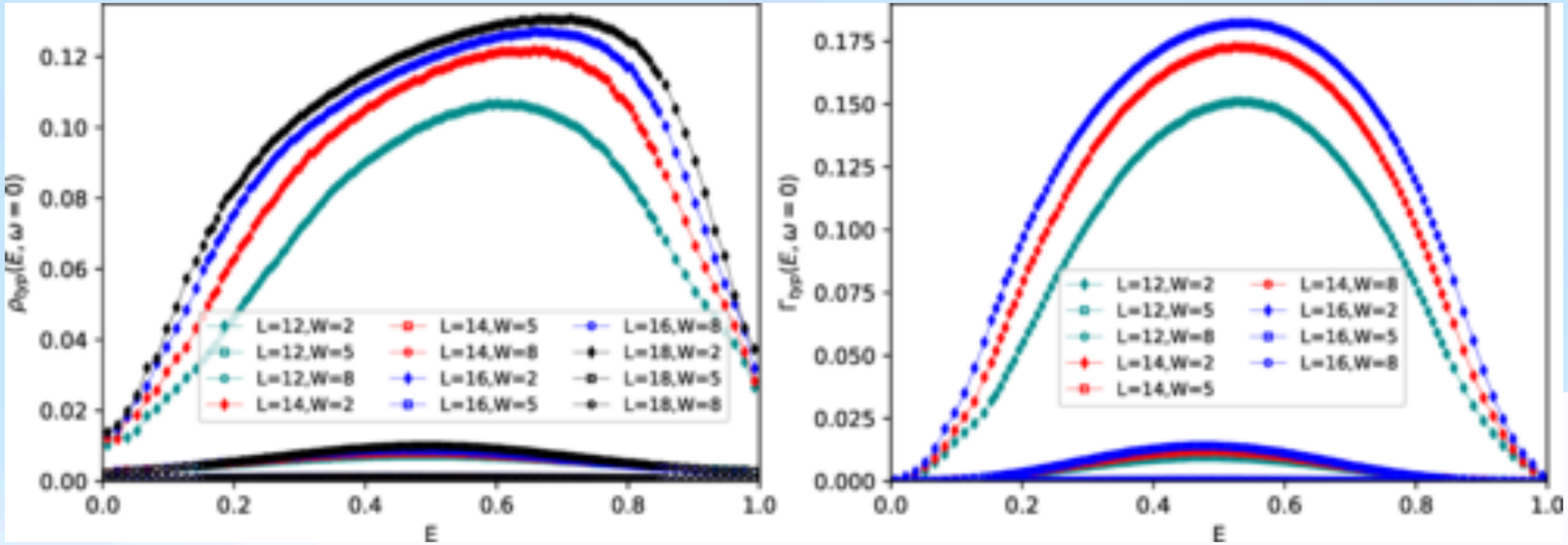






Single-particle excitations





A. Jana, V.R. Chandra, AG PRB(L)(2022).

- $\rho_{typ}(\omega)$ and $\Gamma_{typ}(\omega)$: maximum at $\omega \sim \mu$ and decreases as energy of excitation increases. This is for excitations in mid band states.
- $\rho_{typ}(0)$and $\Gamma_{typ}(0)$ as a function of energy of many-body eigenstate in which excitation is produced. More localized a man-body state is, more localized is the single-particle excitation.

LDOS across MBL transition

$$\rho_i(n,\omega) = \sum_m |\langle \Psi_m | c_i^\dagger | \Psi_n \rangle|^2 \delta(\omega - E_m + E_n)$$
$$+ |\langle \Psi_m | c_i | \Psi_n \rangle|^2 \delta(\omega + E_m - E_n)$$

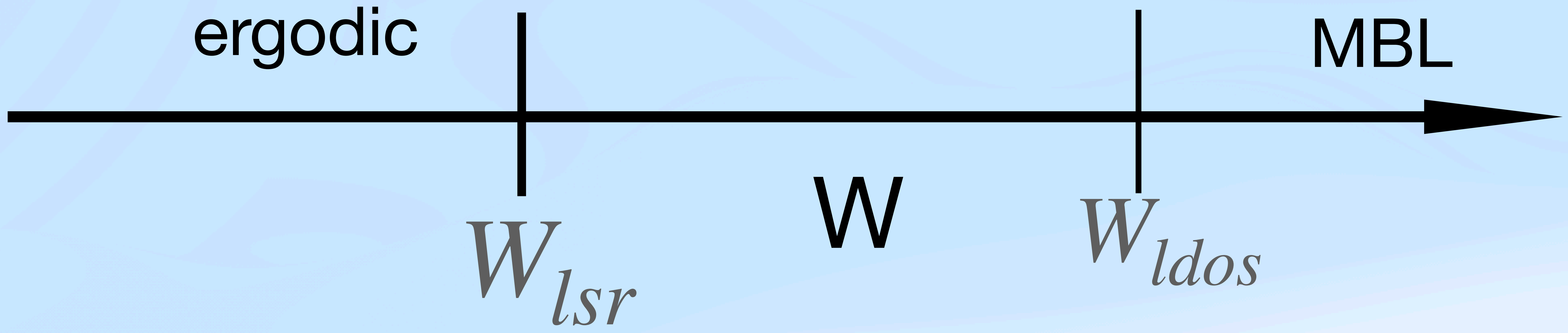
Create a particle-hole pair on top of $|\Psi_n\rangle$. Excited state $|\Psi_{ex,n}\rangle = C_i^\dagger |\Psi_n\rangle = \sum_m a_m |\Psi_m\rangle$

- If $|\Psi_n\rangle$ is localized, number of eigenstates contributing to $|\Psi_{ex,n}\rangle$ is of measure zero $\Rightarrow$ excitation can not propagate over all eigenstates allowed by the energy conservation. Hence $\rho_{typ}(\omega)$ is vanishingly small.

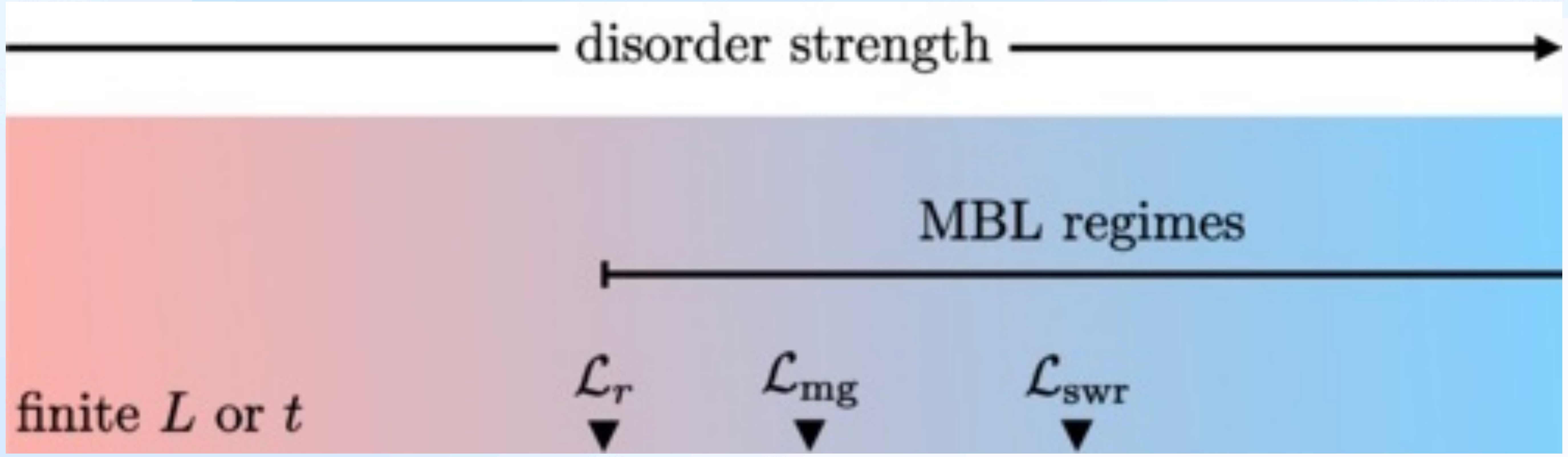
- If $|\Psi_n\rangle$ is extended, $|\Psi_{ex,n}\rangle$ will get contribution from a significant fraction of many-body eigenstates making $\rho_{typ}(\omega)$ finite in the delocalised phase.

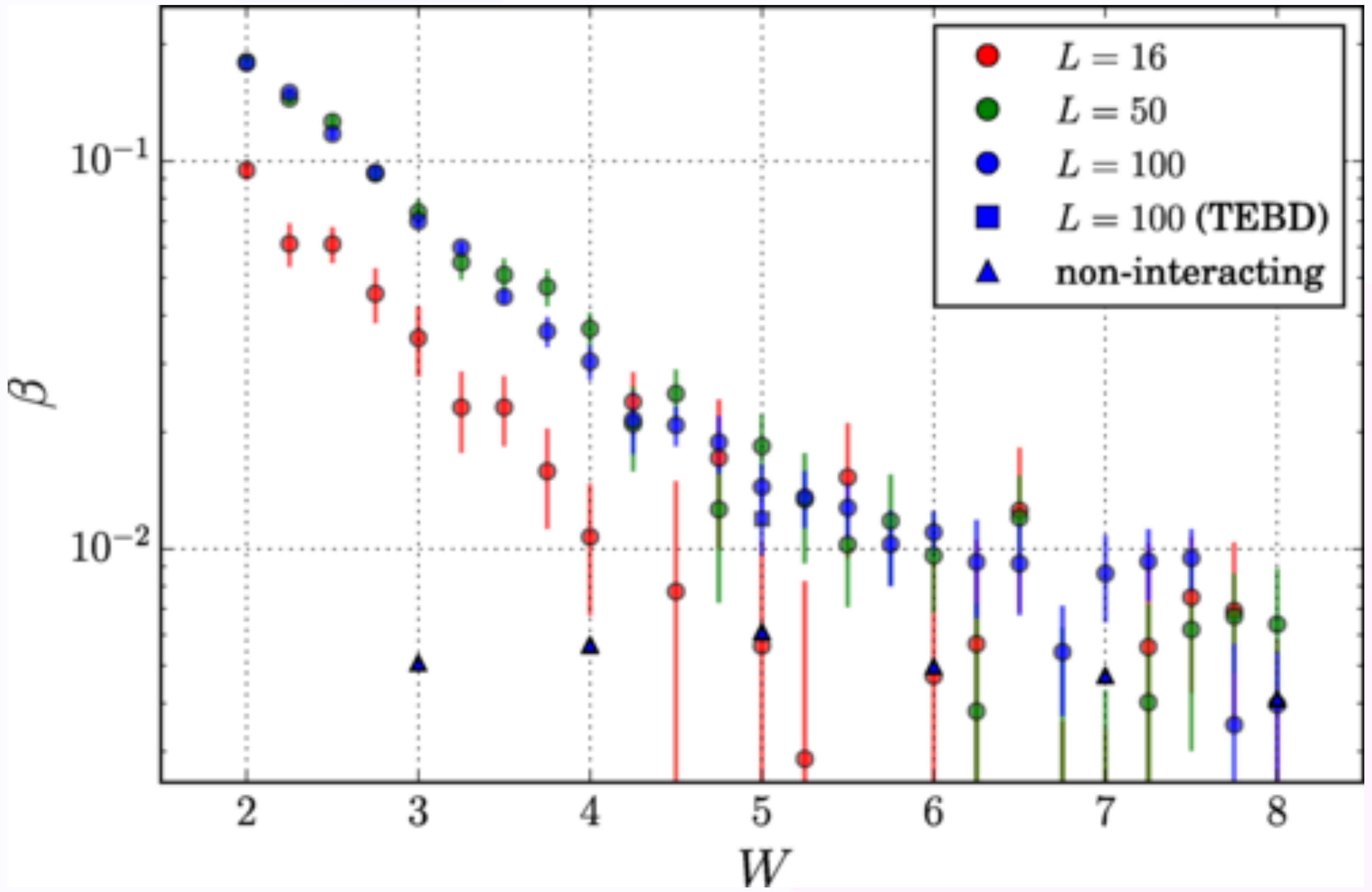
- **From LDOS and Scattering Rate** $7.1t \le W_c^G \le 7.9t$

- **From Level Spacing Ratio** $W_c^{lsr} \sim 5.3t$



- Similar conclusions from analysis of density imbalance and mean square displacement (Mirlin, F. Evers)

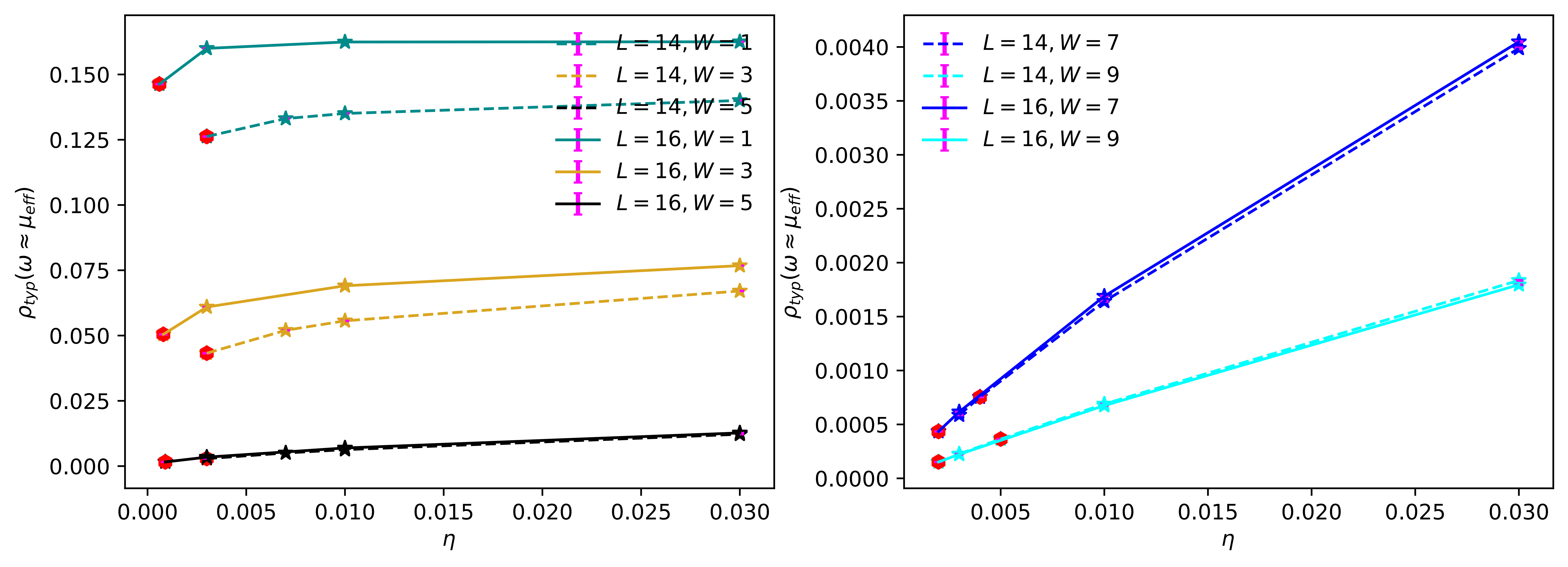




Morningstar et. al PRB (2022)

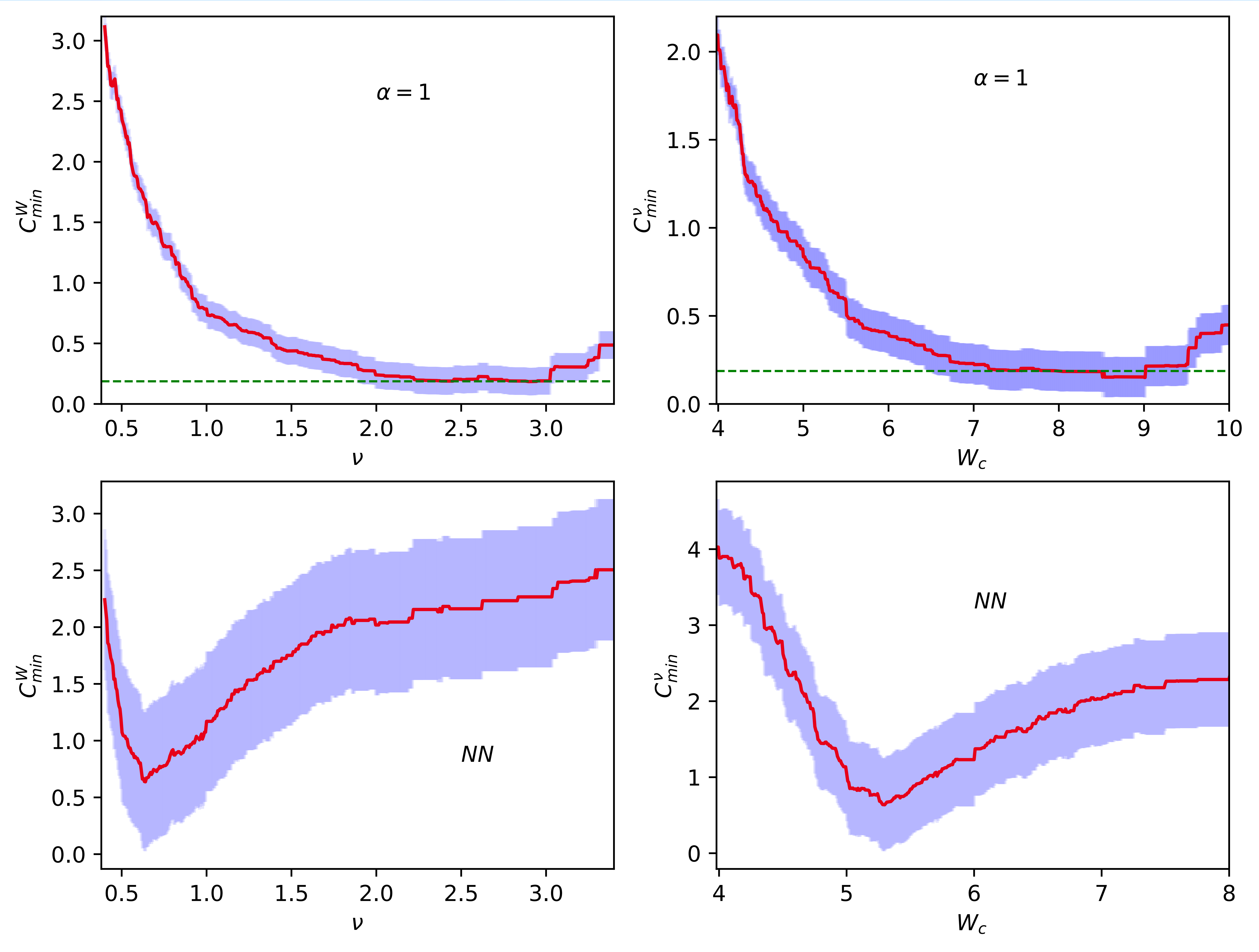
Thank You!

Broadening used in Green's Functions



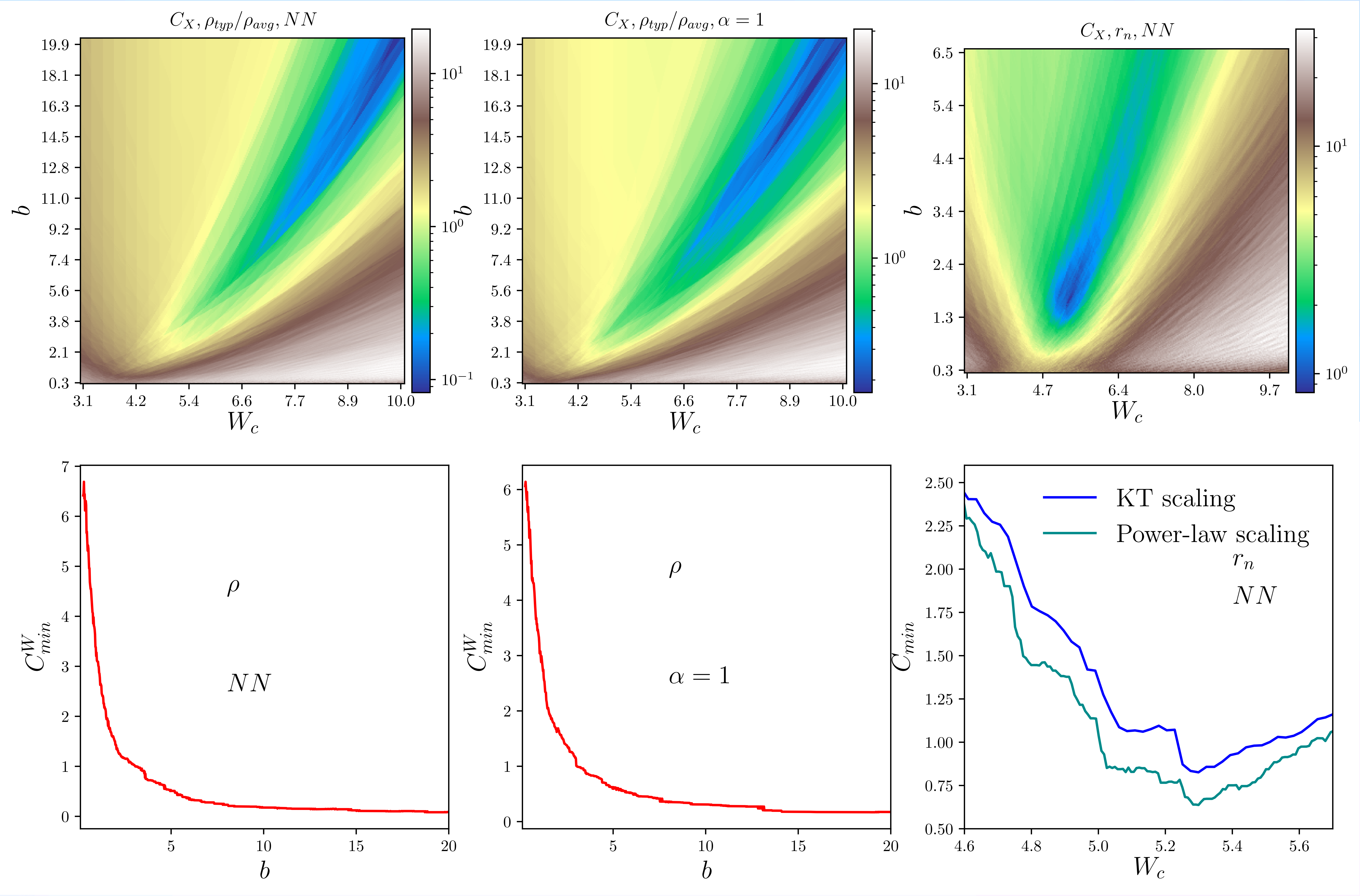
In thermodynamic limit, delocalized phase ρ_{typ} is independent of η, in localized phase ρ_{typ} increases with η. For finite-size systems this happens for $\Delta < \eta < \Delta_2$ with Δ average spacing and Δ_2 spacing for system of size of correlation length.

Cost-function Minima



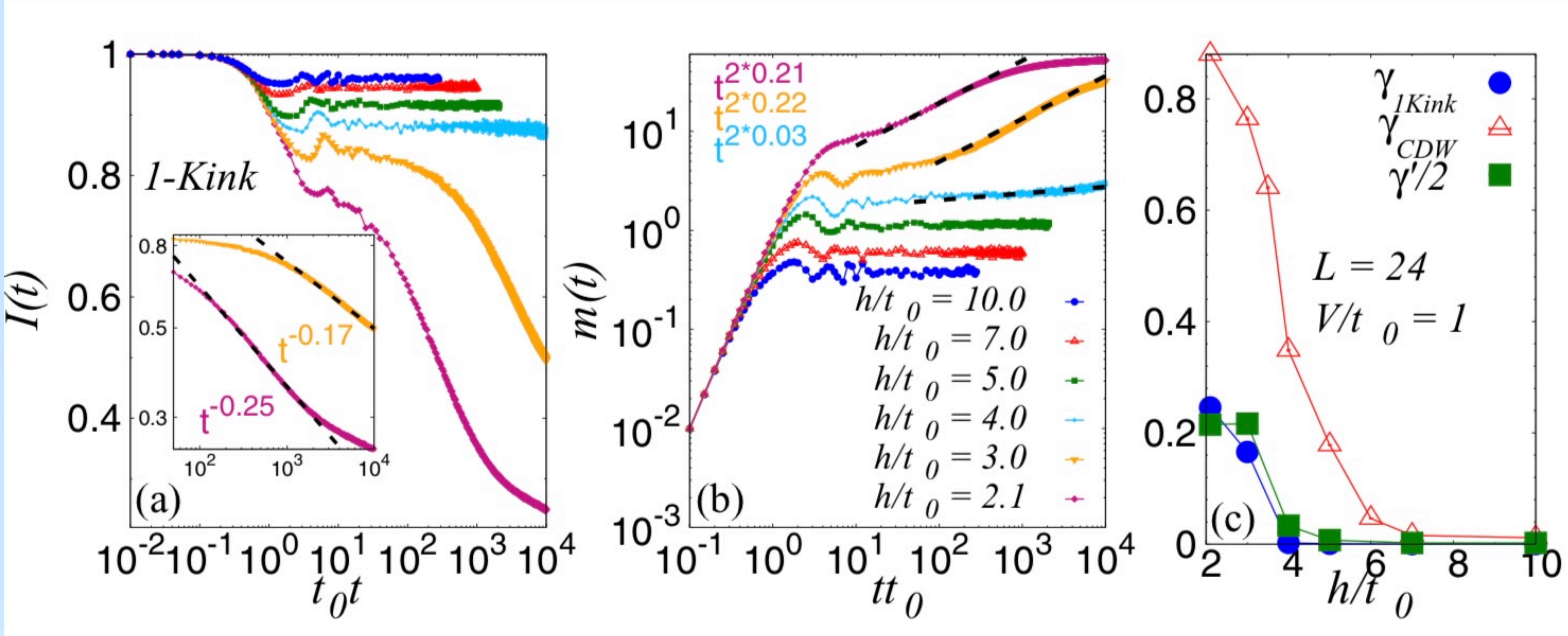
KT Scaling

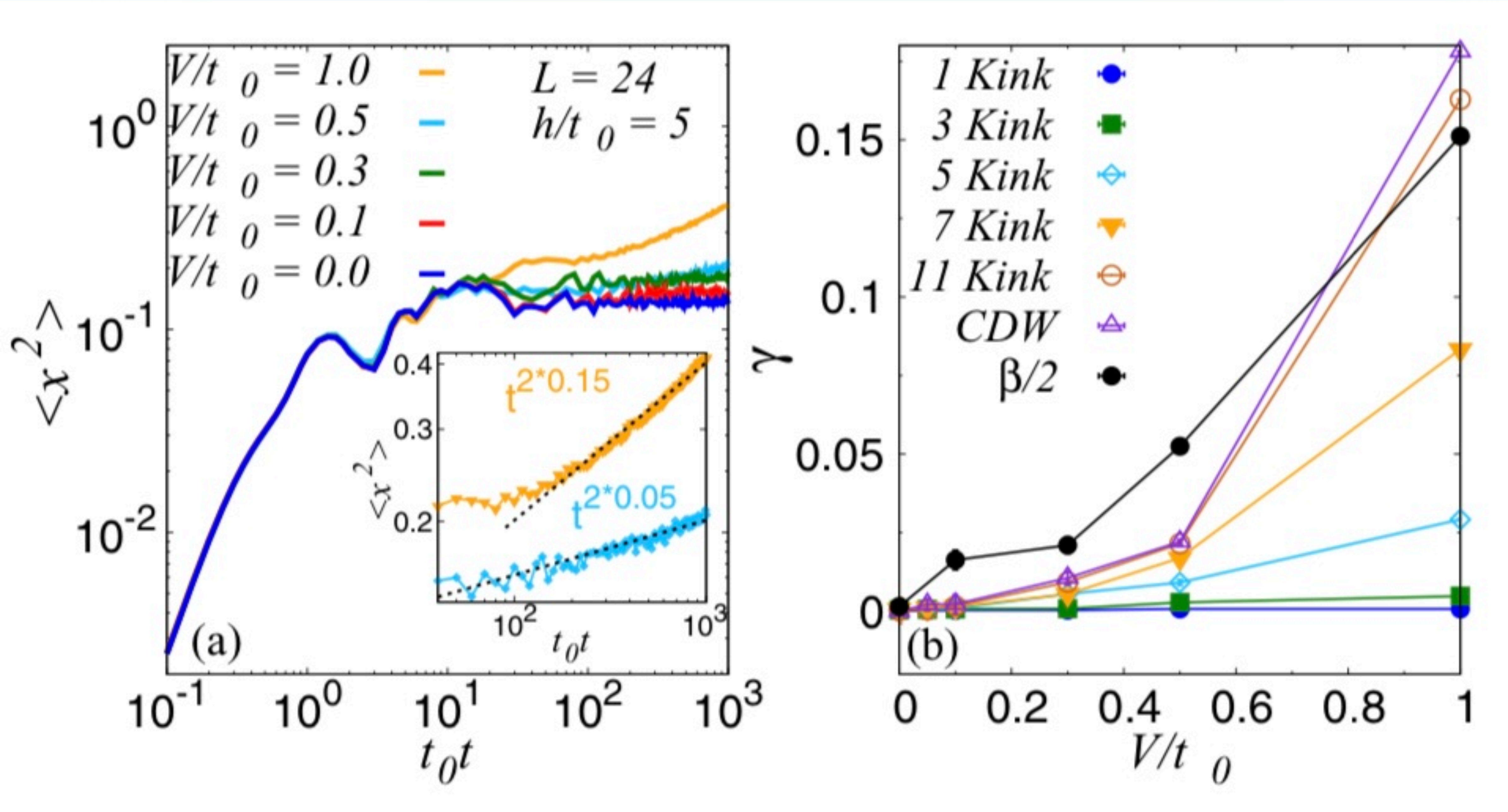
$$\xi_{KT} = \exp \frac{b_\pm}{\sqrt{|W - W_c|}}$$



One-kink state

- $m(t) = \sum_i i[n_i(t) - n_i(0)]$





- $G(x, t) = \sum_i \langle n_{i+x}(t) n_i(0) \rangle$
- $\langle x^2(t) \rangle = \sum_x x^2[\overline{G(x, t)} - \overline{G(x, 0)}]$