# The inhomogeneous multispecies PushTASEP

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(joint with J. Martin, arXiv:2310.09740)

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- Let $0 \leq t \leq 1$. Transitions are:

$$10 \xrightarrow{1} 01, \quad 01 \xrightarrow{t} 10.$$

**Proposition**

*The ASEP on $L$ sites with $n_1$ particles has the uniform stationary distribution, i.e.*

$$\pi(\omega) = \frac{1}{\binom{L}{n_1}}.$$

# Multispecies ASEP

- Now suppose there are particles labelled $1, \ldots, s$ with strength order: $s > s-1 > \cdots > 1$ and vacancies labelled 0.
- Consider a ring of $L$ sites, with particle content given by the partition

$$\lambda = (\underbrace{s, \ldots, s}_{m_s}, \ldots, \underbrace{1, \ldots, 1}_{m_1}, \underbrace{0, \ldots, 0}_{m_0}),$$

where $\sum_i m_i = L$.
- The multispecies ASEP is defined by transitions

$$ij \xrightarrow{1} ji, \quad ji \xrightarrow{t} ij, \quad \text{provided } i > j.$$

# Stationary distribution

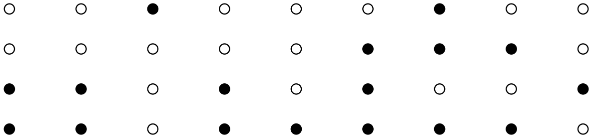
**Theorem (Ferrari–Martin (*Ann. Prob.* 2007))**

*Consider the multispecies TASEP ($t = 0$) with content $\lambda$. Let $M_i = m_i + \cdots + m_s$ for $1 \leq i \leq s$. Then the stationary probability of any configuration is a positive integer divided by*

$$\prod_{i=1}^{s} \binom{L}{M_i}.$$

Moreover, this positive integer has a combinatorial interpretation!

# Bully-path projection example for $(4, 4, 3, 2, 2, 1, 1, 0, 0)$



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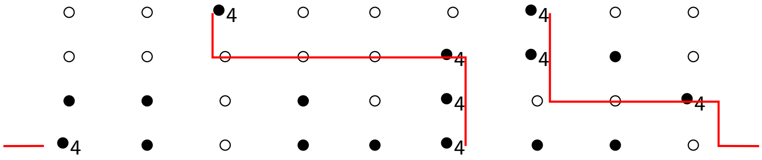

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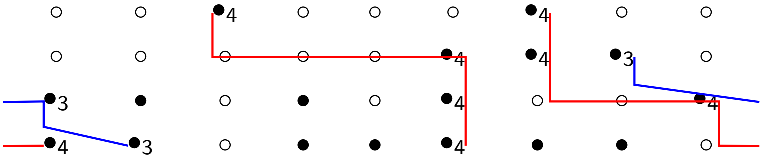

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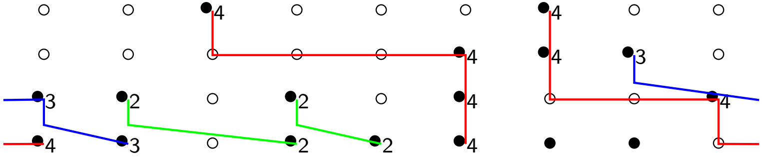

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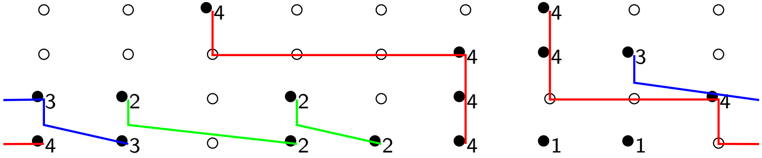

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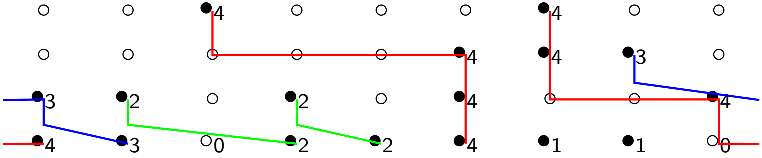


Projected configuration: 430224110

- The positive integer for a configuration is precisely the number of multiline queues that bully-path project to that configuration.

# Partition function for multispecies ASEP

Recall that $[n] \equiv [n]_t := 1 + \cdots + t^{n-1}$ and $[n]_t! := [1][2]\cdots[n]$.

**Theorem (Martin, *Elec. J. Prob.* 2020)**

*Consider the multispecies ASEP with content $\lambda$. Then the common denominator of the stationary probabilities is given by*

$$\prod_{i=1}^{s} \binom{L}{M_i} \frac{[M_i]_t!}{[n_i]_t!}.$$

The proof uses a multiline TASEP with rejection that projects to the multispecies ASEP.

# Connection to ASEP polynomials

**Theorem (Cantini–de Gier–Wheeler (*J. Phys. A, 2015*))**

*The steady state probability that the multispecies ASEP is in configuration $\mu \in S_\lambda$ is*

$$\frac{\mathrm{f}_\mu(1,\dots,1;q=1,t)}{P_\lambda(1,\dots,1;q=1,t)},$$

*where $\mathrm{f}_\mu$ is the ASEP polynomial, and $P_\lambda(1,\dots,1;q=1,t)$ is the Macdonald polynomial.*

Later on, Chen–de Gier–Wheeler (*IMRN*, 2018) found duality functions for the multispecies ASEP on a ring using ASEP polynomials at $x_i = 1$ and $q = 1$.

# Linked multiline diagrams and ASEP polynomials

**Theorem (Corteel–Mandelshtam–Williams (*Amer. J. Math.*, 2022))**

*The ASEP polynomial can be written as*

$$\mathrm{f}_\omega(\beta_1, \ldots, \beta_n; q, t) = \sum_M \mathrm{wt}(M),$$

*where the sum is over all linked multiline diagrams and* $\mathrm{wt}(M)$ *is a complicated weight function.*

Until now, no stochastic process is known for which the stationary probabilities are the ASEP polynomials.

## Single species PushTASEP

- Ring of size $L$ with $n_1 < L$ particles.
- From site $i$,

$$\cdots \underset{i}{\underline{1}}\ \underset{i+1}{\underline{1}} \cdots \underset{j-1}{\underline{1}}\ \underset{j}{\underline{0}} \cdots \longrightarrow \cdots \underset{i}{\underline{0}}\ \underset{i+1}{\underline{1}} \cdots \underset{j-1}{\underline{1}}\ \underset{j}{\underline{1}} \cdots \quad \text{with rate } \frac{1}{\beta_i},$$

- Also called the long-range exclusion process and isomorphic to the Hammersley–Aldous–Diaconis (HAD) process.

# Stationary distribution

- Recall that the elementary symmetric polynomial of degree $m$ in indeterminates $x_1, \dots, x_k$ is

$$e_m(x_1, \dots, x_k) = \sum_{1 \leq i_1 < \cdots < i_m \leq k} x_{i_1} \dots x_{i_k},$$

- Let $\eta = (\eta_1, \dots, \eta_L)$ be a configuration.

**Proposition**

*The stationary probability of a configuration $\eta$ is*

$$\frac{1}{e_{n_1}(\beta_1, \dots, \beta_L)} \prod_{\substack{i=1 \\ \eta_i = 1}}^{L} \beta_i.$$

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  1. Particle at site $i$ moves clockwise,
  2. finds the first weakest particle and displaces it,
  3. which in turn does the same.
  4. Continue this way ending at a vacancy.
- The homogeneous version of this process is the multispecies HAD process (Ferrari and Martin, *AIHP B*, 2009).

# Examples

- $\lambda = (2, 2, 1, 1, 1)$ so that $n = 8, s = 5$.

$$2\ 4\ 3\ 0\ 2\ 4\ 1\ 3 \xrightarrow{1/\beta_2} 2\ 0\ 4\ 3\ 2\ 4\ 1\ 3$$

$$2\ 4\ 3\ 0\ 2\ 4\ 1\ 3 \xrightarrow{1/\beta_4} 2\ 4\ 3\ 0\ 2\ 4\ 1\ 3$$

$$2\ 4\ 1\ 0\ 2\ 4\ 1\ 3 \xrightarrow{1/\beta_5} 2\ 4\ 1\ 1\ 0\ 4\ 2\ 3$$

# Stationary distribution

**Theorem (A.–Martin, 2024+)**

*The stationary distribution $\pi$ of the multispecies PushTASEP with content $\lambda$ is given by*

$$\pi(\eta) = \frac{\mathrm{f}_\eta(\beta_1, \dots, \beta_L; q = 1, t = 0)}{P_\lambda(\beta_1, \dots, \beta_L; 1, 0)},$$

*where* $\mathrm{f}_\eta$ *is the ASEP polynomial and*

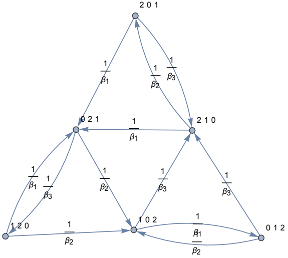
$$P_\lambda(\beta_1, \dots, \beta_L; 1, 0) = \prod_{i=1}^{s} e_{M_i}(\beta_1, \dots, \beta_L)$$

*is the* partition function.

# Connection to the multispecies TASEP

- Recall that the partition function for the multispecies TASEP is
$$\prod_{i=1}^{s} \binom{L}{M_i}.$$
- If we set $\beta_1 = \cdots = \beta_L = 1$ in the PushTASEP, we obtain not only the same partition function, but the same stationary distribution!

# Example: $\lambda = (2, 1, 0)$




- Order the configurations as $\{210, 201, 120, 102, 021, 012\}$.
- The stationary weights turn out to be

$$v = \left(\beta_1\beta_2(\beta_1+\beta_3), \beta_1^2\beta_3, \beta_1\beta_2^2, \beta_1\beta_3(\beta_2+\beta_3), \beta_2\beta_3(\beta_1+\beta_2), \beta_2\beta_2^3\right).$$

- $$Z = (\beta_1+\beta_2+\beta_3)(\beta_1\beta_2+\beta_1\beta_3+\beta_2\beta_3) = e_{(2,1)}(\beta_1, \beta_2, \beta_3).$$

# Idea of proof

- Follow the overall strategy of P. Ferrari and J. Martin (*Ann. Prob.* 2007) for the multispecies TASEP.
- We construct a multiline PushTASEP which projects to the multispecies PushTASEP.
- The projection is the same bully-path projection as for the TASEP.

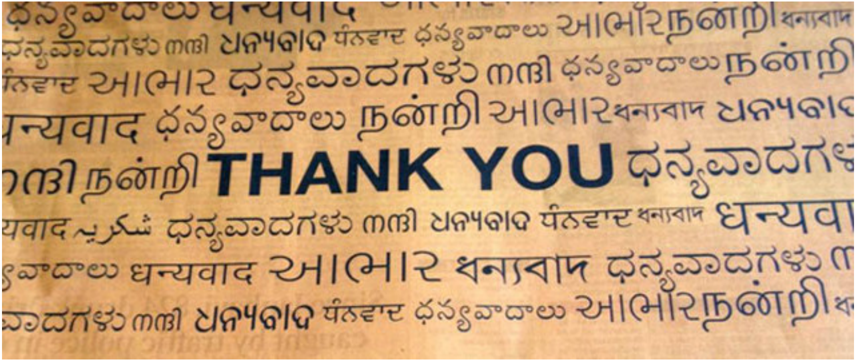
# A nonequilibrium theorem

**Theorem (A.–Martin, 2024+)**

*Run the multispecies PushTASEP on the ring with content $\lambda$, starting either in the stationary distribution, or in any starting configuration with $\eta$ in which $\eta_{k+1} \geq \eta_{k+2} \geq \dots \eta_L$.*
*Then the distribution of the path of the process observed on sites $1, 2, \dots, k$ is invariant under permutations of $\beta_{k+1}, \dots, \beta_L$.*

# Open questions

- It turns out that we can insert the $t$ parameter, see `arXiv:2403.10485`, joint with J. Martin and L. Williams.
- But inserting the $q$ parameter is difficult; $P_\lambda(x; q, t)$ does not factorise in general.
- The intuition is that $q$ should be a parameter in the transition involving sites $n$ and $1$.
- Therefore, we lose translation invariance.
- Insights from integrable models might play a role in defining such a model.


THANK YOU