An exact solution for a quasi-two-dimensional exclusion process

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Motivation

- Exact solutions of nonequilibium statistical mechanical models have proven useful in developing fundamental laws.
- For example, the asymmetric simple exclusion process (ASEP) in one-dimension has had remarkable success.
- The stationary distribution of the open ASEP was determined exactly by Derrida, Evans, Hakim and Pasquier (*J. Phys. A*, 1993) using the matrix ansatz.
- The additivity principle of Bodineau and Derrida has come out of a thorough study of the ASEP.

Motivation

- Several generalisations of the ASEP have also been solved exactly.
- For example, the steady state of the TASEP on a ring (i.e. with periodic boundary conditions) with multiple species was determined by Ferrari and Martin (*Ann. Prob.*,2007).
- The steady state of a disordered open long-range exclusion process (LREP) was solved by A. (*J. Phys. A*, 2016).
- However, all of these are one-dimensional models.
- Very few (if any) two-dimensional models have been solved exactly.
- Very few models with disorder have been solved exactly.

Disordered ASEP

- Evans (Europhys. Lett, 1996) considered an ASEP on a ring where the hopping rates are disordered.
- Ring of size L with n particles.
- The k'th particle performs transitions

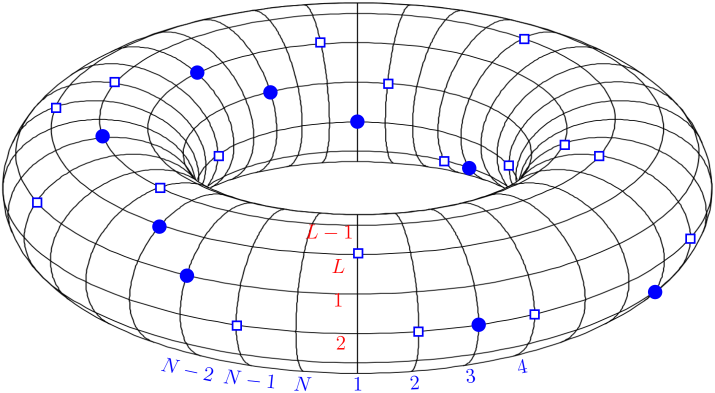
$$\bullet \, \square \to \square \, \bullet \quad \text{with rate } p_k,$$
$$\square \, \bullet \to \bullet \, \square \quad \text{with rate } q_k.$$

- Since particles cannot cross each other, we label the particles $\bullet_1, \dots, \bullet_n$.

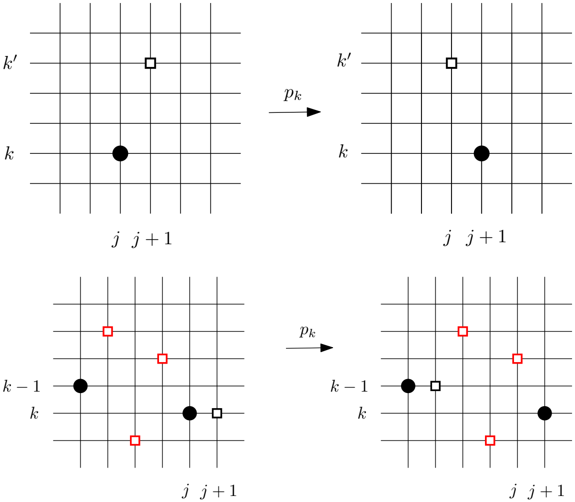
The two-dimensional exclusion process

- Discrete $L \times n$ torus with two kinds of particles and vacancies.
- Denote first class particles by $\bullet$ and second class particles by $\square$.
- Let $\hat{\Omega}_{L,n}$ consist of configurations such that:
  - Each row contains exactly one $\bullet$.
  - Each column contains exactly one particle (either $\bullet$ or $\square$).
  - The row indices of $\bullet$'s read from left to right form a cyclically increasing sequence.
- Thus, we have n $\bullet$'s and $L - n$ $\square$'s.
- $|\hat{\Omega}_{L,n}| = n\binom{L}{n}n^{L-n}$.

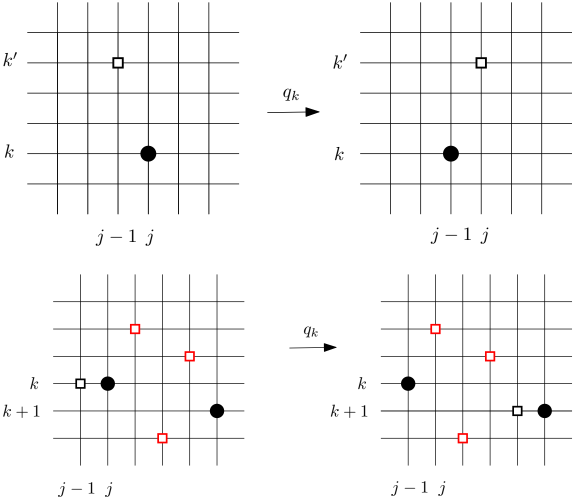
Illustration



Forward transitions: • in row k, column j



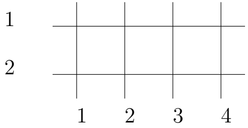
Backward transitions: • in row k, column j

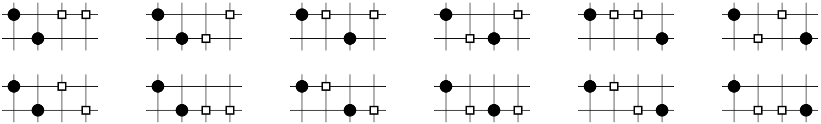


Translation invariance

- The transitions are such that the process is invariant under horizontal translations.
- Therefore, it is enough to focus on $\omega \in \hat{\Omega}_{L,n}$ with $\omega_{1,1} = \bullet$.
- We call such configurations restricted configurations.
- For restricted configurations, the column indices of $\bullet$'s in ω must be a strictly increasing sequence.

Example: $L = 4, n = 2$





Projection

- Look at the columns, replace $\bullet_k$'s by $\bullet$ and $\square_j$'s by $\square$.
- We then get a configuration for the 1D disordered ASEP.
- The forward and backward transitions match exactly.

Proposition

The two-dimensional exclusion process on $\hat{\Omega}_{L,n}$ lumps to the disordered PASEP on the ring with L sites and n particles.

Irreducibility

Lemma

Let $L \geq 1$ and $1 \leq n < L$. If all parameters $p_k, q_k > 0$, the exclusion process on $\hat{\Omega}_{L,n}$ is irreducible.

As a consequence, the steady state is unique.

Weights of configurations

- Let $\omega \in \hat{\Omega}_{L,n}$ be a restricted configuration.
- Let the locations of the 1's in ω by $((1, a_1), \ldots, (n, a_n))$, where $1 = a_1 < \cdots < a_n$.
- Let $C_k \equiv C_k(\omega)$ be the set of those positions (i, j) with $a_k < j < a_{k+1}$ such that $\omega(i, j) = \square$.
- We will assign a weight to every 0 lying in such a column.
- This weight will either be p_j or q_j if the 0 is in row j.

Weights of configurations

- Suppose $(i,j) \in C_k$.
- Two possibilities, depending on the relative order of i with respect to k:

$$\underset{i \le k}{\begin{pmatrix} p_1 \\ \vdots \\ p_{i-1} \\ \square \\ q_{i+1} \\ \vdots \\ q_k \\ p_{k+1} \\ \vdots \\ p_n \end{pmatrix}} \quad \text{or} \quad \underset{i > k}{\begin{pmatrix} q_1 \\ \vdots \\ q_k \\ p_{k+1} \\ \vdots \\ p_{i-1} \\ \square \\ q_{i+1} \\ \vdots \\ q_n \end{pmatrix}}$$

Weights of configurations

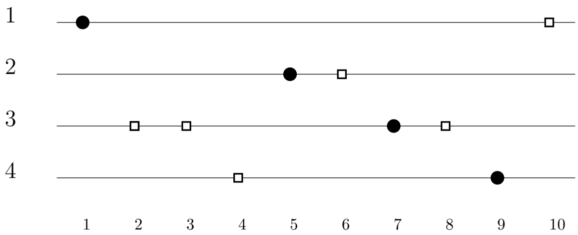
- The weight associated to this $\square$ is

$$w_{\square}(i,k) = \begin{cases} p_1 \dots p_{i-1} q_{i+1} \dots q_k p_{k+1} \dots p_n & 1 \le i \le k, \\ q_1 \dots q_k p_{k+1} \dots p_{i-1} q_{i+1} \dots q_n & k < i \le n. \end{cases}$$

- The weight $\mathrm{wt}(\omega)$ of $\omega \in \hat{\Omega}_{L,n}$ is

$$\mathrm{wt}(\omega) = \prod_{k=1}^{n} \prod_{(i,j)\in C_k} w_{\square}(i,k).$$

Example



- The weight of the configuration in the above figure is

$$\underbrace{(q_4 q_1 p_2)^2 (q_1 p_2 p_3)}_{C_1} \underbrace{(p_3 p_4 p_1)}_{C_2} \underbrace{(p_4 p_1 p_2)}_{C_3} \underbrace{(q_2 q_3 q_4)}_{C_4}$$

$$= p_1^2 p_2^4 p_3^2 p_4^2 q_1^3 q_2 q_3 q_4^3.$$

Steady state

Let the steady state probabilities in $\hat{\Omega}_{L,n}$ be denoted by $\hat{\pi}$.

Theorem (A & Nadeau, 2022)

- *Suppose* $p_k, q_k > 0$ *for* $1 \leq k \leq n$.
- *Then the stationary probability of the configuration* ω *for the exclusion process on* $\hat{\Omega}_{L,n}$ *given by*

$$\hat{\pi}(\omega) = \frac{\mathrm{wt}(\omega)}{L\, Z_{L,n}}.$$

- *Here* $Z_{L,n}$ *is the restricted (nonequilibrium) partition function,*

$$Z_{L,n} = \sum_{\substack{\omega \in \hat{\Omega}_{L,n} \\ \omega_{1,1}=1}} \mathrm{wt}(\omega).$$

Idea of proof: Verify the master equation.

Restricted partition function

Set

$$W_{\square}(k) = \sum_{j=1}^{n} w_{\square}(j,k).$$

Corollary (A & Nadeau, 2022)

The restricted partition function $Z_{L,n}$ is given by:

$$\sum_{L=n}^{\infty} Z_{L,n} x^{L-n} = \prod_{k=1}^{n} \frac{1}{1 - W_{\square}(k)x}.$$

Current of •'s

- Since particles of type • only travel horizontally, we can only talk about horizontal currents for these.
- Let $J_\bullet$ denote the current for the particle of type • on the i'th row in the steady state.
- By particle conservation, this is independent of the choice of edge.
- Since •'s in successive rows cannot overtake each other, $J_\bullet$ is independent of i.
- Formally, the current between sites (i,j) and $(i,j+1)$ is given by

$$J_\bullet = p_i \left\langle \tau_{i,j} \sum_{k=1}^{n} \eta_{k,j+1} \right\rangle - q_i \left\langle \tau_{i,j+1} \sum_{k=1}^{n} \eta_{k,j} \right\rangle .$$

Current of •'s

Theorem (Evans 1995, A & Nadeau 2022)

For $1 \le i \le n$, *we have*

$$J_\bullet = (p_1 \dots p_n - q_1 \dots q_n)\frac{Z_{L-1,n}}{LZ_{L,n}}.$$

Evans gave the same formula for the 1D ASEP (in slightly different language).

Horizontal current of □'s

- The □'s travel both horizontally and vertically.
- So we can talk about two kinds of currents.
- In the horizontal direction, their motion is local.
- For any fixed horizontal edge e, we let $J_\square^e$ be the current of □'s along e.
- For $e = ((i,j),(i,j+1))$,

$$J_\square^e = \left\langle \eta_{i,j} \sum_{\substack{k=1 \\ k \neq i}}^{n} q_k \tau_{k,j+1} \right\rangle - \left\langle \eta_{i,j+1} \sum_{\substack{k=1 \\ k \neq i}}^{n} p_k \tau_{k,j} \right\rangle .$$

Horizontal current of □'s

Theorem (A & Nadeau, 2022)

Across any horizontal edge e, we have

$$J^e_{\square} = 0.$$

Vertical current of □'s

- In the vertical direction, the motion of □'s is always nonlocal.
- So, we cannot talk about the current across any one vertical edge.
- We will instead define the upward current $J_{\square}^{i+}$ between rows i and $i-1$,

$$J_{\square}^{i+} = \sum_{j=1}^{L} p_i \left\langle \eta_{i,j} \tau_{i,j-1} \right\rangle .$$

- Similarly, the downward current $J_{\square}^{i-}$ between rows i and $i+1$,

$$J_{\square}^{i-} = \sum_{j=1}^{L} q_i \left\langle \eta_{i,j} \tau_{i,j+1} \right\rangle .$$

- The net vertical current between rows i and $i+1$ is $J_{\square}^{i} = J_{\square}^{i+} - J_{\square}^{(i+1)-}$.

Vertical current of □'s

Theorem (A & Nadeau, 2022)

We have

$$J_{\square}^{i+} = p_1 \dots p_n \frac{Z_{L-1,n}}{L Z_{L,N}}, \quad J_{\square}^{i-} = q_1 \dots q_n \frac{Z_{L-1,n}}{L Z_{L,N}},$$

Corollary

The vertical current of □'s between rows i and $i+1$ is the same as the horizontal current of 1's, i.e.

$$J_{\square}^{i} = J_{\bullet}.$$

Scott Russell linkage phenomenon

- In our 2D ASEP, horizontal motion of •'s gives rise to 'purely' vertical motion of □'s.
- We call this the microscopic Scott Russell (linkage) phenomenon.
- This is a manifestly two-dimensional phenomenon.
- A Scott Russell linkage is a mechanism for transferring linear motion in one direction to a perpendicular direction.
- It is named after John Scott Russell, a Scottish civil engineer.
- It is a standard piece of equipment in most cars today.

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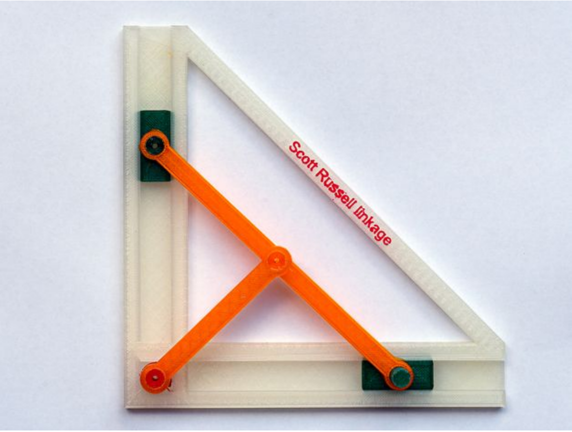
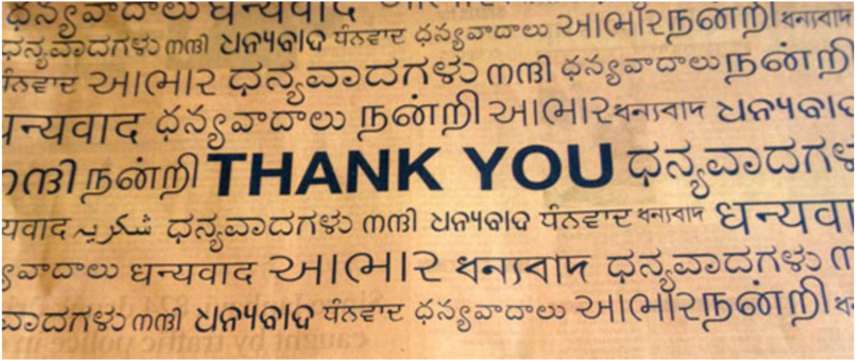
- It is named after John Scott Russell, a Scottish civil engineer.
- It is a standard piece of equipment in most cars today.
- His other claim to fame is the discovery of solitons in 1834.

Scott Russell linkage




THANK YOU