

Classical Gravitational Wave Tails from Quantum Soft Theorem

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Consider a violent explosion in space

D



A stationary system breaks apart into fragments.

This process emits gravitational wave.

Detector D placed far away detects this wave as ripples in space-time.

Examples: Supernova

A more general situation: collision

A set of objects come together, interact, and fly apart, possibly exchanging mass, energy and momentum during this process.

This will also produce gravitational wave.

Example: Bullet cluster

A supercluster of galaxies passing through another supercluster of galaxies, each weighing about 10^{14} times the mass of the sun, at a relative speed of about 1% of the speed of light.

In general, computing gravitational waves produced during such processes is complicated.

1. When the objects are close, they may undergo complicated, non-gravitational interactions, as in the case of explosion of supernova.

2. Gravity is described by non-linear partial differential equations

– even if the interactions were purely gravitational, e.g. in the case of black hole merger, the analysis is complicated.

Surprisingly, some results in quantum theory of gravity, known as **soft graviton theorem**, can be used to get analytical results on some aspects of classical gravitational wave.

This will be the focus of this talk.

Collaborators

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A short review

A.S., arXiv:2408.08851

Introduction

Gravitational radiation

In general theory of relativity, a gravitational field is a symmetric two index field $h_{\mu\nu}$ that measures distortion of space-time

– sourced by mass / energy

Our convention:

$$h_{\mu\nu} \equiv (\mathbf{g}_{\mu\nu} - \eta_{\mu\nu})/2, \quad 0 \leq \mu, \nu \leq 3$$

Gravitational waves involve time and space-varying $h_{\mu\nu}$

Key distinguishing feature of gravitational radiation:

At a distance R from the source, the fields fall off as $1/R$ and not as $1/R^2$ or faster.

$h_{\mu\nu}$ can be measured by a gravitational wave detector like LIGO

Goal of a theorist: For a given process, compute $h_{\mu\nu}$ at a far away detector.

Conventions

c will denote the speed of light.

G will denote Newton's gravitational constant

Space-time coordinates: x^μ for $0 \leq \mu \leq 3$

$x^0 = ct$, t is the time coordinate,

$\vec{x} = (x^1, x^2, x^3)$ are space coordinates

4-momentum p^μ : p^0 is (energy / c), (p^1, p^2, p^3) are momenta

$\eta_{\mu\nu}$ and $\eta^{\mu\nu}$: 4×4 diagonal matrix diag (-1, 1, 1, 1)

$$\mathbf{a} \cdot \mathbf{b} \equiv \sum_{\mu, \nu=0}^3 \eta_{\mu\nu} \mathbf{a}^\mu \mathbf{b}^\nu = -\mathbf{a}^0 \mathbf{b}^0 + \mathbf{a}^1 \mathbf{b}^1 + \mathbf{a}^2 \mathbf{b}^2 + \mathbf{a}^3 \mathbf{b}^3$$

$$\mathbf{p}^2 \equiv \mathbf{p} \cdot \mathbf{p} = -(\mathbf{p}^0)^2 + \vec{\mathbf{p}}^2 = -\mathbf{E}^2/c^2 + \vec{\mathbf{p}}^2 = -\mathbf{m}^2 \mathbf{c}^2$$

Summation convention: An index repeated twice is summed over from 0 to 3, e.g.

$$\eta_{\mu\nu} \mathbf{p}^\mu = \sum_{\mu=0}^3 \eta_{\mu\nu} \mathbf{p}^\mu$$

Indices are raised and lowered by η

$$\mathbf{b}_\mu \equiv \eta_{\mu\rho} \mathbf{b}^\rho, \quad \mathbf{a}^\mu \equiv \eta^{\mu\rho} \mathbf{a}_\rho$$

$$\Rightarrow \quad \mathbf{b}_0 = -\mathbf{b}^0, \quad \mathbf{b}_1 = \mathbf{b}^1, \quad \mathbf{b}_2 = \mathbf{b}^2, \quad \mathbf{b}_3 = \mathbf{b}^3$$

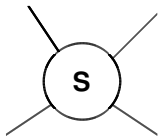
Define $\mathbf{e}_{\mu\nu}$ via:

$$\mathbf{e}_{\mu\nu} = \mathbf{h}_{\mu\nu} - \frac{1}{2} \eta_{\mu\nu} \eta^{\rho\sigma} \mathbf{h}_{\rho\sigma} \quad \Leftrightarrow \quad \mathbf{h}_{\mu\nu} = \mathbf{e}_{\mu\nu} - \frac{1}{2} \eta_{\mu\nu} \eta^{\rho\sigma} \mathbf{e}_{\rho\sigma}$$

Summary of results

Consider a scattering in space

A set of objects of four momenta p'_1, \dots, p'_m come together, interact, and disperse as a set of objects with four momenta p_1, \dots, p_n .



We shall choose the origin of space-time to be inside the region S where the scattering event takes place

The peak of the $h_{\mu\nu}$ signal reaches a detector D placed at a far way point \vec{x} at some time t_0 :

$$t_0 = R/c + \text{correction}, \quad R \equiv |\vec{x}|$$

The correction is due to the gravitational drag on the gravitational wave.

Define retarded time:

$$\mathbf{u} \equiv \mathbf{t} - \mathbf{t}_0$$

Our focus will be on the late and early time tail of the wave – the value of $h_{\mu\nu}$ at \mathbf{D} at large positive \mathbf{u} and large negative \mathbf{u} .

Terms proportional to R^{-1} have the form,

$$\mathbf{e}_{\mu\nu} = \frac{1}{\mathbf{u}} \mathbf{C}_{\mu\nu} + \mathbf{G}_{\mu\nu} \mathbf{u}^{-2} \ln |\mathbf{u}| + \mathcal{O}(\mathbf{u}^{-2}), \quad \text{for large negative } \mathbf{u}$$

$$\mathbf{e}_{\mu\nu} = \mathbf{A}_{\mu\nu} + \frac{1}{\mathbf{u}} \mathbf{B}_{\mu\nu} + \mathbf{F}_{\mu\nu} \mathbf{u}^{-2} \ln \mathbf{u} + \mathcal{O}(\mathbf{u}^{-2}), \quad \text{for large positive } \mathbf{u}$$

$\mathbf{A}_{\mu\nu}, \mathbf{B}_{\mu\nu}, \mathbf{C}_{\mu\nu}, \mathbf{F}_{\mu\nu}, \mathbf{G}_{\mu\nu}$ are given solely by the 4-momenta of the ingoing and outgoing objects without requiring any knowledge of the details of the scattering process.

$$\mathbf{A}^{\mu\nu} = \frac{2G}{Rc^3} \left[- \sum_{a=1}^n \mathbf{p}_a^\mu \mathbf{p}_a^\nu \frac{1}{\mathbf{n} \cdot \mathbf{p}_a} + \sum_{a=1}^m \mathbf{p}'_a{}^\mu \mathbf{p}'_a{}^\nu \frac{1}{\mathbf{n} \cdot \mathbf{p}'_a} \right], \quad \mathbf{R} \equiv |\vec{\mathbf{x}}|, \quad \mathbf{n} \equiv (1, \vec{\mathbf{x}}/R)$$

$$\mathbf{B}^{\mu\nu} = - \frac{4G^2}{Rc^7} \left[\sum_{a=1}^n \sum_{\substack{b=1 \\ b \neq a}}^n \frac{\mathbf{p}_a \cdot \mathbf{p}_b}{\{(\mathbf{p}_a \cdot \mathbf{p}_b)^2 - m_a^2 m_b^2 c^4\}^{3/2}} \left\{ \frac{3}{2} m_a^2 m_b^2 c^4 - (\mathbf{p}_a \cdot \mathbf{p}_b)^2 \right\} \right. \\ \left. \times \frac{\mathbf{p}_a^\mu}{\mathbf{n} \cdot \mathbf{p}_a} (\mathbf{n} \cdot \mathbf{p}_b \mathbf{p}_a^\nu - \mathbf{n} \cdot \mathbf{p}_a \mathbf{p}_b^\nu) \right. \\ \left. - \left\{ \sum_{b=1}^n \mathbf{p}_b \cdot \mathbf{n} \sum_{a=1}^n \frac{1}{\mathbf{p}_a \cdot \mathbf{n}} \mathbf{p}_a^\mu \mathbf{p}_a^\nu - \sum_{b=1}^m \mathbf{p}'_b \cdot \mathbf{n} \sum_{a=1}^m \frac{1}{\mathbf{p}'_a \cdot \mathbf{n}} \mathbf{p}'_a{}^\mu \mathbf{p}'_a{}^\nu \right\} \right]$$

$$\mathbf{C}^{\mu\nu} = \frac{4G^2}{Rc^7} \left[\sum_{a=1}^m \sum_{\substack{b=1 \\ b \neq a}}^m \frac{\mathbf{p}'_a \cdot \mathbf{p}'_b}{\{(\mathbf{p}'_a \cdot \mathbf{p}'_b)^2 - m_a'^2 m_b'^2 c^4\}^{3/2}} \left\{ \frac{3}{2} m_a'^2 m_b'^2 c^4 - (\mathbf{p}'_a \cdot \mathbf{p}'_b)^2 \right\} \right. \\ \left. \times \frac{\mathbf{p}'_a{}^\mu}{\mathbf{n} \cdot \mathbf{p}'_a} (\mathbf{n} \cdot \mathbf{p}'_b \mathbf{p}'_a{}^\nu - \mathbf{n} \cdot \mathbf{p}'_a \mathbf{p}'_b{}^\nu) \right].$$

$$\begin{aligned}
F^{\mu\nu} &= 2 \frac{G^3}{R c^{11}} \left[4 \left\{ \sum_{b=1}^n p_b \cdot n \sum_{d=1}^n p_d \cdot n \sum_{a=1}^n \frac{p_a^\mu p_a^\nu}{p_a \cdot n} - \sum_{b=1}^m p'_b \cdot n \sum_{d=1}^n p'_d \cdot n \sum_{a=1}^m \frac{p'_a{}^\mu p'_a{}^\nu}{p'_a \cdot n} \right\} \right. \\
&+ 4 \sum_{d=1}^n p_d \cdot n \sum_{a=1}^n \sum_{\substack{b=1 \\ b \neq a}}^n \frac{1}{p_a \cdot n} \frac{p_a \cdot p_b}{\{(p_a \cdot p_b)^2 - m_a^2 m_b^2 c^4\}^{3/2}} \{2(p_a \cdot p_b)^2 - 3m_a^2 m_b^2 c^4\} \{n \cdot p_b p_a^\mu p_a^\nu - n \cdot p_a p_a^\mu p_b^\nu\} \\
&+ 2 \sum_{d=1}^n p'_d \cdot n \sum_{a=1}^m \sum_{\substack{b=1 \\ b \neq a}}^m \frac{1}{p'_a \cdot n} \frac{p'_a \cdot p'_b}{\{(p'_a \cdot p'_b)^2 - m_a'^2 m_b'^2 c^4\}^{3/2}} \{2(p'_a \cdot p'_b)^2 - 3m_a'^2 m_b'^2 c^4\} \{n \cdot p'_b p'_a{}^\mu p'_a{}^\nu - n \cdot p'_a p'_a{}^\mu p'_b{}^\nu\} \\
&+ \sum_{a=1}^n \sum_{\substack{b=1 \\ b \neq a}}^n \sum_{\substack{d=1 \\ d \neq a}}^n \frac{1}{p_a \cdot n} \frac{p_a \cdot p_b}{\{(p_a \cdot p_b)^2 - m_a^2 m_b^2 c^4\}^{3/2}} \{2(p_a \cdot p_b)^2 - 3m_a^2 m_b^2 c^4\} \frac{p_a \cdot p_d}{\{(p_a \cdot p_d)^2 - m_a^2 m_d^2 c^4\}^{3/2}} \\
&\quad \left. \{2(p_a \cdot p_d)^2 - 3m_a^2 m_d^2 c^4\} \{n \cdot p_b p_a^\mu - n \cdot p_a p_b^\mu\} \{n \cdot p_d p_a^\nu - n \cdot p_a p_d^\nu\} \right],
\end{aligned}$$

$$\begin{aligned}
G^{\mu\nu} &= -2 \frac{G^3}{R c^{11}} \left[2 \sum_{d=1}^m p'_d \cdot n \sum_{a=1}^m \sum_{\substack{b=1 \\ b \neq a}}^m \frac{1}{p'_a \cdot n} \frac{p'_a \cdot p'_b}{\{(p'_a \cdot p'_b)^2 - m_a'^2 m_b'^2 c^4\}^{3/2}} \{2(p'_a \cdot p'_b)^2 - 3m_a'^2 m_b'^2 c^4\} \right. \\
&\quad \left. \{n \cdot p'_b p'_a{}^\mu p'_a{}^\nu - n \cdot p'_a p'_a{}^\mu p'_b{}^\nu\} \right. \\
&- \sum_{a=1}^m \sum_{\substack{b=1 \\ b \neq a}}^m \sum_{\substack{d=1 \\ d \neq a}}^m \frac{1}{p'_a \cdot n} \frac{p'_a \cdot p'_b}{\{(p'_a \cdot p'_b)^2 - m_a'^2 m_b'^2 c^4\}^{3/2}} \{2(p'_a \cdot p'_b)^2 - 3m_a'^2 m_b'^2 c^4\} \frac{p'_a \cdot p'_d}{\{(p'_a \cdot p'_d)^2 - m_a'^2 m_d'^2 c^4\}^{3/2}} \\
&\quad \left. \{2(p'_a \cdot p'_d)^2 - 3m_a'^2 m_d'^2 c^4\} \{n \cdot p'_b p'_a{}^\mu - n \cdot p'_a p'_b{}^\mu\} \{n \cdot p'_d p'_a{}^\nu - n \cdot p'_a p'_d{}^\nu\} \right].
\end{aligned}$$

$$\mathbf{e}_{\mu\nu} = \mathbf{A}_{\mu\nu} + \frac{1}{u} \mathbf{B}_{\mu\nu} + \mathbf{F}_{\mu\nu} u^{-2} \ln u + \mathcal{O}(u^{-2}), \quad \text{for large positive } u$$

$$\mathbf{e}_{\mu\nu} = \frac{1}{u} \mathbf{C}_{\mu\nu} + \mathbf{G}_{\mu\nu} u^{-2} \ln |u| + \mathcal{O}(u^{-2}), \quad \text{for large negative } u$$

$\mathbf{A}_{\mu\nu}$: memory term

– a permanent change in the state of the detector after the passage of gravitational wave

Zeldovich, Polnarev; Braginsky, Grishchuk; Braginsky, Thorne; Strominger; . . .

$\mathbf{B}_{\mu\nu}, \mathbf{C}_{\mu\nu}, \mathbf{F}_{\mu\nu}, \mathbf{G}_{\mu\nu}$: tail terms

Laddha, A.S.; Sahoo, A.S.; Saha, Sahoo, A.S.; Sahoo

1. $A_{\mu\nu}, B_{\mu\nu}, C_{\mu\nu}, F_{\mu\nu}, G_{\mu\nu}$ can be expressed in terms of the momenta of incoming and outgoing objects without knowing what forces operated and how the objects moved during the scattering

– consequence of ‘soft graviton theorem’

In contrast, if we were to compute $h_{\mu\nu}$ at finite time, it will depend on the details of the scattering process and will involve very complicated calculations.

2. In the expressions for $A^{\mu\nu}$, $B^{\mu\nu}$ and $F^{\mu\nu}$, the sum over final state particles a,b includes integration over outgoing flux of gravitational radiation, regarded as a flux of massless particles.

For $A^{\mu\nu}$ this gives the ‘non-linear memory’ term

Christodoulou; Thorne; Blanche, Damour; Bieri, Garfinkle; . . .

Due to some miraculous cancellation, in $B^{\mu\nu}$ and $F^{\mu\nu}$ the contribution from massless final states can be expressed in terms of massive state momenta.

In $B_{\mu\nu}$, drop massless particles / radiation contribution in the sum over final states, and add

$$-\frac{4G^2}{Rc^7} [P_F^\mu P_F^\nu - P_I^\mu P_I^\nu]$$

P_I : total incoming momentum

P_F : total outgoing momentum carried by massive particles

In $F_{\mu\nu}$, drop massless particles / radiation contribution in the sum over final states, and add

$$-\frac{8G^3}{Rc^{11}} [n \cdot P_F P_F^\mu P_F^\nu - n \cdot P_I P_I^\mu P_I^\nu]$$

Note: These are not new formulæ but follow from manipulating the results shown earlier

3. If the incoming or outgoing objects carry charge then there are additional terms in the formula (known)

4. Explosion can be regarded as a special case of scattering when the initial state has just one object.

In this case $C_{\mu\nu}$ and $G_{\mu\nu}$ vanish and $e_{\mu\nu}$ takes the form:

$$e_{\mu\nu} = A_{\mu\nu} + \frac{1}{u} B_{\mu\nu} + F_{\mu\nu} u^{-2} \ln u + \mathcal{O}(u^{-2}), \quad \text{for large positive } u$$

$$e_{\mu\nu} = 0, \quad \text{for large negative } u$$

5. The results are statements in classical theory of gravity.

However they are easier to understand as classical limits of some results in quantum theory of gravity

– known as quantum soft graviton theorem.

In fact the tail terms were first predicted from quantum soft graviton theorem and then a fully classical derivation was found.

From now onwards, we set $c=1$

Outline of the derivation from soft graviton theorem

In quantum theory of fields the scattering amplitude A gives the probability amplitude for transition of one state to another.

$|A|^2$: transition probability from one state to another.

Suppose we know the scattering amplitude A for transition from some incoming state P to some outgoing state Q .

Both P and Q contain sets of particles, in general different

Soft graviton theorem gives the transition amplitude for

$$P \Rightarrow Q + \text{some low energy gravitons}$$

in terms of the original transition amplitude A .

Weinberg; . . .

Gravitons are massless particles representing quanta of gravitational wave

– just like photons are quanta of electromagnetic wave.

1. Take the states **P** and **Q** to consist of very massive objects.
2. Compute the transition probability for

P \Rightarrow **Q** + **M** low energy gravitons, each of energy $\hbar\omega$ and momentum $\hbar\omega\hat{n}$

– takes the form

$$R S^M / M! \longrightarrow \text{bose statistics}$$

- **R** depends on the details of the scattering process
- **S** is the known ‘soft factor’ that depends only on the properties of the incoming and outgoing objects.

3. This has a sharp maximum at **M=S** for large **S**

\rightarrow **S** is the expected ‘classical number’ of low energy gravitons.

\Rightarrow energy carried by these gravitons is **S** $\hbar\omega$

4. Classical flux of energy carried by gravitational wave of frequency ω is

$$S\hbar\omega$$

The energy flux can be translated into the time Fourier transform of the gravitational wave-form $h_{\mu\nu}$, expressed as function of ω .

Inverse Fourier transform gives the late and early time profile, leading to the quoted results

Some shortcomings in $D=4$:

1. The usual soft factor S is infrared divergent

– we need to regulate the infrared divergence by physical principles

Laddha, A.S.; Sahoo, A.S.

2. Since this analysis proceeds via computation of the energy flux in different frequencies, it is insensitive to the frequency dependent ‘phase’ of the wave-form

– needs to be determined by separate computation

Nevertheless these difficulties can be overcome, leading to the results quoted above.

Direct classical analysis is more complicated, but does not have these shortcomings

Direct classical derivation

Goal: Find $h_{\mu\nu}$, or equivalently $e_{\mu\nu}$, generated during the collision process.

In de Donder gauge, Einstein's equation determining $e_{\mu\nu}$ can be written as:

$$\square e^{\mu\nu}(\mathbf{x}) = -8\pi \mathbf{G} \mathbf{T}^{\mu\nu}(\mathbf{x})$$

$T^{\mu\nu}$ on the right hand side includes matter energy momentum tensor and all the non-linear terms in the Einstein equation

– has explicit dependence on $e^{\rho\sigma}$, and also the trajectories of the various objects involved in the scattering

The trajectories of the objects in turn depend on $e^{\rho\sigma}$

– set of complicated non-linear partial differential equations

– may not even be known if the interactions are unknown

Define

$$\tilde{e}^{\mu\nu}(\omega, \vec{x}) = e^{-i\omega|\vec{x}|} \int dt e^{i\omega t} e^{\mu\nu}(t, \vec{x})$$

$e^{\mu\nu} \rightarrow$ constant for large u corresponds to $\tilde{e}^{\mu\nu} \sim \omega^{-1}$ for small ω

$e^{\mu\nu} \sim 1/u$ for large u corresponds to $\tilde{e}^{\mu\nu} \sim \ln \omega$ for small ω

$e^{\mu\nu} \sim \ln u/u^2$ corresponds to $\tilde{e}^{\mu\nu} \sim \omega(\ln \omega)^2$

These are non-analytic at $\omega = 0$

□ $e^{\mu\nu} = -8\pi G T^{\mu\nu}$ can be formally 'solved' as:

$$e^{\mu\nu}(\mathbf{x}) = -8\pi G \int d^4y G_r(\mathbf{x}, \mathbf{y}) T^{\mu\nu}(\mathbf{y})$$

$G_r(\mathbf{x}, \mathbf{y})$: retarded Green's function in flat space-time

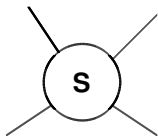
Using the explicit expression for G_r , one finds that for large $|\vec{x}|$

$$\tilde{e}^{\mu\nu}(\omega, \vec{x}) = \frac{2G}{|\vec{x}|} \int d^4y e^{-ik \cdot y} T^{\mu\nu}(\mathbf{y}), \quad \mathbf{k} = \omega(1, \vec{x}/|\vec{x}|)$$

$$\tilde{\mathbf{e}}^{\mu\nu}(\omega, \vec{\mathbf{x}}) = \frac{2G}{|\vec{\mathbf{x}}|} \int d^4\mathbf{y} e^{-i\mathbf{k}\cdot\mathbf{y}} \mathbf{T}^{\mu\nu}(\mathbf{y}), \quad \mathbf{k} = \omega(\mathbf{1}, \vec{\mathbf{x}}/|\vec{\mathbf{x}}|)$$

We divide the integration region over \mathbf{y} into two parts:

1. **Scattering region: A region S of large size around $\mathbf{y}=0$.**

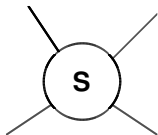


2. **Asymptotic region: Complement of S**

Non-analytic terms in $\tilde{\mathbf{e}}^{\mu\nu}$ cannot come from the integral of a finite quantity over a finite region

\Rightarrow **the behaviour of $\mathbf{e}^{\mu\nu}$ for large $|\mathbf{u}|$ is insensitive to the contribution from the finite region S in $\int d^4\mathbf{y}$.**

$$\tilde{\mathbf{e}}^{\mu\nu}(\omega, \vec{\mathbf{x}}) = \frac{2\mathbf{G}}{|\vec{\mathbf{x}}|} \int \mathbf{d}^4\mathbf{y} \mathbf{e}^{-i\mathbf{k}\cdot\mathbf{y}} \mathbf{T}^{\mu\nu}(\mathbf{y}), \quad \mathbf{k} = \omega(\mathbf{1}, \vec{\mathbf{x}}/|\vec{\mathbf{x}}|)$$



One can restrict the \mathbf{y} integral in the region outside \mathbf{S} .

In this region, only long range forces are important.

Furthermore, $\mathbf{e}^{\mu\nu}$ is small and as a result $\mathbf{T}^{\mu\nu}$ simplifies.

The equations can be solved iteratively and give the result quoted earlier.

$$\mathbf{T}^{\mathbf{X}\mu\nu}(\mathbf{x}) \equiv \sum_{\mathbf{a}=1}^n m_{\mathbf{a}} \int_0^{\infty} d\tau \delta^{(4)}(\mathbf{x} - \mathbf{X}_{\mathbf{a}}(\tau)) \frac{d\mathbf{X}_{\mathbf{a}}^{\mu}}{d\tau} \frac{d\mathbf{X}_{\mathbf{a}}^{\nu}}{d\tau} \rightarrow \text{outgoing}$$

$$+ \sum_{\mathbf{a}=1}^m m'_{\mathbf{a}} \int_{-\infty}^0 d\tau \delta^{(4)}(\mathbf{x} - \mathbf{X}'_{\mathbf{a}}(\tau)) \frac{d\mathbf{X}'_{\mathbf{a}}{}^{\mu}}{d\tau} \frac{d\mathbf{X}'_{\mathbf{a}}{}^{\nu}}{d\tau} \leftarrow \text{incoming},$$

$$\mathbf{T}^{\mu\nu}(\mathbf{x}) = \mathbf{T}^{\mathbf{X}\mu\nu}(\mathbf{x}) + \mathbf{T}^{\text{h}\mu\nu}(\mathbf{x}),$$

$$\square \mathbf{e}^{\mu\nu} = -8\pi \mathbf{G} \mathbf{T}^{\mu\nu},$$

$$\frac{d^2 \mathbf{X}_{\mathbf{a}}^{\mu}}{d\tau^2} = -\Gamma_{\nu\rho}^{\mu}(\mathbf{X}(\tau)) \frac{d\mathbf{X}_{\mathbf{a}}^{\nu}}{d\tau} \frac{d\mathbf{X}_{\mathbf{a}}^{\rho}}{d\tau}, \quad \frac{d^2 \mathbf{X}'_{\mathbf{a}}{}^{\mu}}{d\tau^2} = -\Gamma_{\nu\rho}^{\mu}(\mathbf{X}'(\tau)) \frac{d\mathbf{X}'_{\mathbf{a}}{}^{\nu}}{d\tau} \frac{d\mathbf{X}'_{\mathbf{a}}{}^{\rho}}{d\tau},$$

Boundary conditions:

$$\lim_{\tau \rightarrow \infty} \frac{d\mathbf{X}_{\mathbf{a}}^{\mu}}{d\tau} = \mathbf{v}_{\mathbf{a}}^{\mu} = \frac{1}{m_{\mathbf{a}}} \mathbf{p}_{\mathbf{a}}^{\mu},$$

$$\lim_{\tau \rightarrow -\infty} \frac{d\mathbf{X}'_{\mathbf{a}}{}^{\mu}}{d\tau} = \mathbf{v}'_{\mathbf{a}}{}^{\mu} = \frac{1}{m'_{\mathbf{a}}} \mathbf{p}'_{\mathbf{a}}{}^{\mu}.$$

Comments:

The classical analysis is straightforward and does not involve the complications of infrared divergence and phase ambiguity that the quantum soft theorems suffer from.

However, it is the quantum theory that teaches us the right question for which we have simple answer

In the usual classical set up, we specify the initial configuration and then evolve it to find the final state and the radiation emitted during the process

– a very complicated problem with no universal answer

Quantum theory teaches us to specify both the initial and final state of matter and then ask for radiation emitted during the process

Possible generalizations

There is reason to believe that the coefficients of $u^{-n-1}(\ln |u|)^n$ terms may also have universal properties, i.e. they depend only on the momenta of the particles

Sahoo; Alessio, Di Vecchia, Heissenberg

Can we compute them?

Soft theorem is not useful any more for $n > 1$ but direct classical analysis may be possible

Debanjan Karan, Babli Khatun, Biswajit Sahoo, A.S., in progress

Ultimate goal: Use these results in reverse

By measuring the gravitational wave-form in a detector, we can measure the coefficient of $u^{-n-1}(\ln u)^n$ terms.

Can we use this data to reconstruct the initial and final states of the event that led to the production of the gravitational wave? 34