Lectures on Symplectic Singularities

Monopole formula, Magnetic Quiver, Phase diagram

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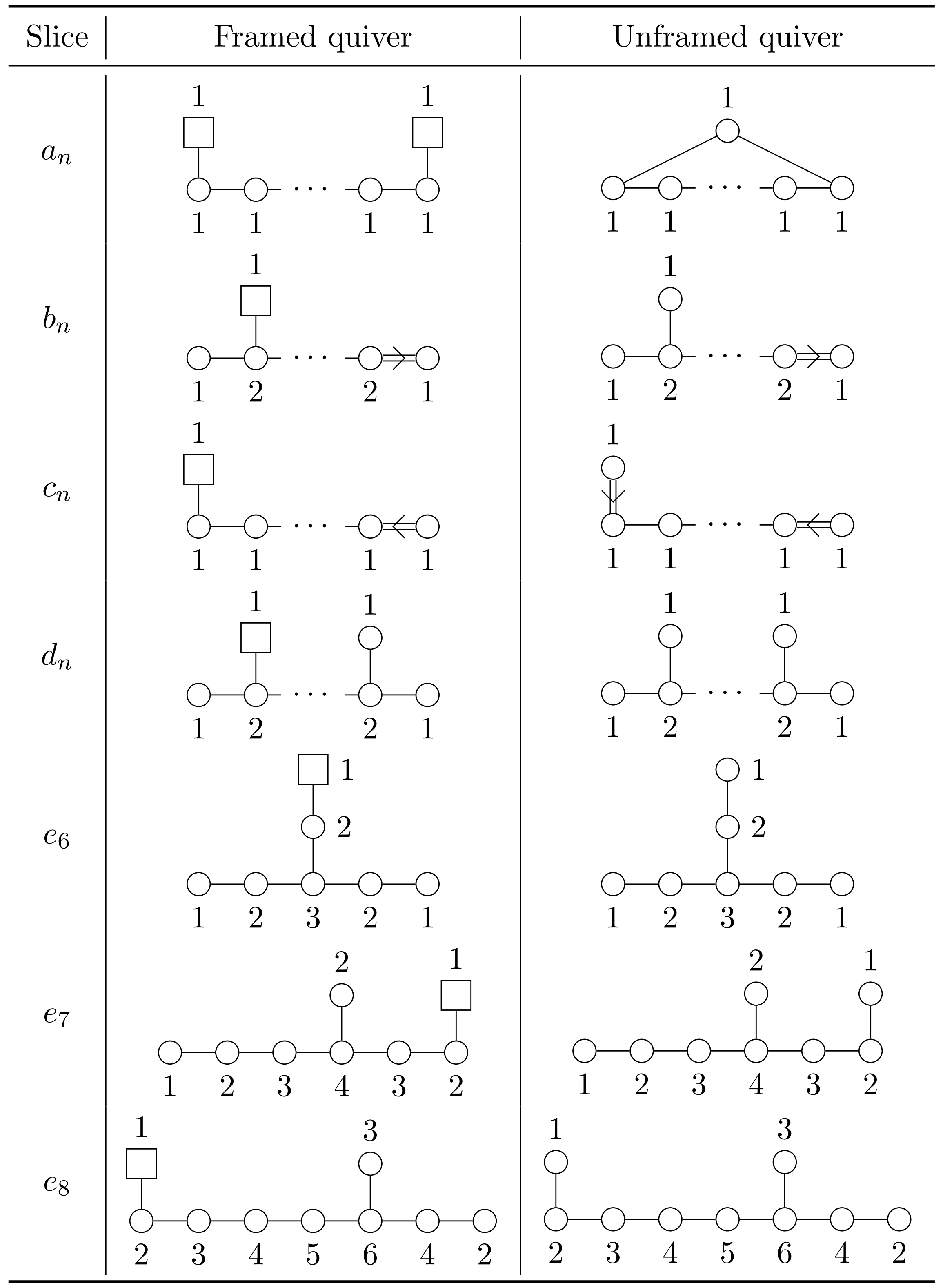
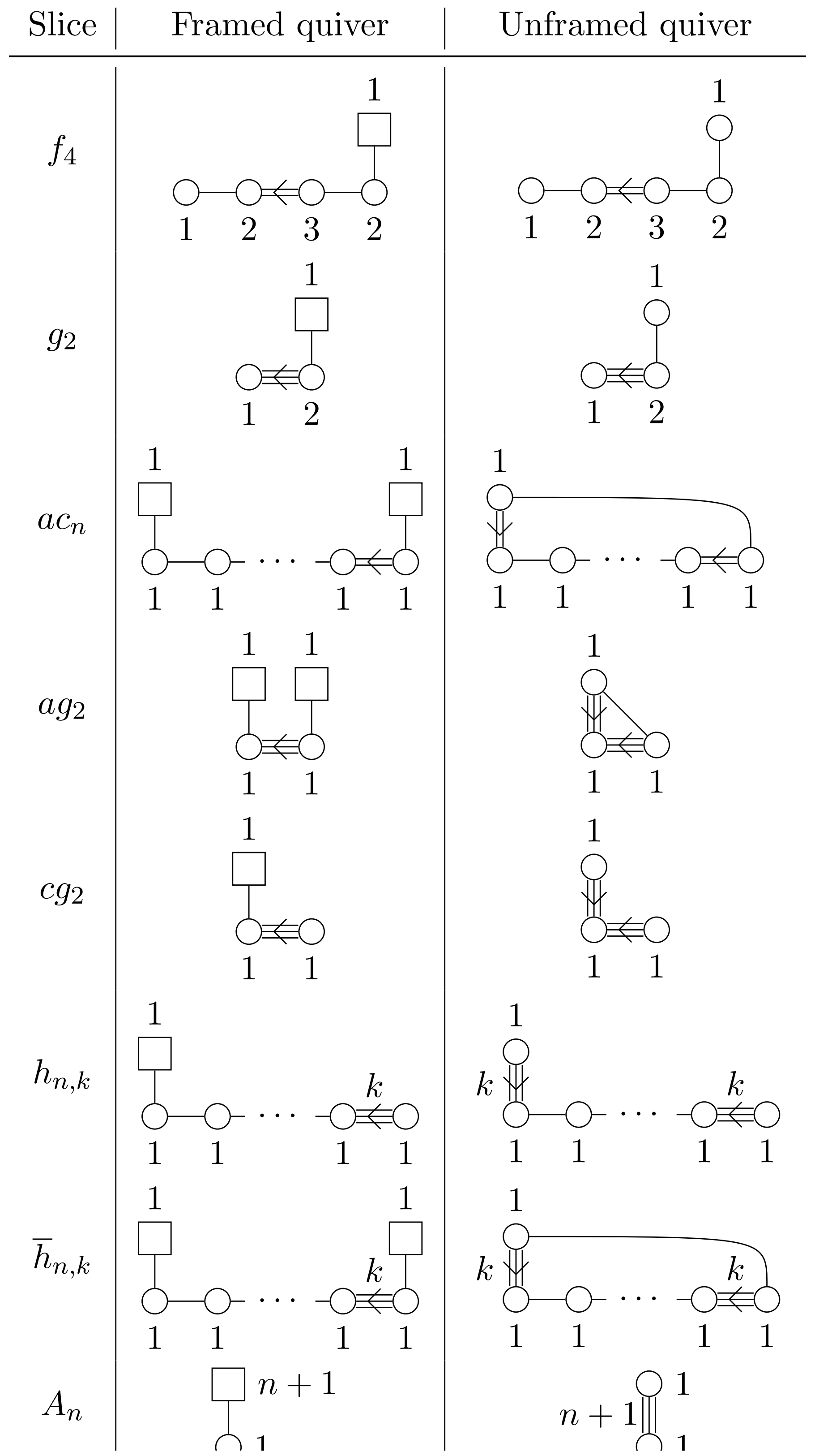
Status Report

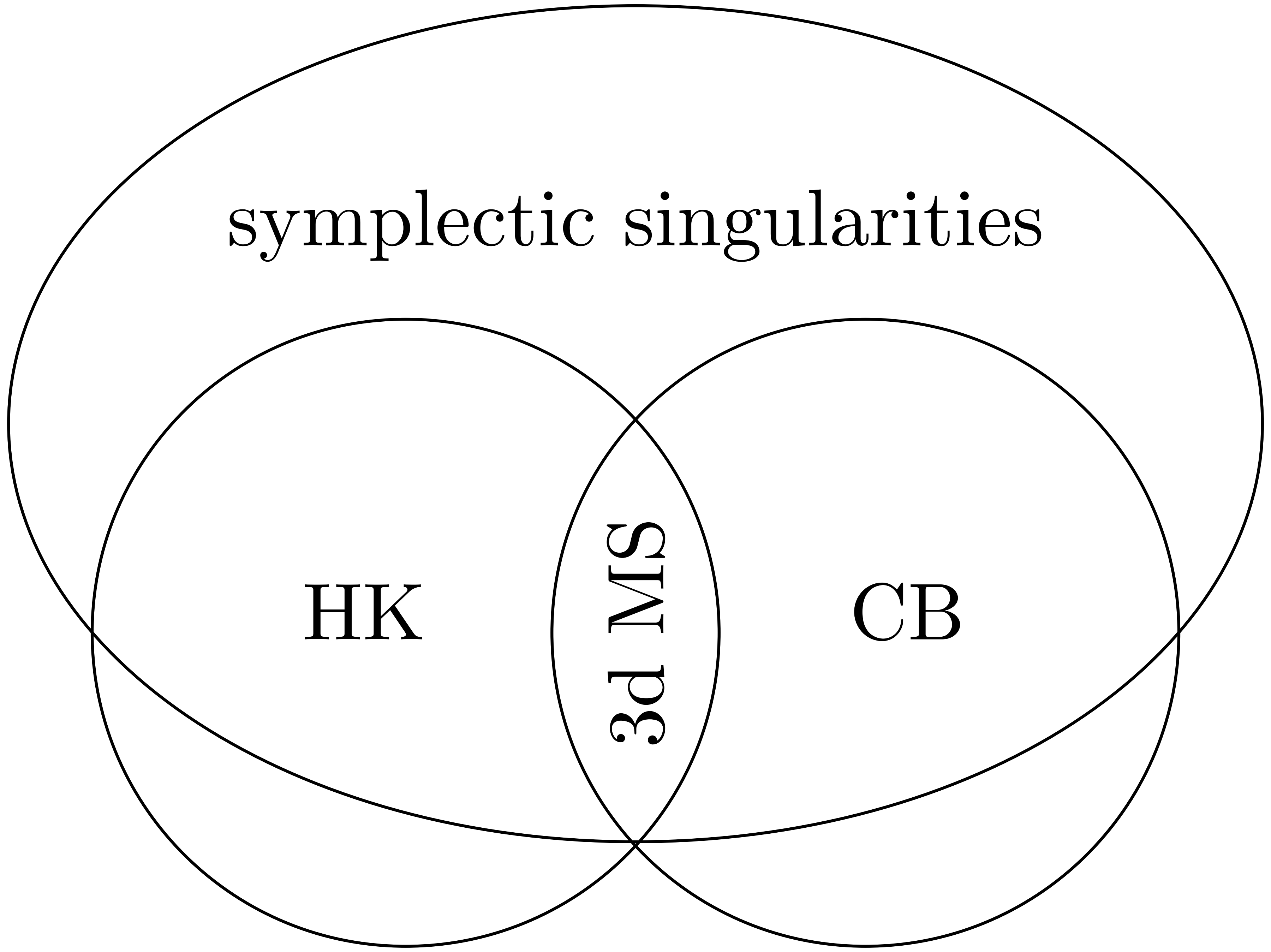
Symplectic singularities

- These lectures review the current understanding we have of moduli spaces for theories with 8 supercharges
- The focus is on
- Higgs branches in 3, 4, 5, 6 dimensions
- Coulomb branch in 3 dimensions





Symplectic singularities & Physics



Characterization of Symplectic Singularities

physical quantities — ordered by ease of computation

- Dimension (quaternionic)
- Global symmetry
- Phase (Hasse) diagram
- Representation content of the chiral ring (Hilbert Series)
- Highest weight generating function (HWG)
- Chiral ring - generators and relations

Dimension

Higgs branch

- If there is complete Higging
- $dim\mathscr{H} = H - V$
- H is number of hyper multipets
- V is number of vector multipets

Dimension of Higgs branch

Kibble 1967

- A theory with gauge group G and matter R
- If the gauge group G is broken to a subgroup H
- Decompose $R = \sum_i a_i r_i$;   Adj = adj + $\sum_i b_i r_i$
- r_i — irreducible representations of H, a_i, b_i multiplicities, r_0 — trivial representation
- New theory with gauge group H and matter $R' = \sum_{i \neq 0} (a_i - b_i) r_i$
- $a_i - b_i \geq 0$
- $dim \mathscr{H} = a_0 - b_0$

Dimension of 3d Coulomb branch

- $dim\mathscr{C} = r$
- rank of G

Global symmetry
symplectic singularity

- $SU(2)_R$ acts on the moduli space
- rotates complex structures
- Pick one — holomorphic functions
- U(1) inside gives weight to holmorphic functions
- weight n is the highest weight in the representation with spin $\frac{n}{2}$ under $SU(2)_R$
- functions of weight 2 are closed when paired under the symplectic form
- Form Lie Algebra
- adj of global symmetry is given by the set of all functions of weight 2

Global symmetry

Higgs branch

- In a quiver with flavor nodes of rank N_i the global symmetry is
$$S\left(\prod_i U(N_i)\right)$$

- As quarks have weight 1, on the Higgs branch we need to find all possible mesons

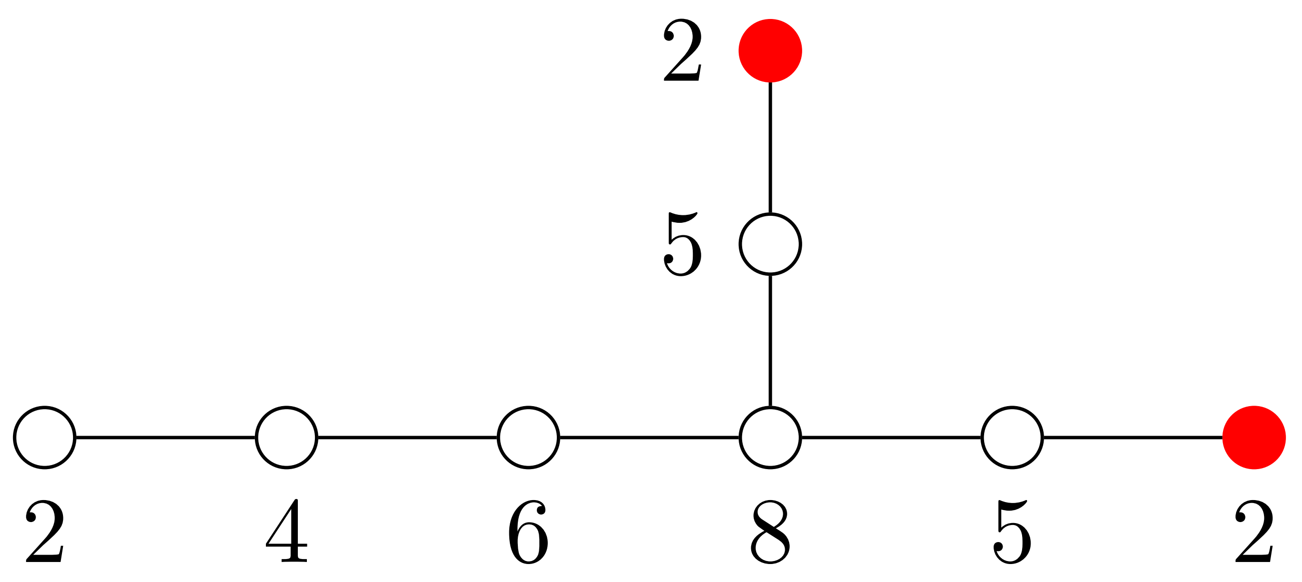
- They transform in the adjoint representation of the global symmetry

Global symmetry
Coulomb branch

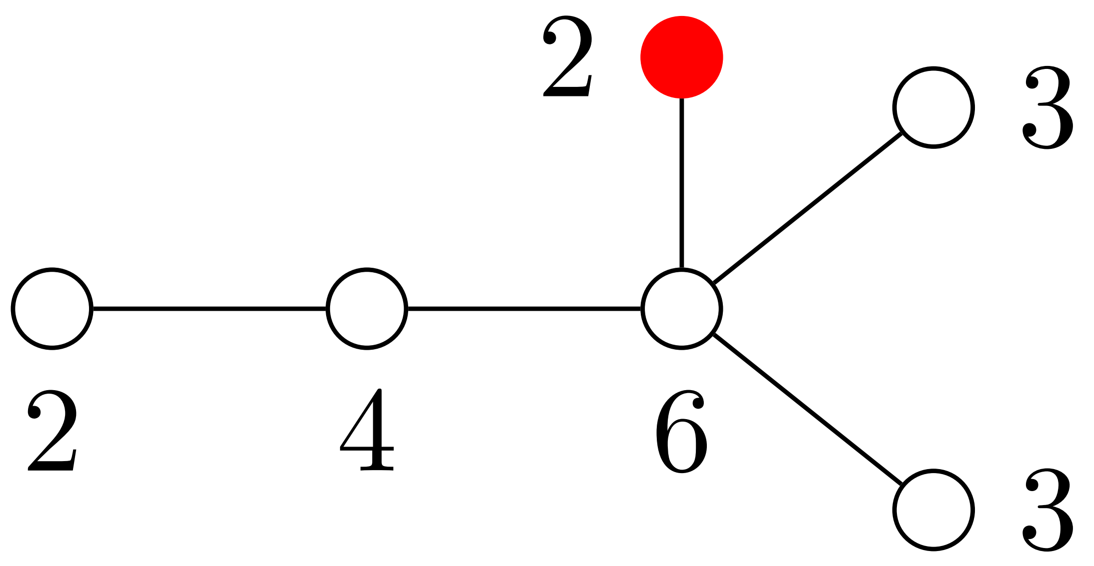
- Balance of a gauge node — sum of node ranks connected to it minus twice its rank
- For a large class of quivers the subset of balanced nodes forms the Dynkin diagram for the non Abelian part of the global symmetry
- For the remaining n unbalanced nodes there is an additional $U(1)^{n-1}$ contribution to the global symmetry
- These combinatorial criteria need to be tested with an explicit evaluation of all functions of weight 2

Global symmetry

Coulomb branch exercise



- Find the global symmetry for the quivers
- Show it is bigger than the symmetry expected by balance
- to be published in a paper with K. Gledhill



Phase (Hasse) diagrams

massless fields

- We characterize different phases by identifying the set of massless fields of the theory
- Theories with massive fields that have the same massless content are considered to be equivalent.
- They are in the same phase.

Massive fields

- Masses are functions of moduli in the theory
- As we move along the moduli space, masses of massive states vary.
- At some critical points some states become massless
- In such cases we say that the phase of the theory changes
- It contains more massless states.

Natural questions

- As we move from phase A to phase B with more massless fields:
- Characterise each phase — give some names
- How many moduli are tuned to get from A to B?
- What is the geometry of these moduli?
- These are called transition moduli, as they move from phase A to phase B.

Transition moduli

- Necessarily conical
- As we scale these moduli massive states remain massive
- massless states remain massless
- At the origin new massless states show up

Example: Free scalar field

- Consider a scalar field with mass m
- There are two phases
- $m \neq 0$ one dimensional phase with 0 massless states
- $m = 0$ zero dimensional phase with 1 massless state
- The transition modulus m parametrizes $\mathbb{R}^+$ which is conical, one dimensional

Minimal transitions

- Given a phase, a minimal transition is a minimal set of tuned moduli for moving to a new phase
- The Hasse (phase) diagram for such moduli consists of 2 points connected by one edge.
- Two phases
- The origin
- Anything else
- An important problem — find such minimal cases

Supersymmetry

- The discussion so far is very generic and can apply to any theory
- With supersymmetry we get a better control.
- Can compute masses with control over quantum corrections
- Can use geometric techniques to get exact results
- Will focus on theories with 8 supercharges
- look at Higgs branches in 3, 4, 5, 6d
- Coulomb branch in 3d

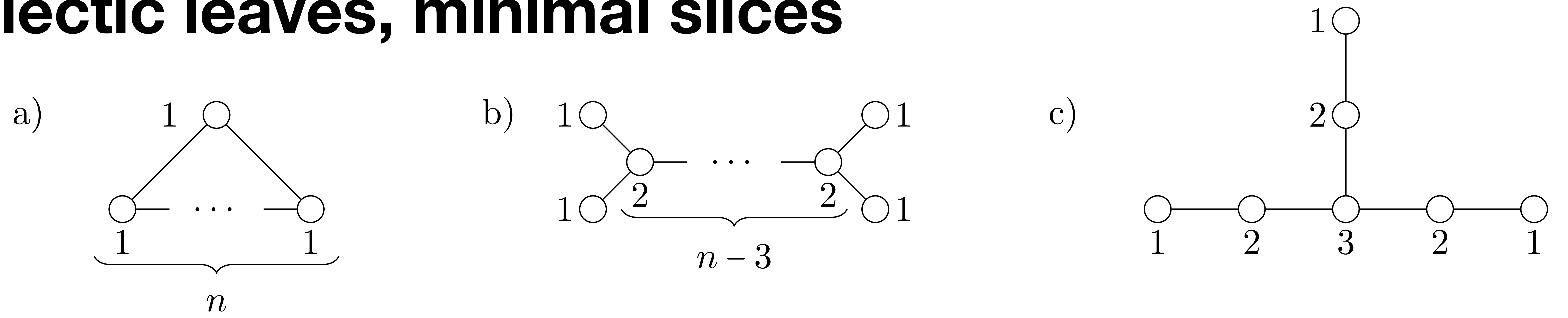
Phase Diagram

Hasse diagram

- We form a diagram with two objects
- nodes and edges
- A node represents a phase (symplectic leaf)
- An edge represents a minimal transition (transverse slice) between a node A with some massless states to a node B with additional massless states

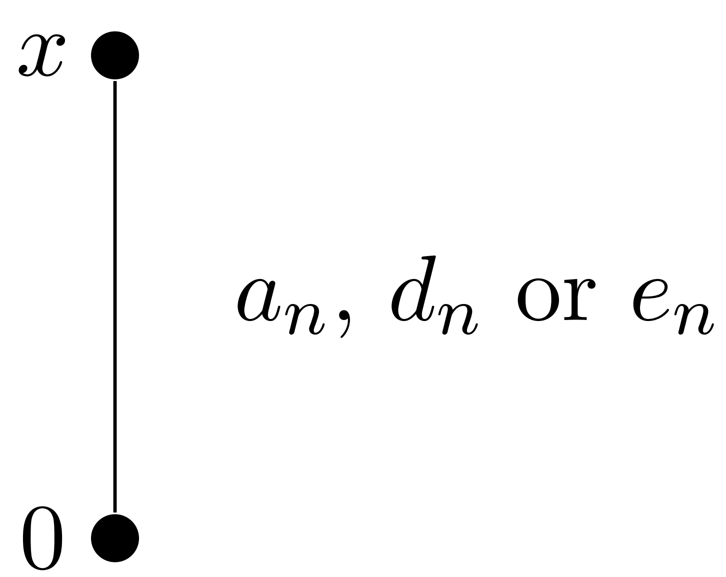
Basic Hasse diagrams - affine ADE quivers

2 symplectic leaves, minimal slices

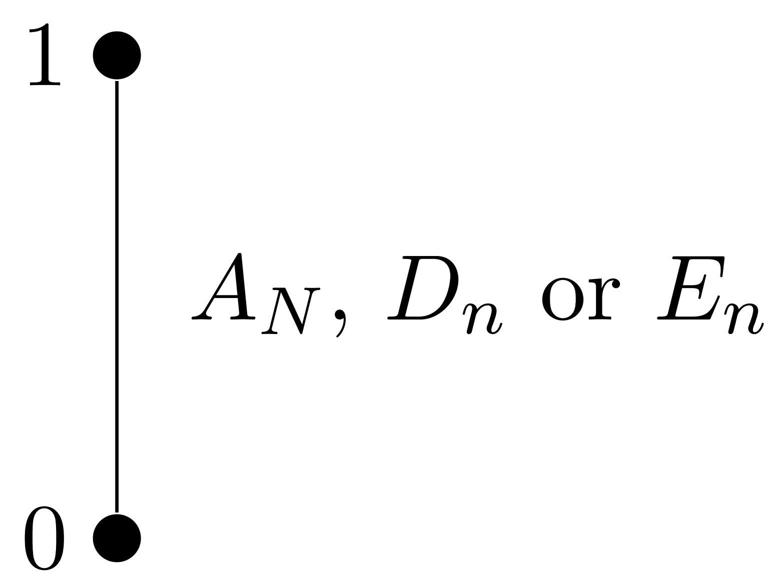


$$x = \begin{cases} n & \text{for } a_n \\ 2n-3 & \text{for } d_n \\ 11 & \text{for } e_6 \\ 17 & \text{for } e_7 \\ 29 & \text{for } e_8 \end{cases}$$

$\mathfrak{H}_C =$



$\mathfrak{H}_H =$



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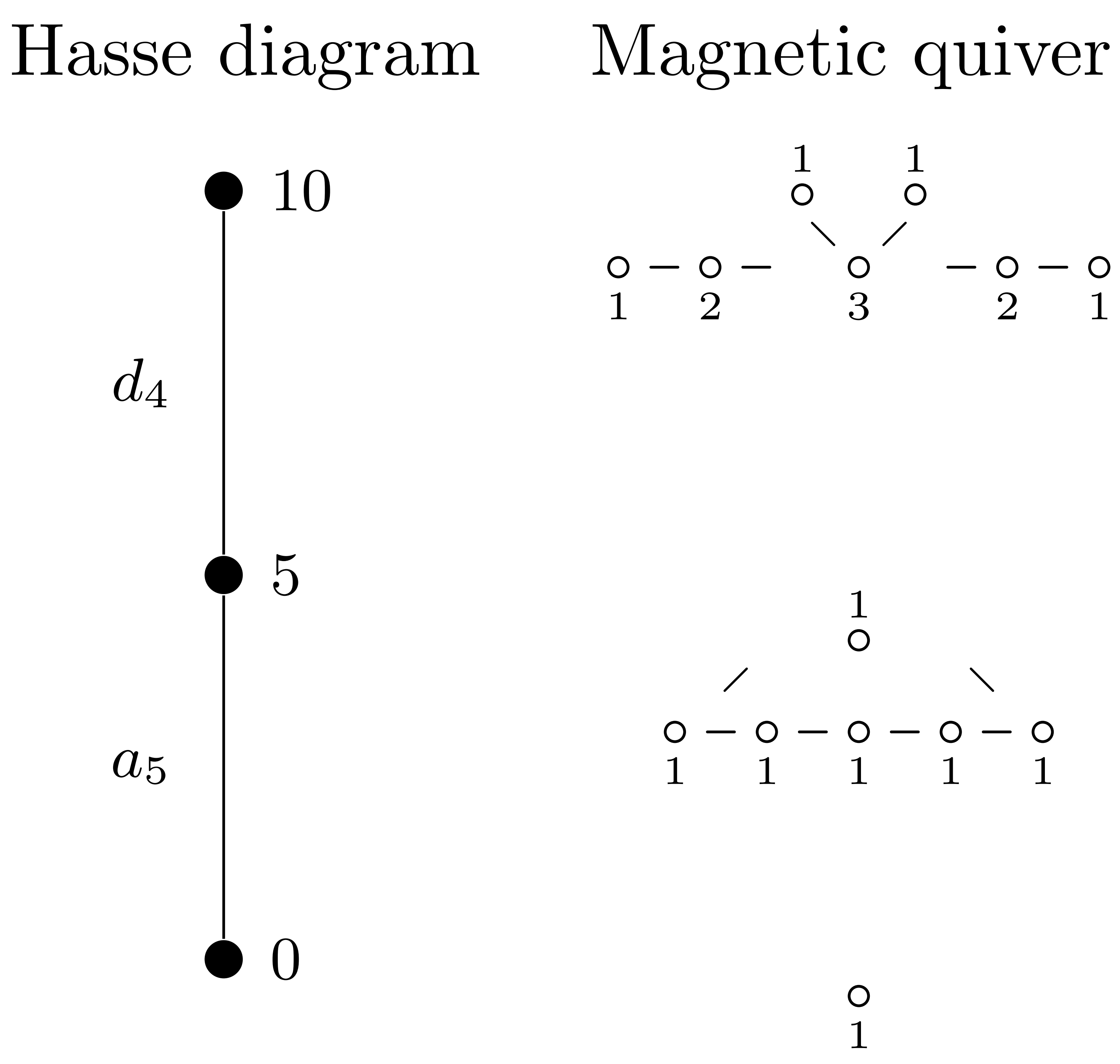
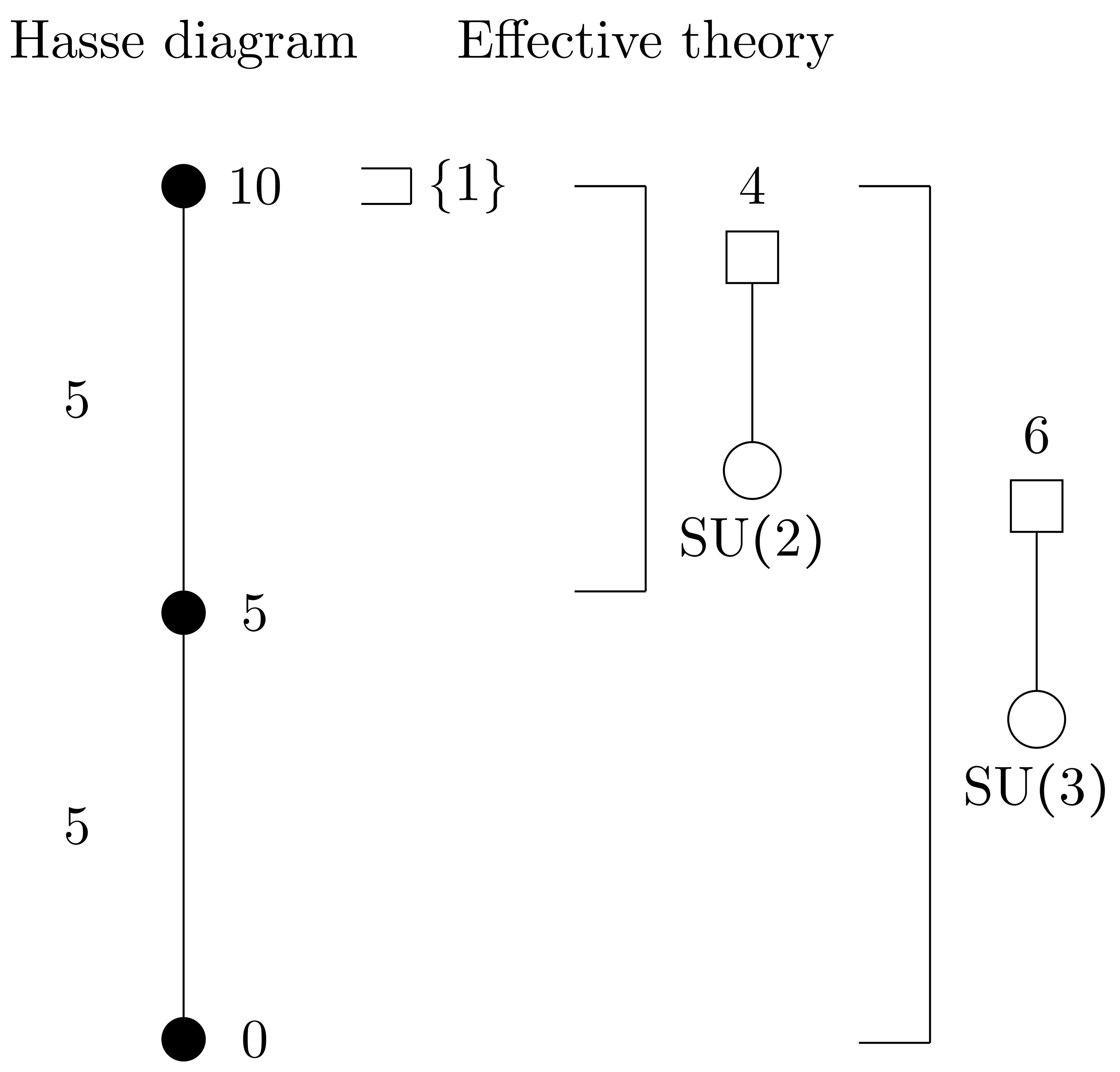
Higgs branch of SU(3) with 6 flavors

3 symplectic leaves, 2 minimal slices





Phase diagram for SU(3) with 6 flavors

Higgs mechanism — recall Kibble's method

- At the origin SU(3) is massless
- Now turn moduli such that SU(2) is massless
- $8 \to 3 + 2 + 2 + 1$
- $6 \times (3 \to 2 + 1)$
- SU(2) with remaining matter $4 \times 2 + 5 \times 1$
- 5 moduli which parametrize the Higgs branch of U(1) with 6 flavors
- Further Higgsing to give masses to SU(2) adds 5 more moduli for the Higgs branch of SU(2) with 4 flavors

Phase diagram for SU(3) with 6 flavors

Coulomb branch — quiver subtraction

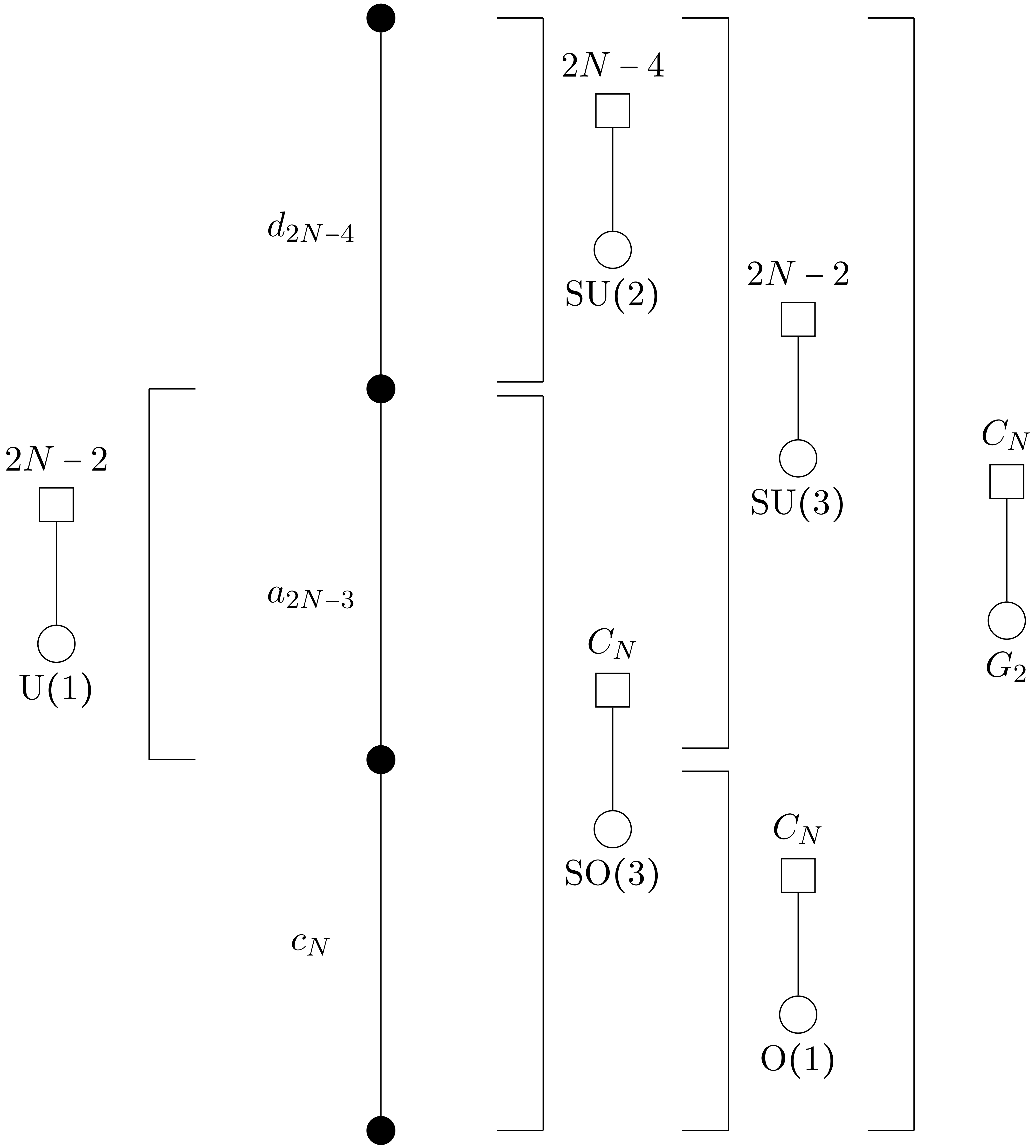
- The moduli space is given by the Coulomb branch of the 4 leg quiver
- Look for a sub quiver which is in the family of the affine Dynkin diagram
- Subtract and add flavors so that balance is preserved

Exercise

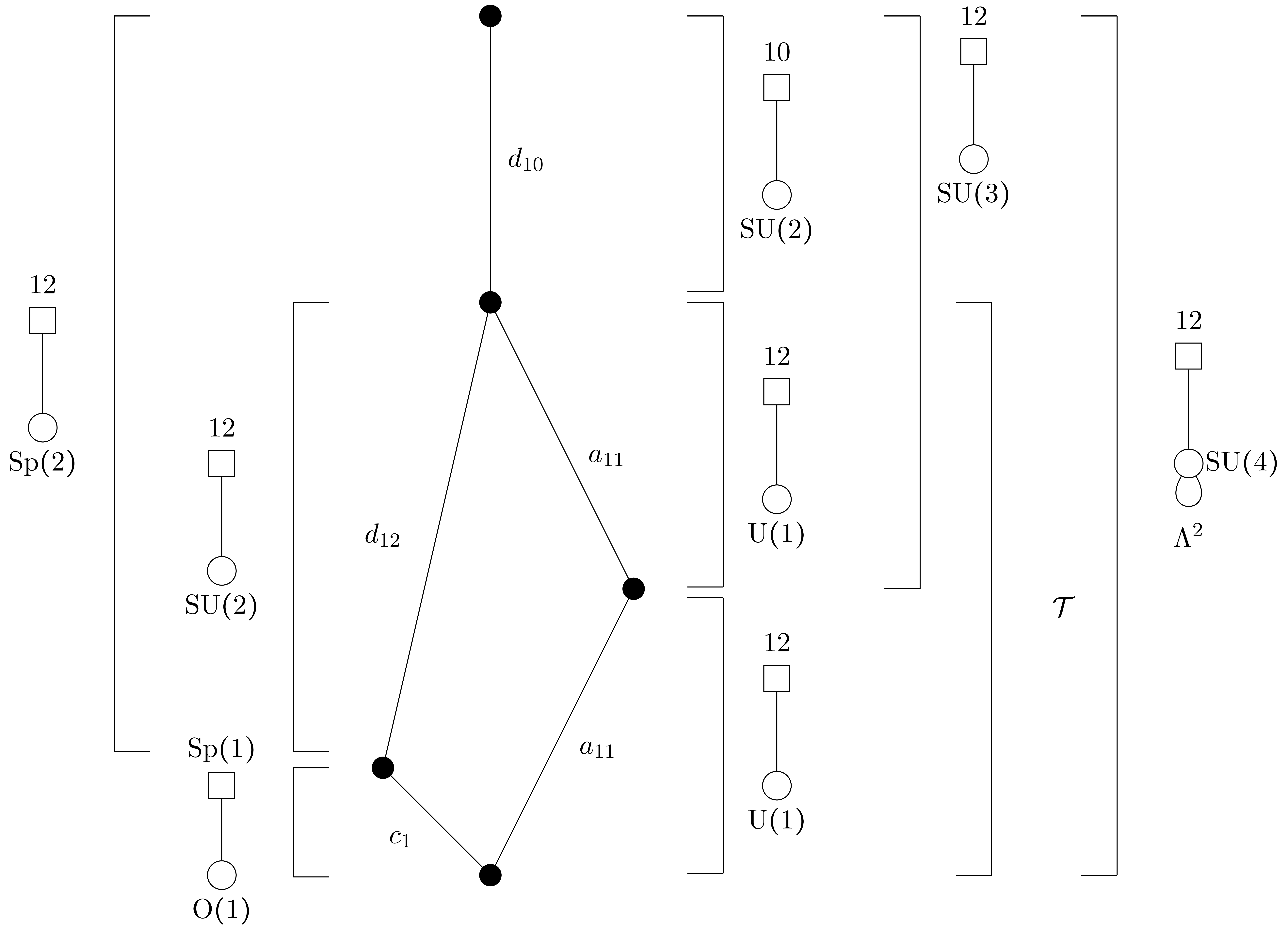
SU(4) with 9 flavors

- Compute the Hasse diagram for the Higgs branch of this theory
- First going bottom up using the Higgs mechanism
- Second going top down using the method of Quiver Subtraction

G2 with N hypers of fundamental matter



SU(4) with 1 antisymm and 12 fundamentals



Monopole formula — the ingredients

per each node of label k

- $W = S_k$ — the Weyl group of $GL(k)$
- $\hat{\Lambda}$ — The (Langlands) dual lattice
- A set of integer numbers $\hat{\Lambda} = \mathbb{Z}^k \ni m = (m_1, \ldots, m_k)$ — magnetic charges
- $\hat{\Lambda}/W$ — Principal Weyl chamber $m_1 \leq \cdots \leq m_k$
- Boundaries of the Weyl chamber — when some m_i coincide
- H_m — stabilizer of m in $GL(k)$ — a Levi subgroup of GL(k)
- d_i^m — degrees of Casimir invariants of H_m

Example

$GL(2)$

- S_2 — the Weyl group of $GL(2)$
- A set of integer numbers $m = (m_1, m_2)$ — magnetic charges
- Principal Weyl chamber $m_1 \leq m_2$
- Boundary of the Weyl chamber: $m_1 = m_2$
- H_m — stabilizer of m in $GL(2)$: $\begin{cases} (\mathbb{C}^*)^2 & \text{for } m_1 \neq m_2 \\ GL(2) & \text{for } m_1 = m_2 \end{cases}$
- d_i^m — degrees of Casimir invariants of H_m : $\begin{cases} (1,1) & \text{for } m_1 \neq m_2 \\ (1,2) & \text{for } m_1 = m_2 \end{cases}$

The gauge group

Quivers with no flavor nodes

- Given a quiver with a set of nodes, each with labels k_a
- The gauge group is $\left[\prod_a GL(k_a)\right]/\mathbb{C}*$
- corresponding dual lattice $\hat{\Lambda}$ and Weyl group W

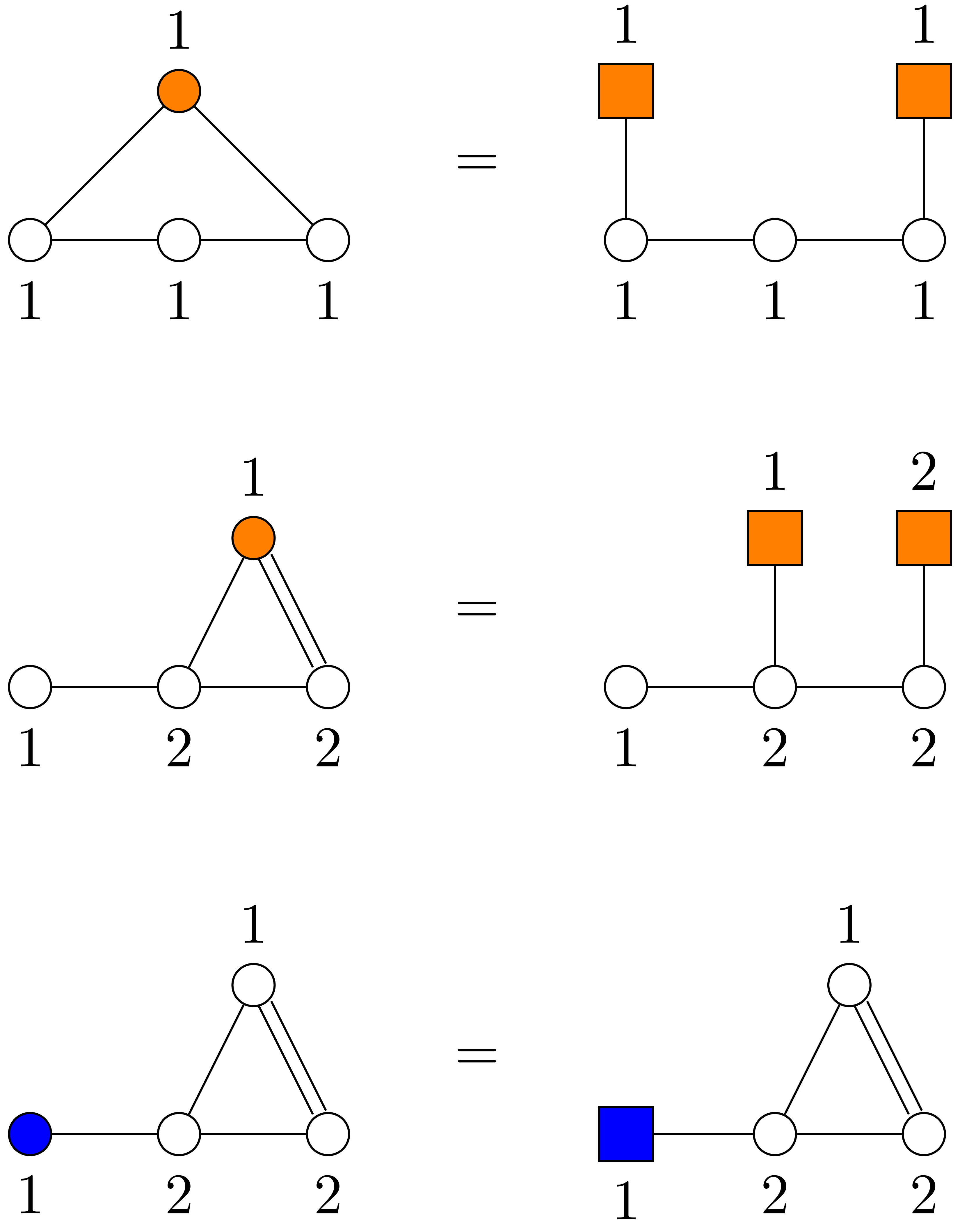
The gauge group

Quivers with flavor nodes

- In the presence of flavor (square) nodes there is no overall $\mathbb{C}^*$ to divide by
- The gauge group is $\prod_a GL(k_a)$ and the product is over gauge (circle) nodes

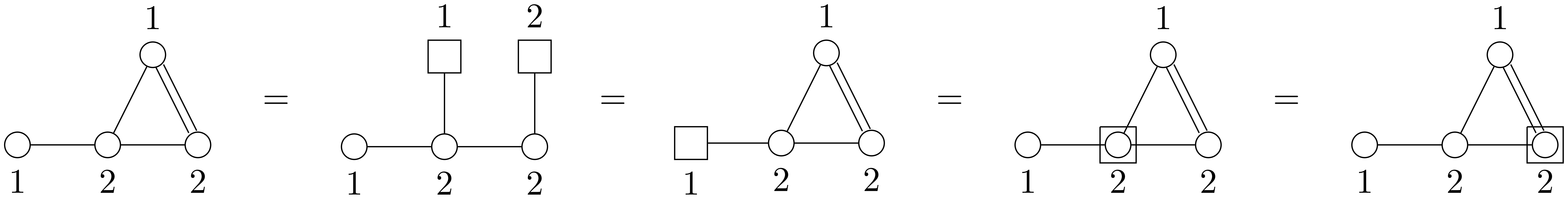
Ungauging

graph equivalence



Ungauging

graph equivalence



The conformal dimension — $\Delta(m)$

$\mathbb{C}^*$ grading on the Coulomb branch

- Given a quiver with a set of nodes, each with labels k_a
- $\Delta(m)$ is a sum of contributions from nodes and edges:
- For each node with magnetic charges $m_i^a, i = 1 \ldots k_a$ there is a negative contribution
- $-\sum_{1 \leq i < j \leq k_a} |m_i^a - m_j^a|$ (associated with positive roots of $GL(k_a)$)
- For each edge connecting nodes a, b with magnetic charges m_i^a and m_j^b a positive contribution
- $\frac{1}{2} \sum_{i=1}^{k_a} \sum_{j=1}^{k_b} |m_i^a - m_j^b|$ (associated with bifundamental representation)

The monopole formula

Hilbert series of the Coulomb branch

- Given a quiver with all the ingredients defined so far
- Introduce a variable t
- The Hilbert series is given by (flavor nodes have fixed m. Set to 0.)
- $H(t) = \sum_{m \in \hat{\Lambda}/W} t^{2\Delta(m)} P_m(t)$
- $P_m(t) = \prod_i \frac{1}{1 - t^{2d_i^m}}$

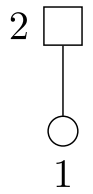
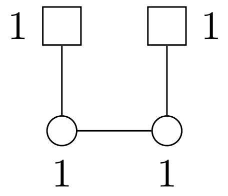
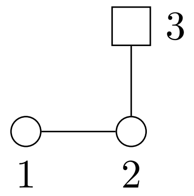
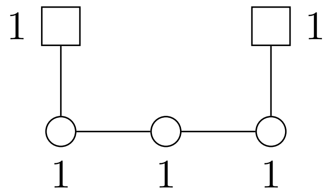
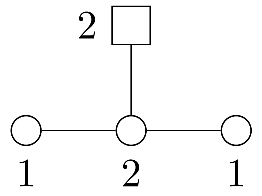
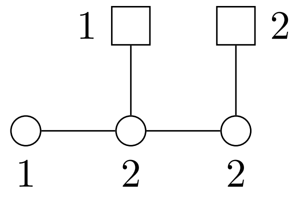
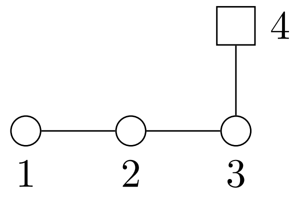
Example — the trivial case

Coulomb branch of $\mathbb{H}^n = \mathbb{C}^{2n}$

- For a finite A type quiver:
- A linear quiver with n+1 gauge nodes, each with label 1, connected by n edges
- The Coulomb branch is $\mathbb{H}^n$
- The Hilbert series is
- $H(t) = \dfrac{1}{(1-t)^{2n}} = 1 + 2n\,t + n(2n+1)t^2 + \dots$

Examples — from the world of nilpotent orbits

Simple quivers and their Hilbert Series

Nilpotent Orbit	$\mathrm{Dim}_{\mathbb{H}}$	Quiver	HS	HWG
$[1,1]$	0	-	1	1
$[2]$	1		$\frac{(1-t^4)}{(1-t^2)^3}$	$\frac{1}{(1-\mu^2t^2)}$
$[1,1,1]$	0	-	1	1
$[2,1]$	2		$\frac{(1+4t^2+t^4)}{(1-t^2)^4}$	$\frac{1}{(1-\mu_1\mu_2t^2)}$
$[3]$	3		$\frac{(1-t^4)(1-t^6)}{(1-t^2)^8}$	$\frac{(1-\mu_1^3\mu_2^3t^{12})}{(1-\mu_1\mu_2t^2)(1-\mu_1\mu_2t^4)(1-\mu_1^3t^6)(1-\mu_2^3t^6)}$
$[1,1,1,1]$	0	-	1	1
$[2,1,1]$	3		$\frac{(1+t^2)(1+8t^2+t^4)}{(1-t^2)^6}$	$\frac{1}{(1-\mu_1\mu_3t^2)}$
$[2,2]$	4		$\frac{(1+t^2)^2(1+5t^2+t^4)}{(1-t^2)^8}$	$\frac{1}{(1-\mu_1\mu_3t^2)(1-\mu_2^2t^4)}$
$[3,1]$	5		$\frac{(1+t^2)(1+4t^2+10t^4+4t^6+t^8)}{(1-t^2)^{10}}$	$\frac{(1-\mu_1^3\mu_2^3\mu_3^3t^{12})}{(1-\mu_1\mu_3t^2)(1-\mu_2^2t^4)(1-\mu_1\mu_3t^4)(1-\mu_1^2\mu_2t^6)(1-\mu_2\mu_3^2t^6)}$
$[4]$	6		$\frac{(1-t^4)(1-t^6)(1-t^8)}{(1-t^2)^{15}}$	messy

Hilbert Series

Dimension of the Coulomb branch

- The complex dimension of the Coulomb branch is the order of the pole of $H(t)$ at $t = 1$

The global symmetry
a Lie algebra

- For each quiver there is an associated finite dimensional Lie algebra F
- Set $H(t) = \sum_{n=0}^{\infty} c_n t^n$
- c_n are dimensions of (reducible) representations of F
- c_2 is the dimension of the adjoint representation of F
- If $c_1 \neq 0$ it is even and $H'(t) = (1-t)^{c_1} H(t)$ is a Hilbert series for $\mathscr{C}'$
- The moduli space factorizes $\mathscr{C} = \mathbb{H}^{\frac{c_1}{2}} \times \mathscr{C}'$ with $\mathbb{H} = \mathbb{C}^2$

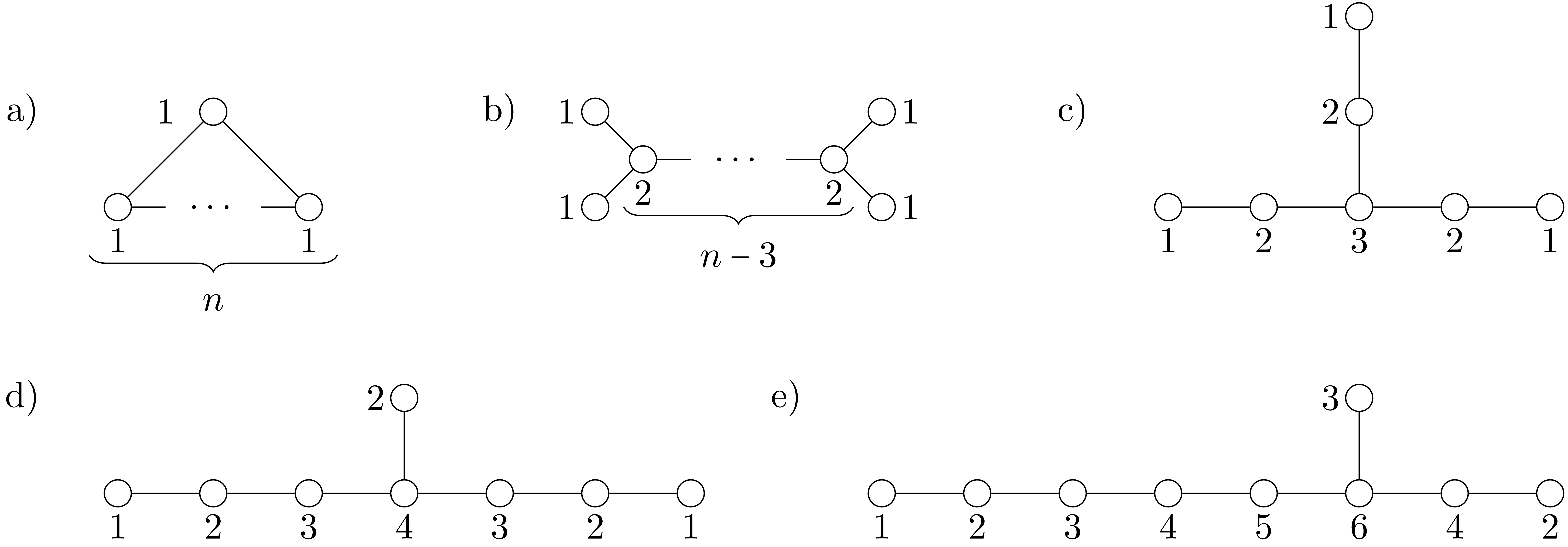
A balanced node

conditions for symmetry

- A gauge node k_a is said to be balanced if the sum of node labels connected to it is $2k_a$
- Set C to be the Cartan matrix
- k the vector of gauge node labels. f the vector of flavor labels
- Then the imbalance of the gauge nodes is the vector
- $b = f - Ck$

Affine ADE quivers

all nodes are balanced



The refined Hilbert Series

another ingredient

- For any node with node number k_a set $\mathbb{C}^*$ gradings
- $J_a(m) = \sum_{i=1}^{k_a} m_i^a$
- Introduce the fugacities z_a
- The refined Hilbert series is
- $H(t, z_a) = \sum_{m \in \hat{\Lambda}/W} t^{2\Delta(m)} P_m(t) \prod_a z_a^{J_a(m)}$

Global symmetry
dimensions are refined to characters

- Set $H(t, z_a) = \sum_{n=0}^{\infty} c_n(z_a) t^n$
- $c_n(z_a)$ are characters of the global symmetry F
- $c_2(z_a)$ is the character of the adj representation of F

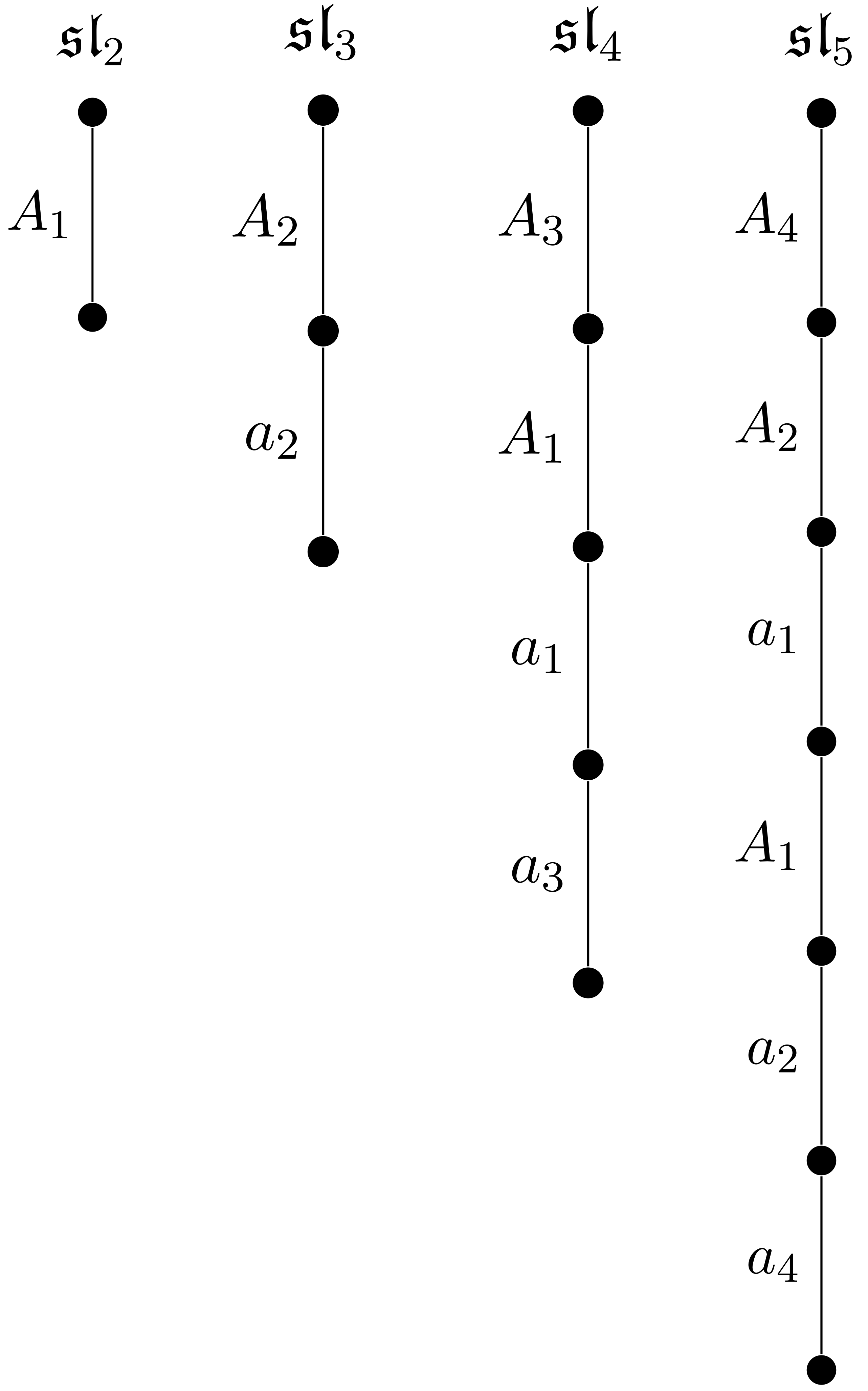
Hasse diagrams

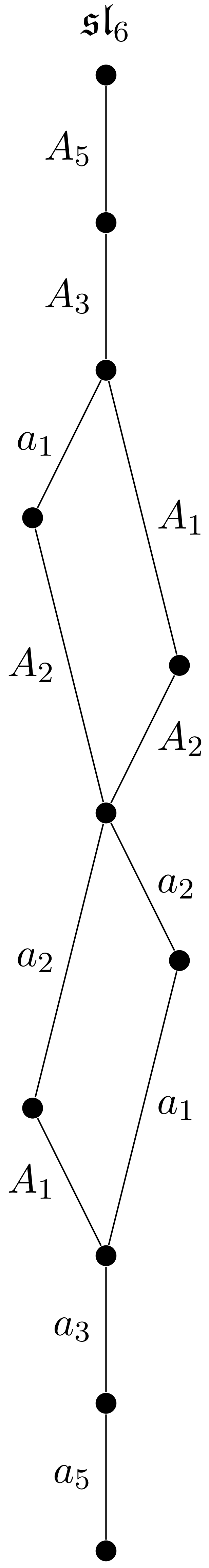
Quiver subtraction

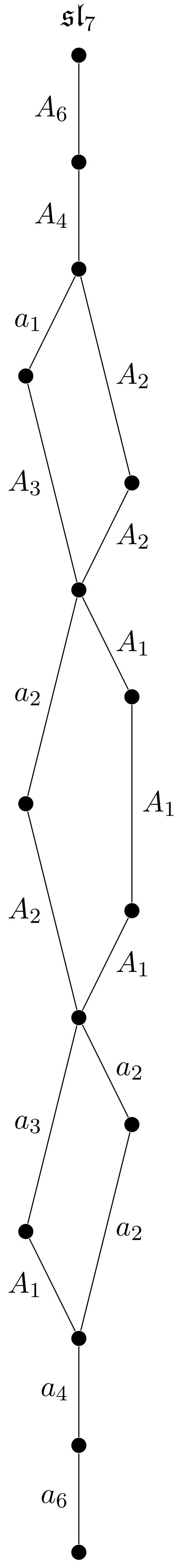
- Recall the work of Kraft and Procesi who classified degenerations in closures of nilpotent orbits
- Minimal degenerations are of two types
- Klein singularity (ADE) — denoted by capital letters
- closure of a minimal nilpotent orbit of some algebra — denoted lower case
- This is reproduced and generalized with the Coulomb branch

Hasse diagrams for nilpotent orbits

taken from KP



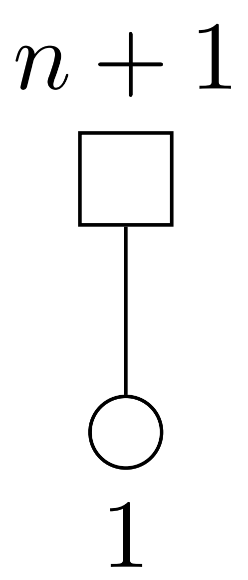




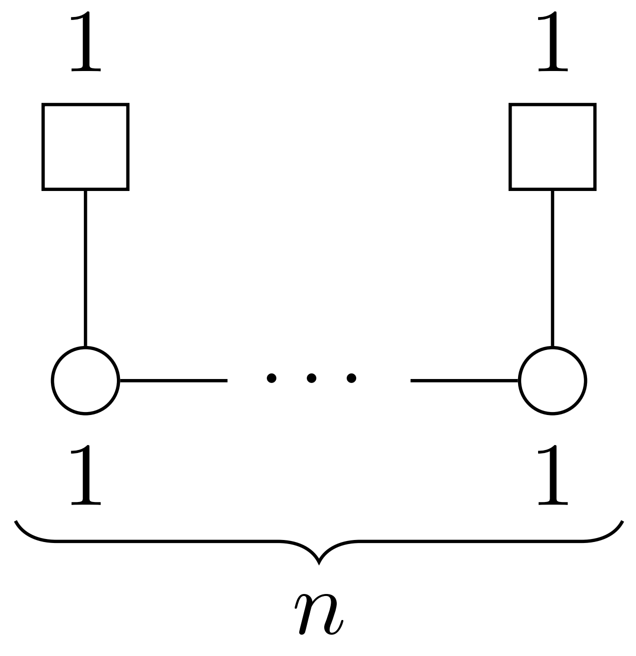
Minimal degenerations

A & a

$A_n =$



$a_n =$



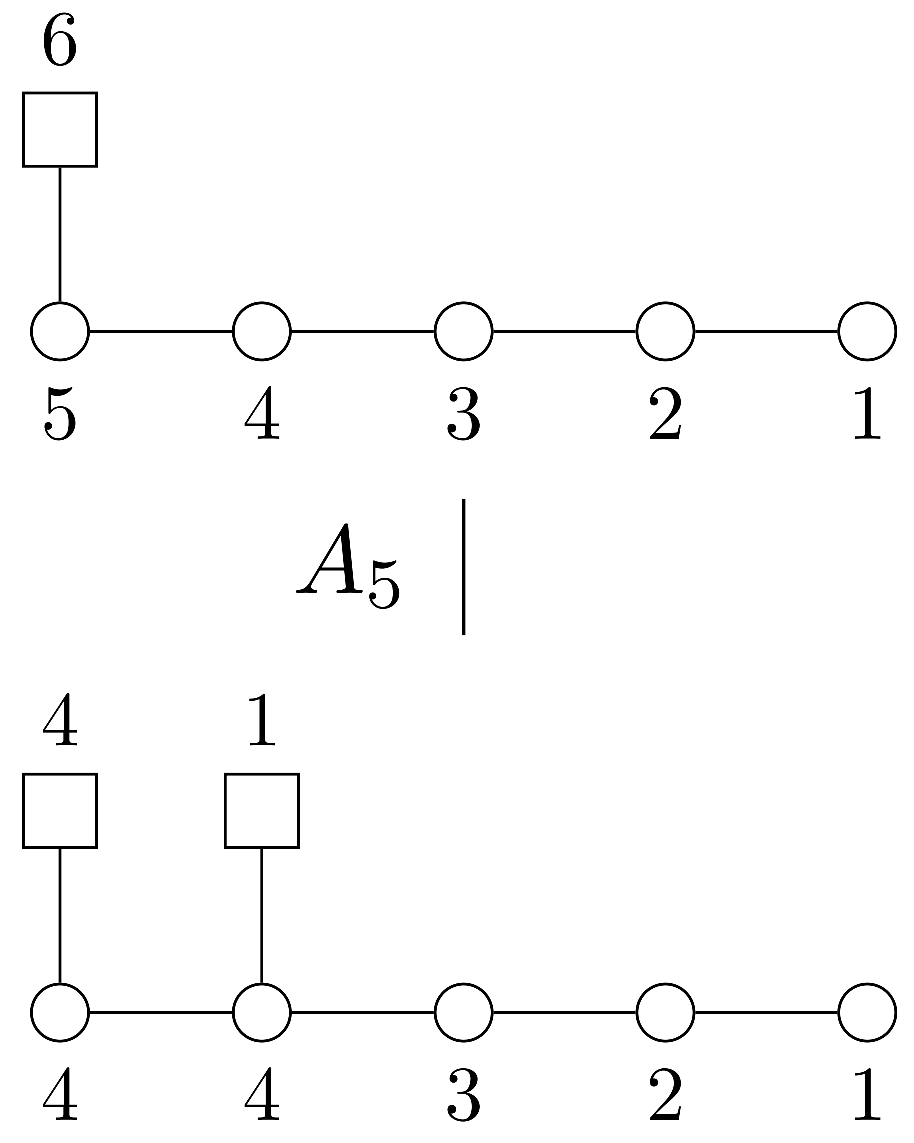
Quiver subtraction

algorithm

- Given a quiver, identify sub quivers which are in the list of minimal degenerations
- align and subtract
- rebalance — add/remove flavors to nodes such that their imbalance is preserved
- get a smaller quiver
- repeat till reaching a minimal degeneration

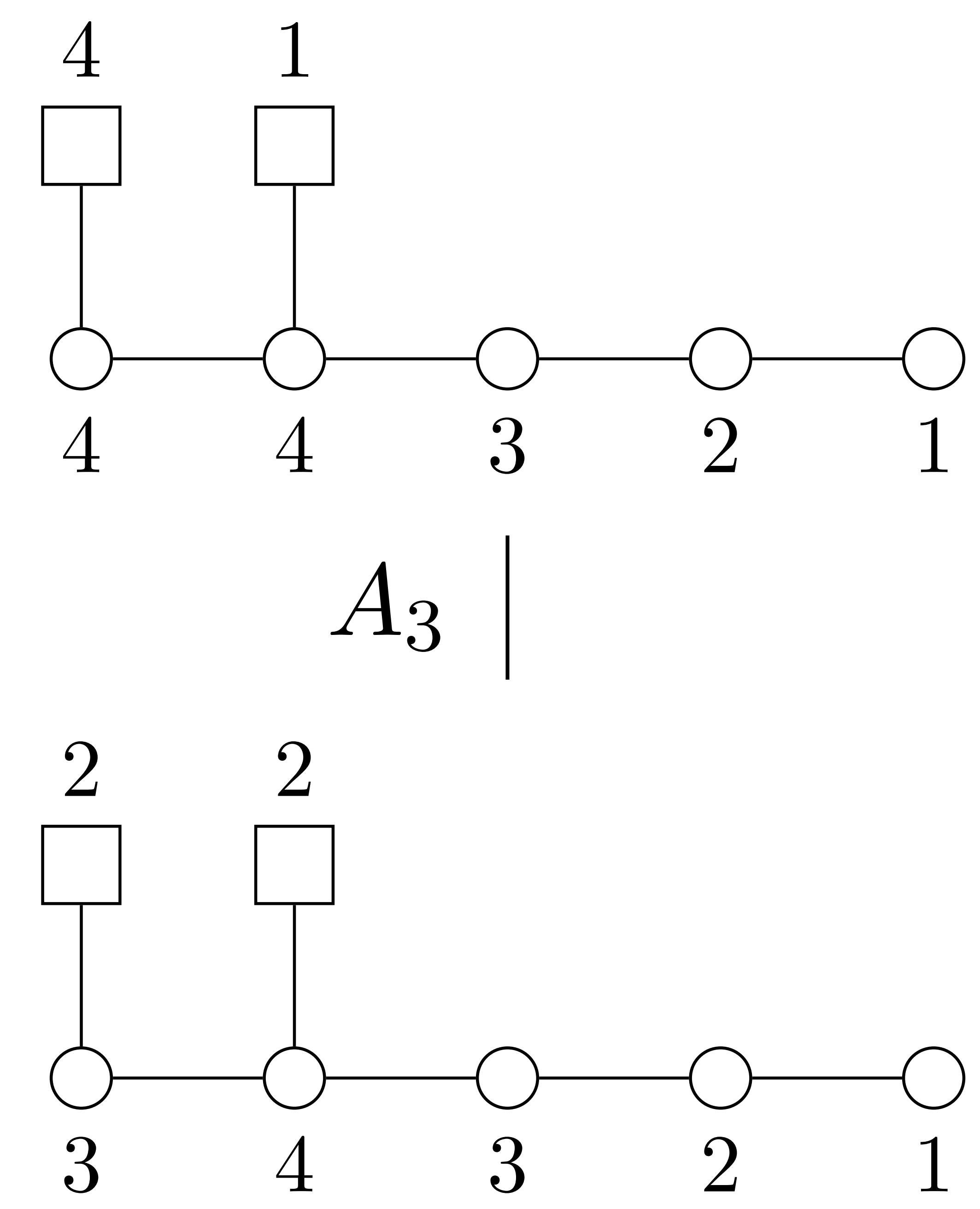
Quiver subtraction

nilpotent cone of A_5 — step 1



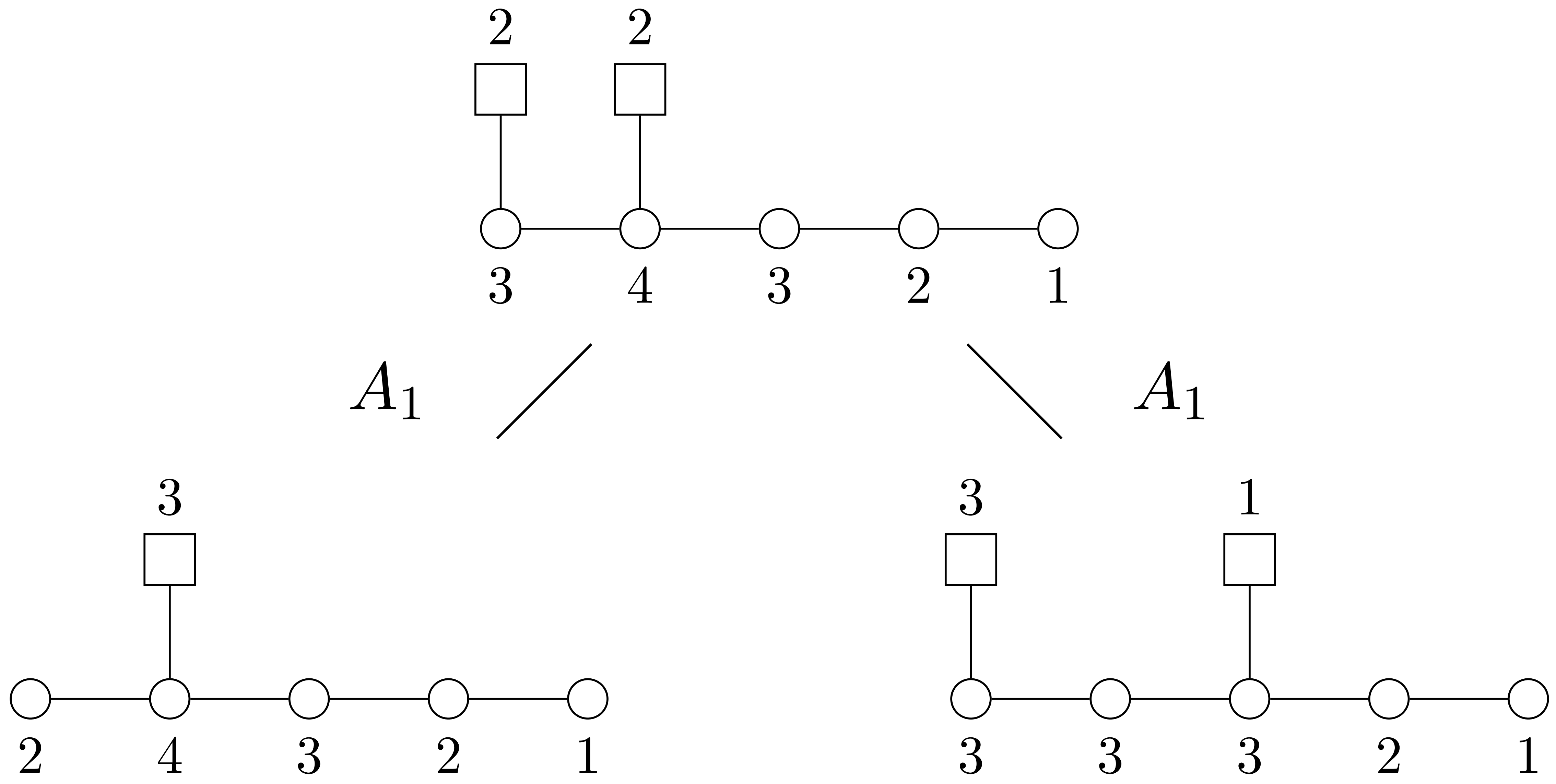
Quiver subtraction

nilpotent cone of A_5 — step 2



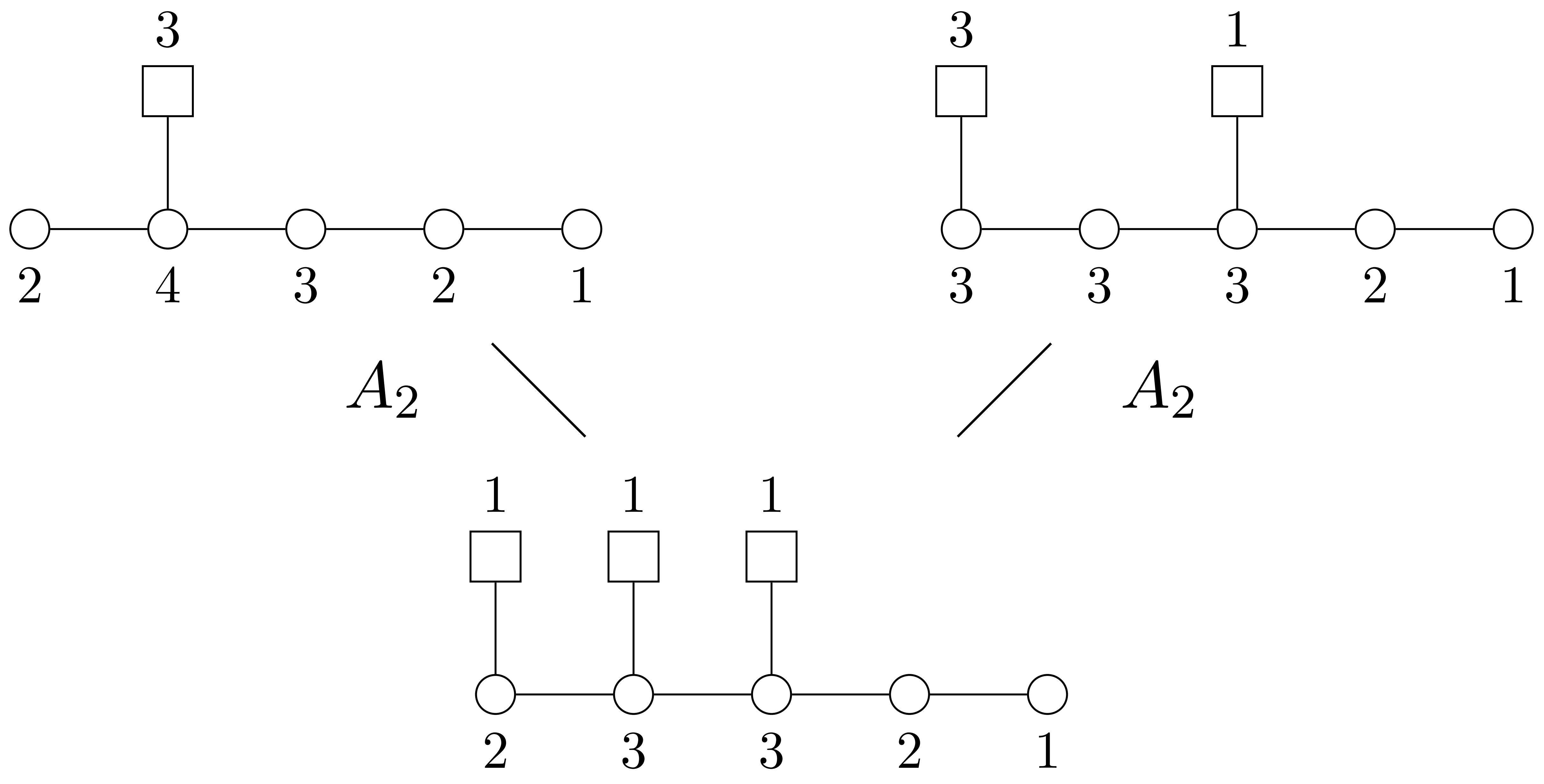
Quiver subtraction

nilpotent cone of A_5 — steps 3



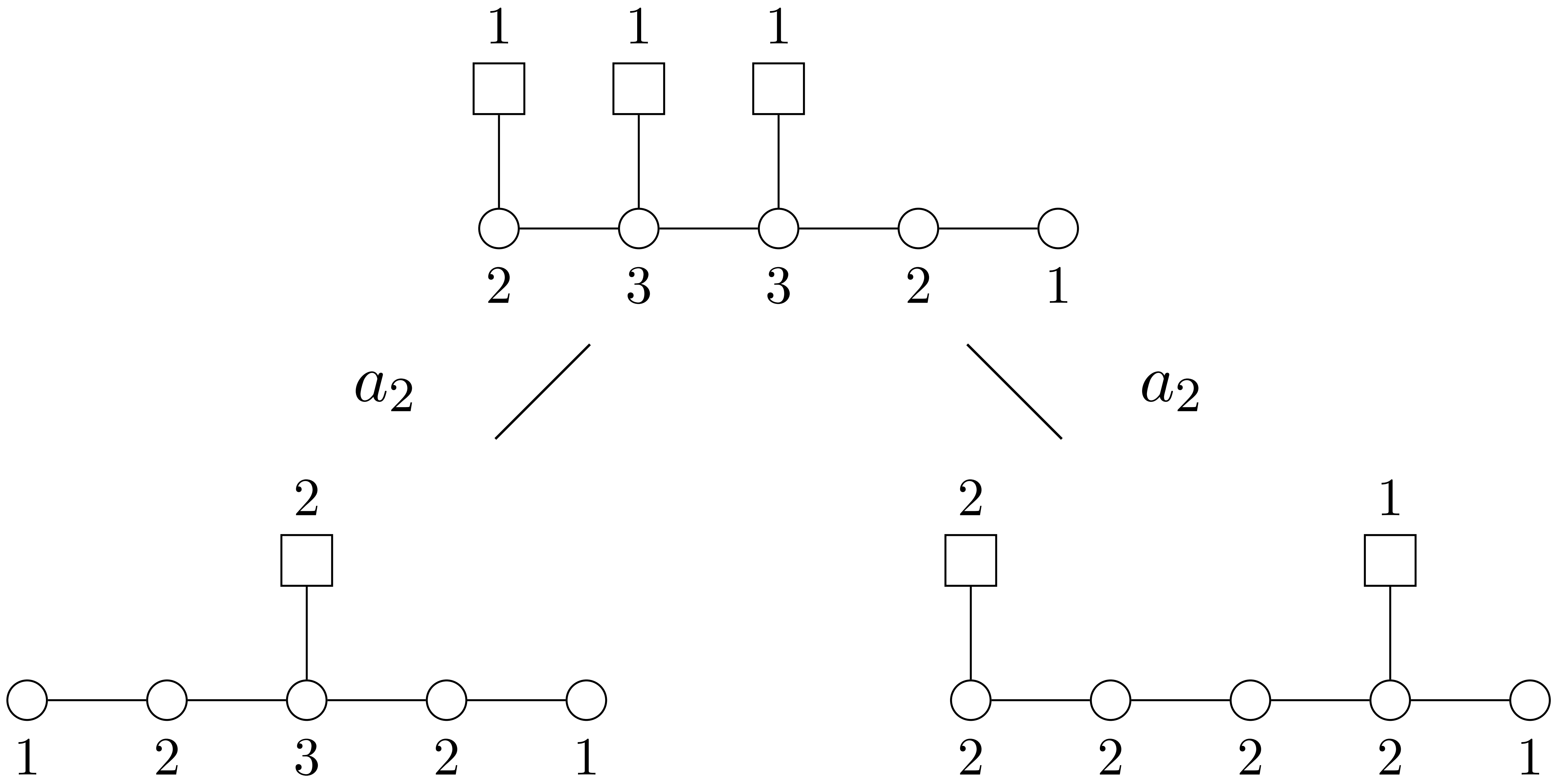
Quiver subtraction

nilpotent cone of A_5 — steps 4



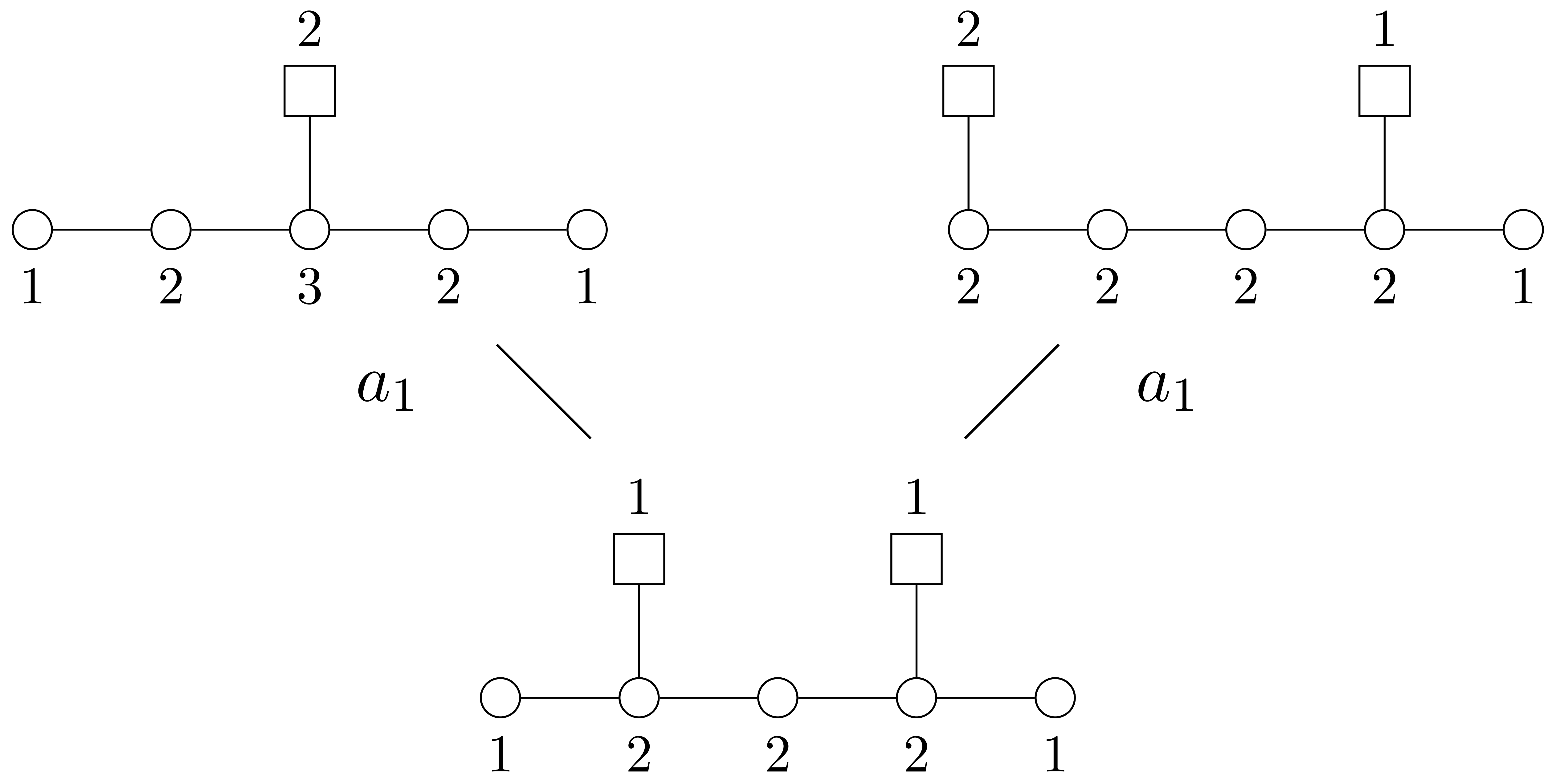
Quiver subtraction

nilpotent cone of A_5 — steps 5



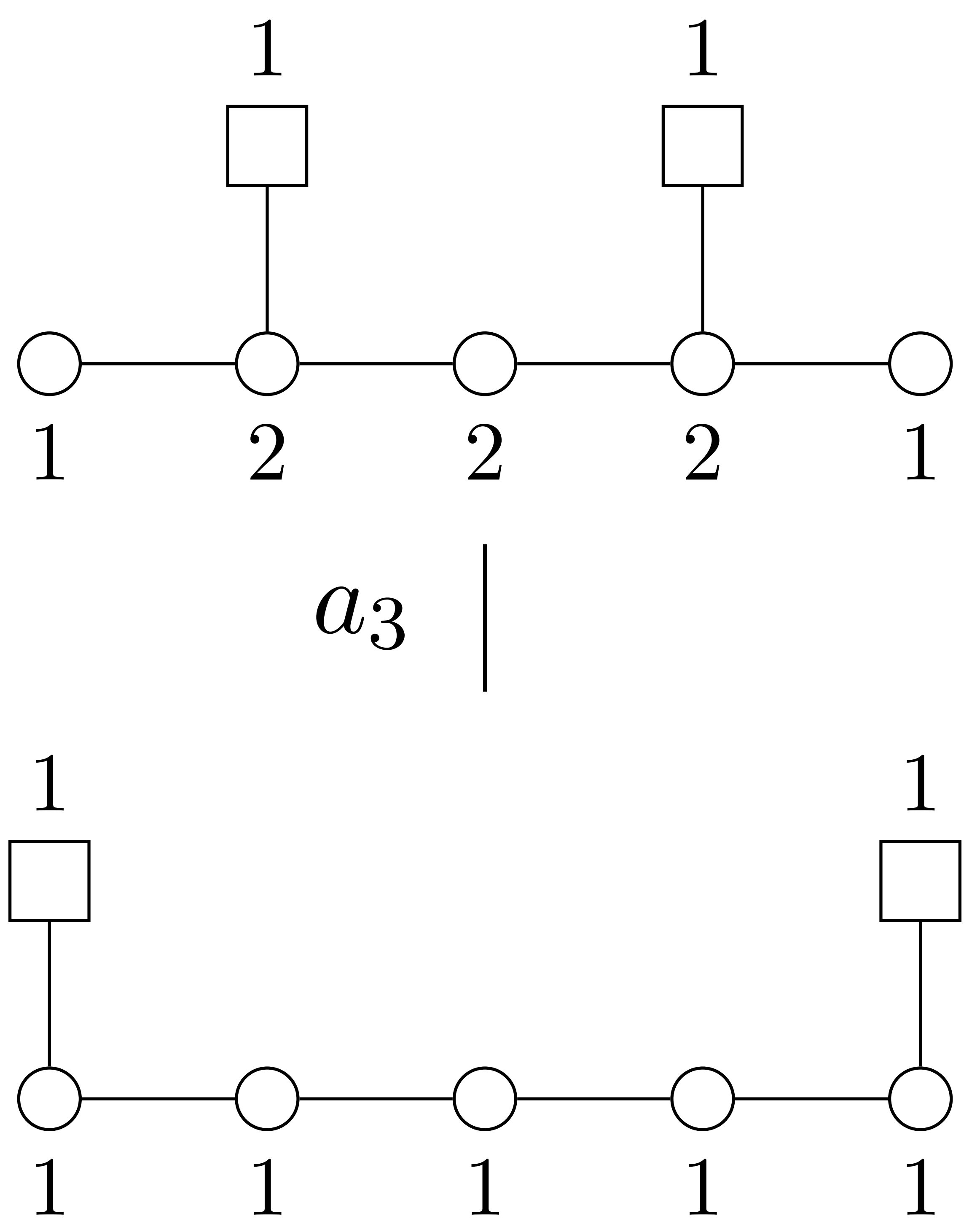
Quiver subtraction

nilpotent cone of A_5 — steps 6



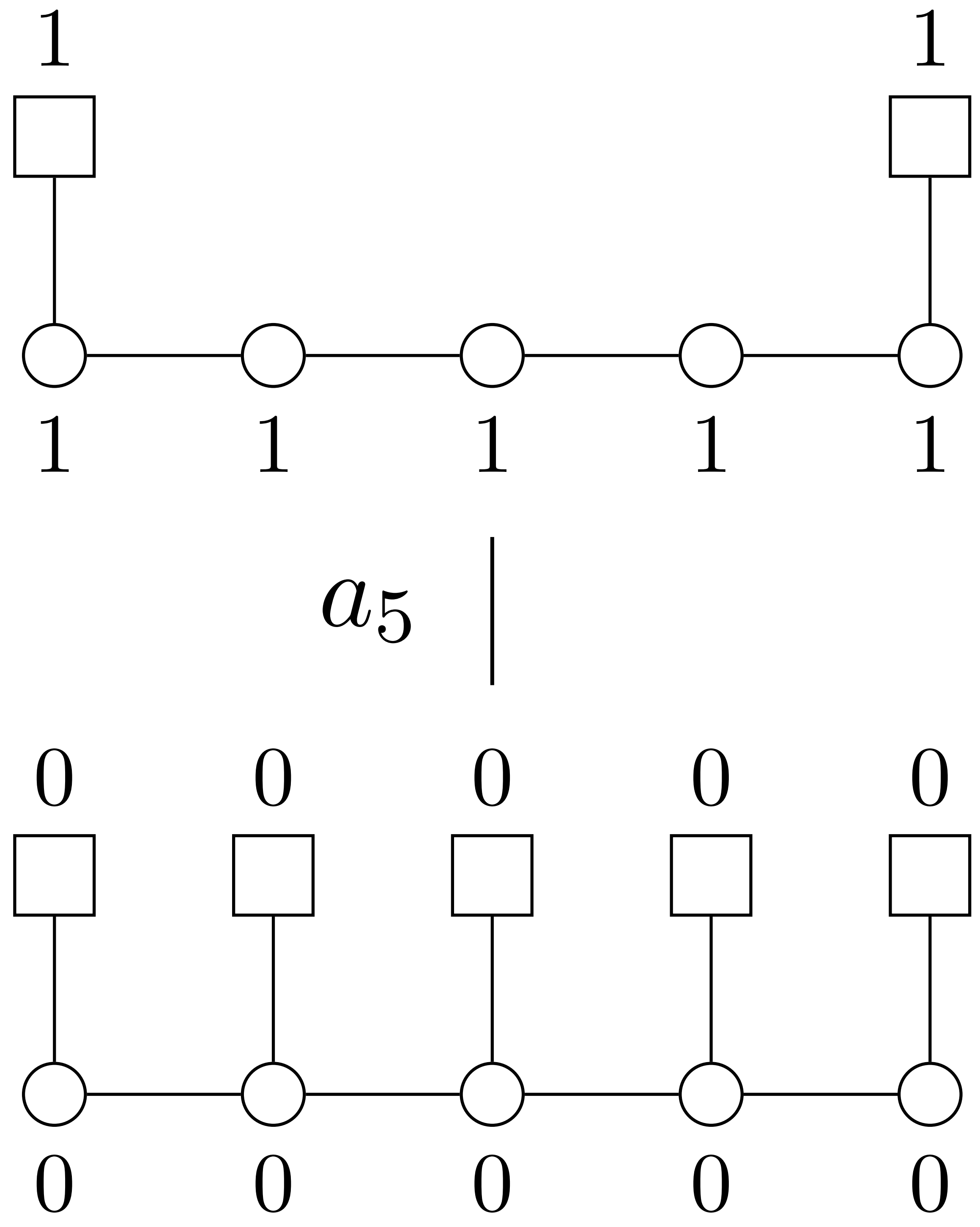
Quiver subtraction

nilpotent cone of A_5 — step 7



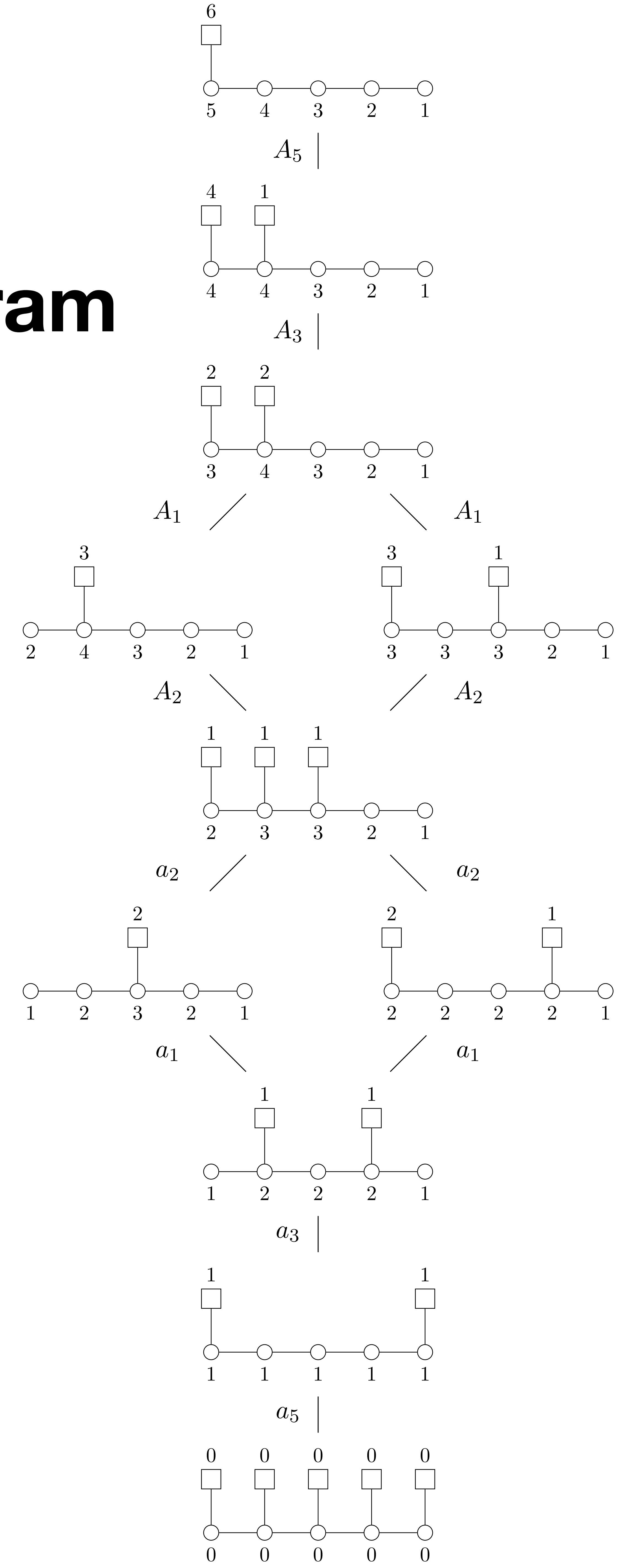
Quiver subtraction

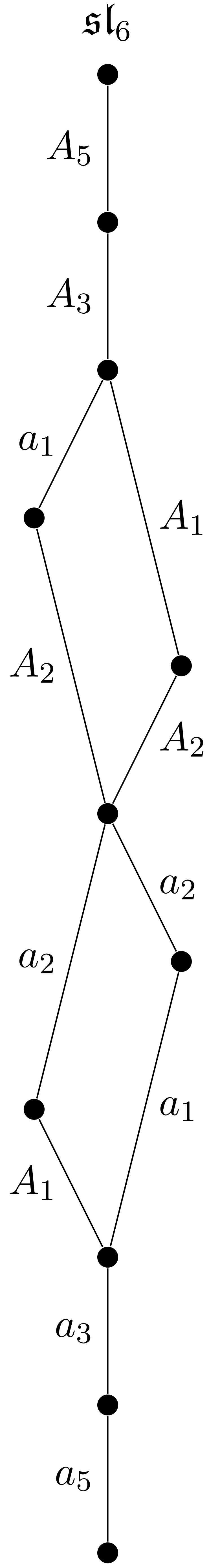
nilpotent cone of A_5 — step 8



Quiver subtraction

nilpotent cone of A_5 — final diagram





Brane Webs and Magnetic Quivers

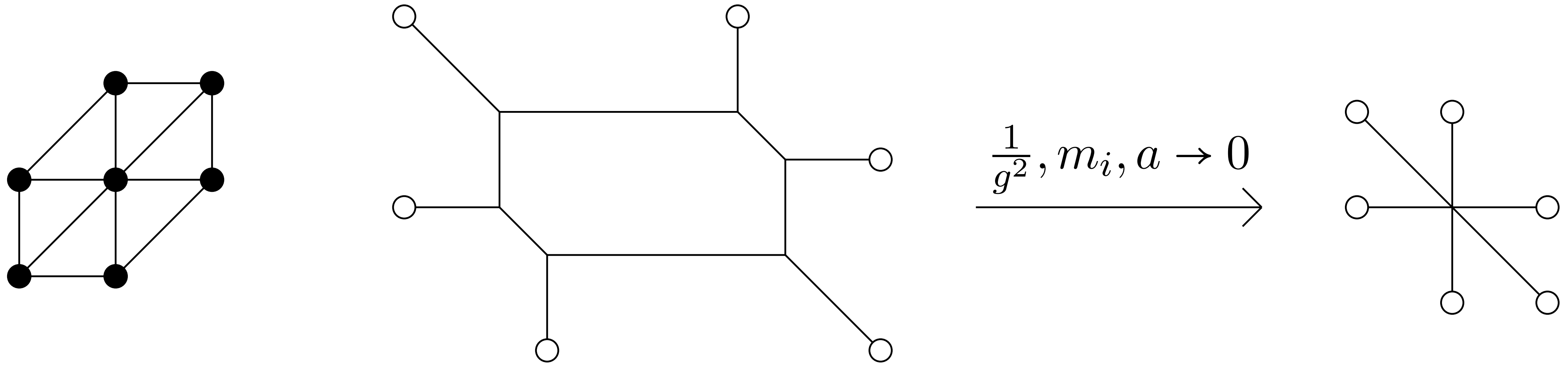
5d Higgs branches

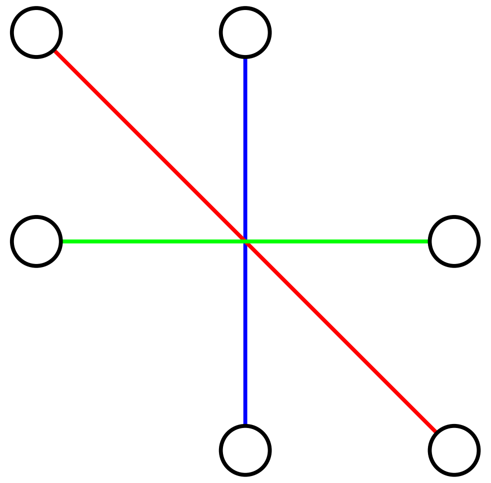
- We turn to a collection of methods to derive quivers from brane systems
- Our first set of examples are brane webs which help deriving many moduli spaces at weak and strong coupling

$E_3 = A_1 \times A_2$

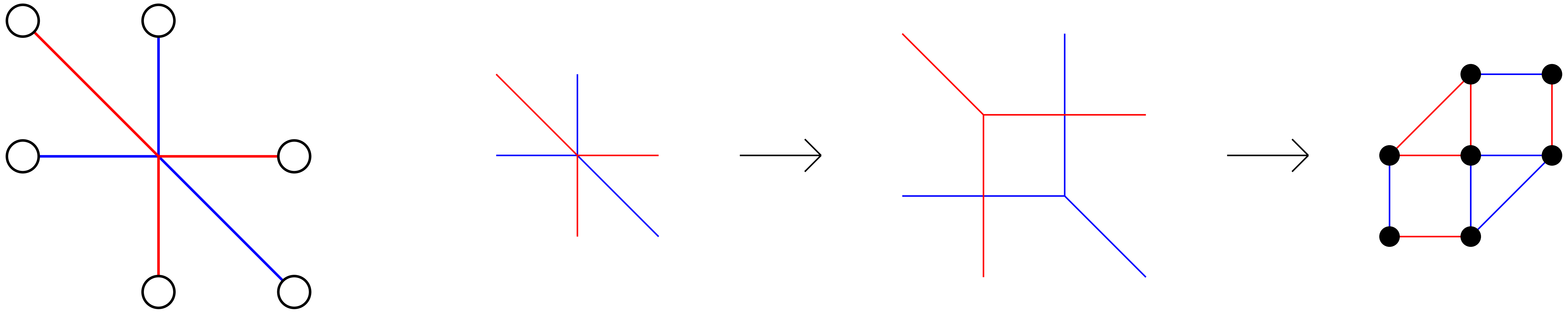
A union of two cones

$$\mathcal{H}_\infty\left(\begin{array}{c} 2 \\ \square \\ | \\ \circ \\ SU(2)_0 \end{array}\right) = \overline{\min_{A_2}} \cup \overline{\min_{A_1}}$$





$$\overline{\min_{A_2}} = \mathcal{C}^{3d}\left(\begin{array}{ccc} & 1 & \\ & \circ & \\ / & & \backslash \\ \circ & - & \circ \\ 1 & & 1 \end{array}\right)$$



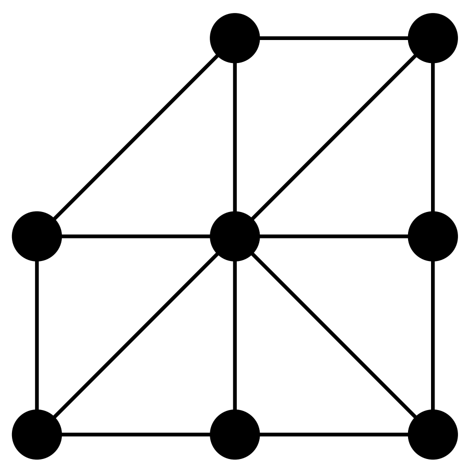
$$\overline{\min_{A_1}} = \mathcal{C}^{3d}\left(\begin{array}{c} 1 \\ \circ \\ \| \\ \circ \\ 1 \end{array}\right)$$

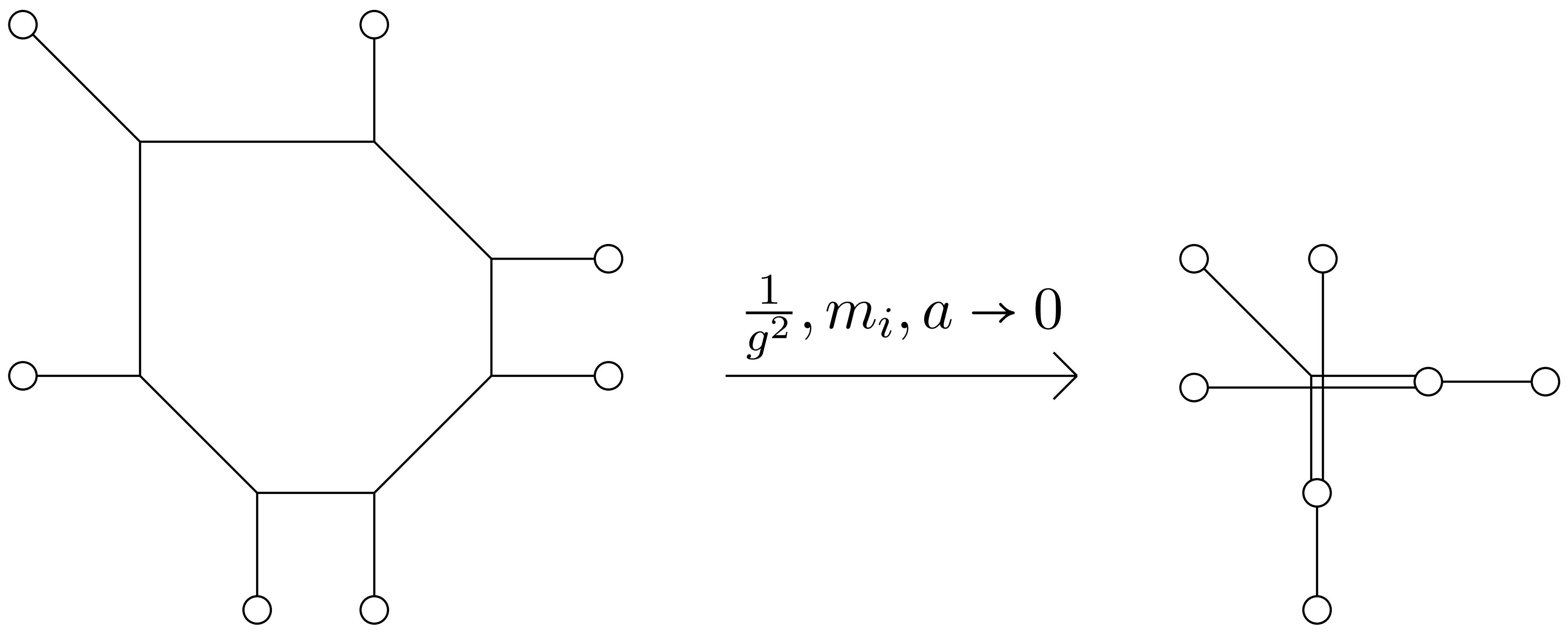
Necklace Quiver

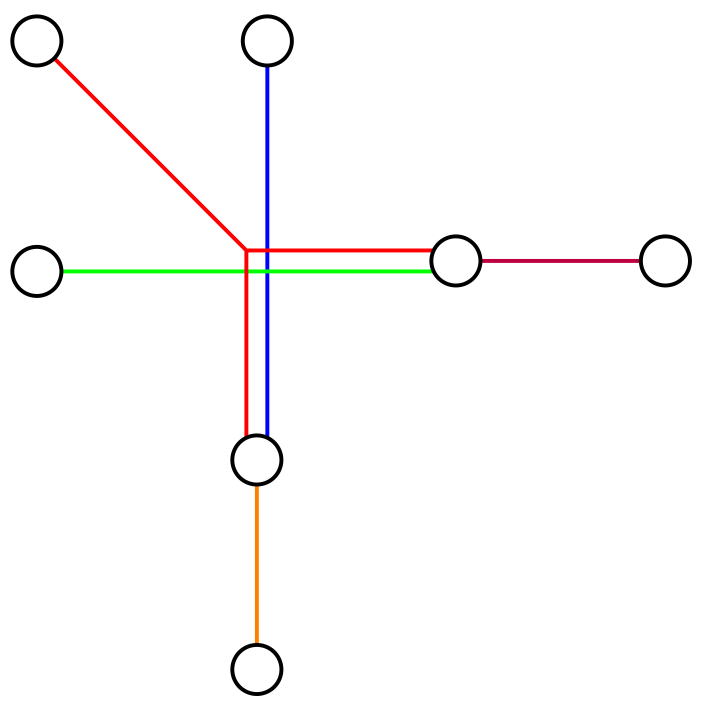
$E_4 = A_4$

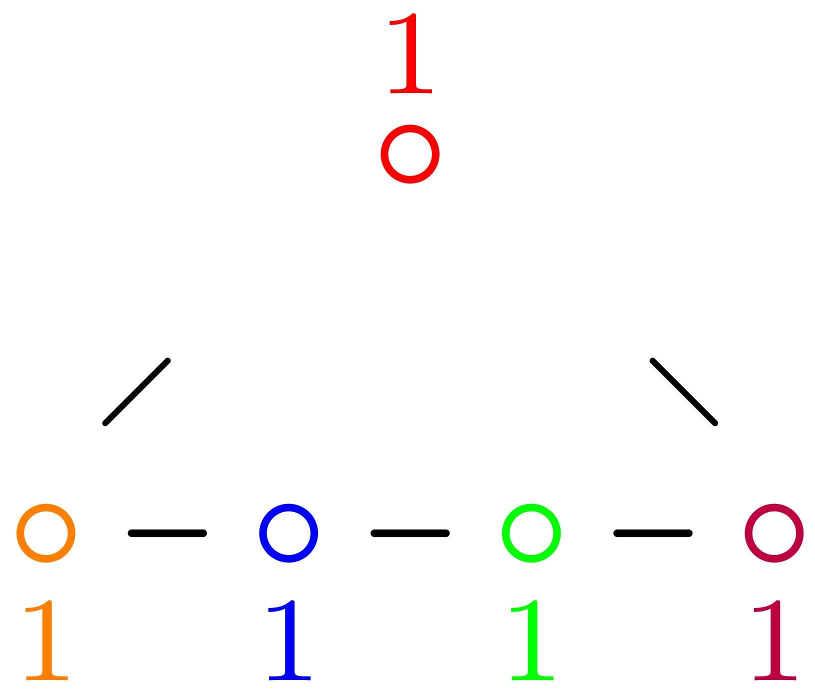
$$\mathcal{H}_\infty\left(\begin{array}{c}3\\ \square\\ | \\ \circ \\ SU(2)\end{array}\right) = \overline{\min_{A_4}}$$

$$\overline{\min_{A_4}} = \mathcal{C}^{3d}\left(\begin{array}{ccccccc} & & & \overset{1}{\circ} & & & \\ & / & & & & \backslash & \\ \underset{1}{\circ} & - & \underset{1}{\circ} & - & \underset{1}{\circ} & - & \underset{1}{\circ}\end{array}\right)$$



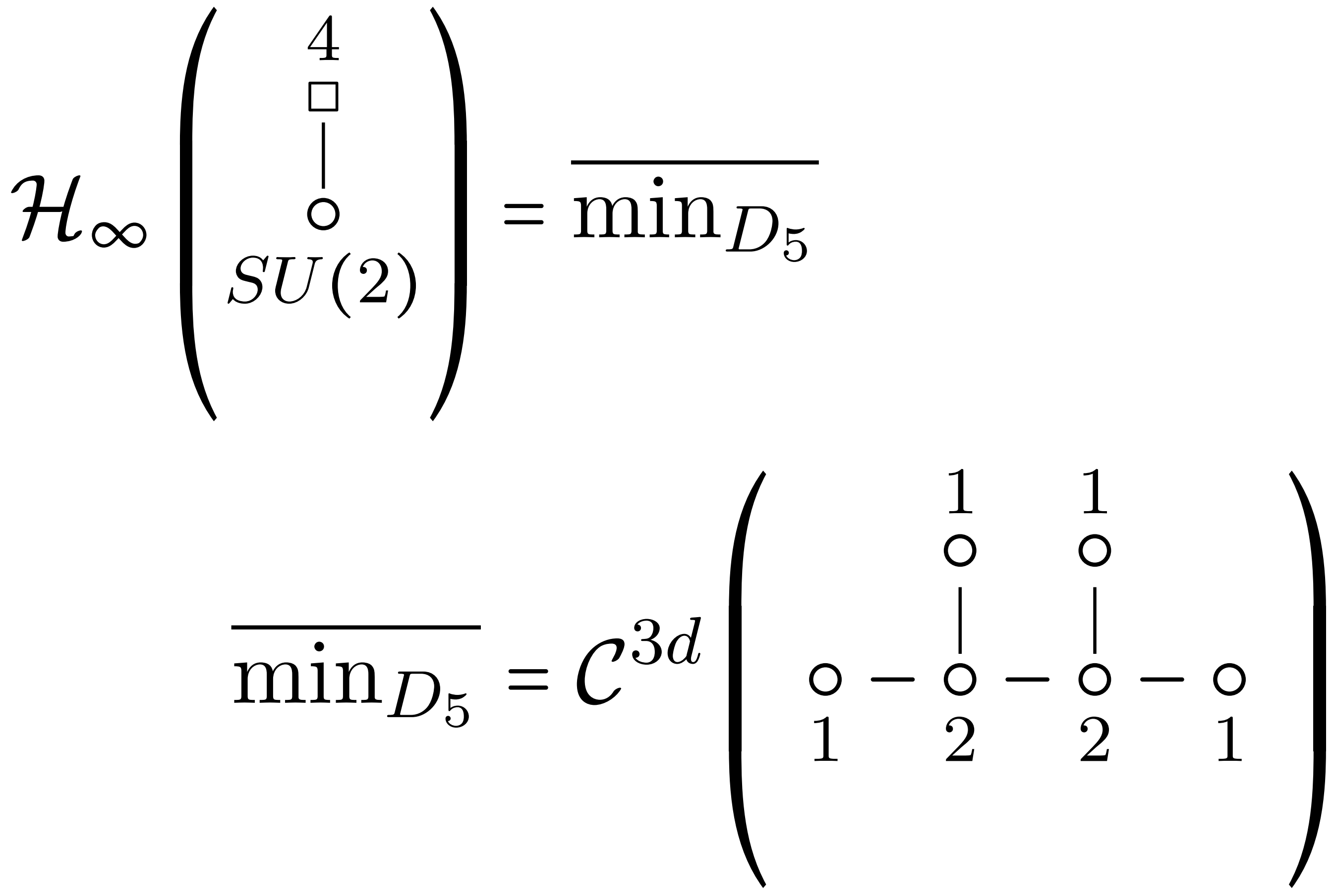


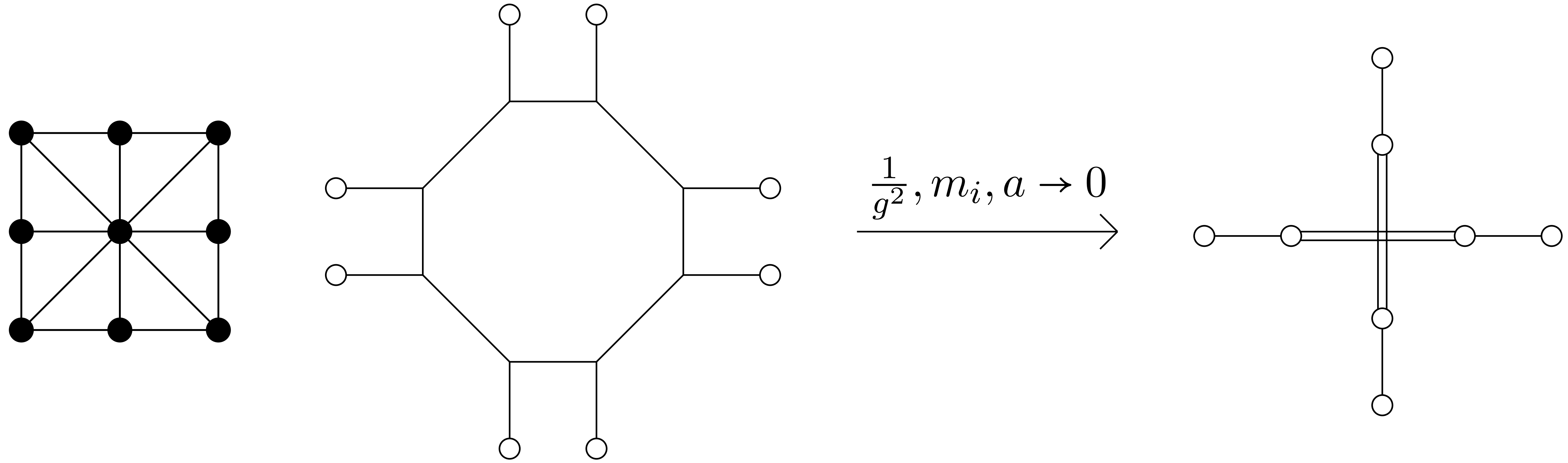


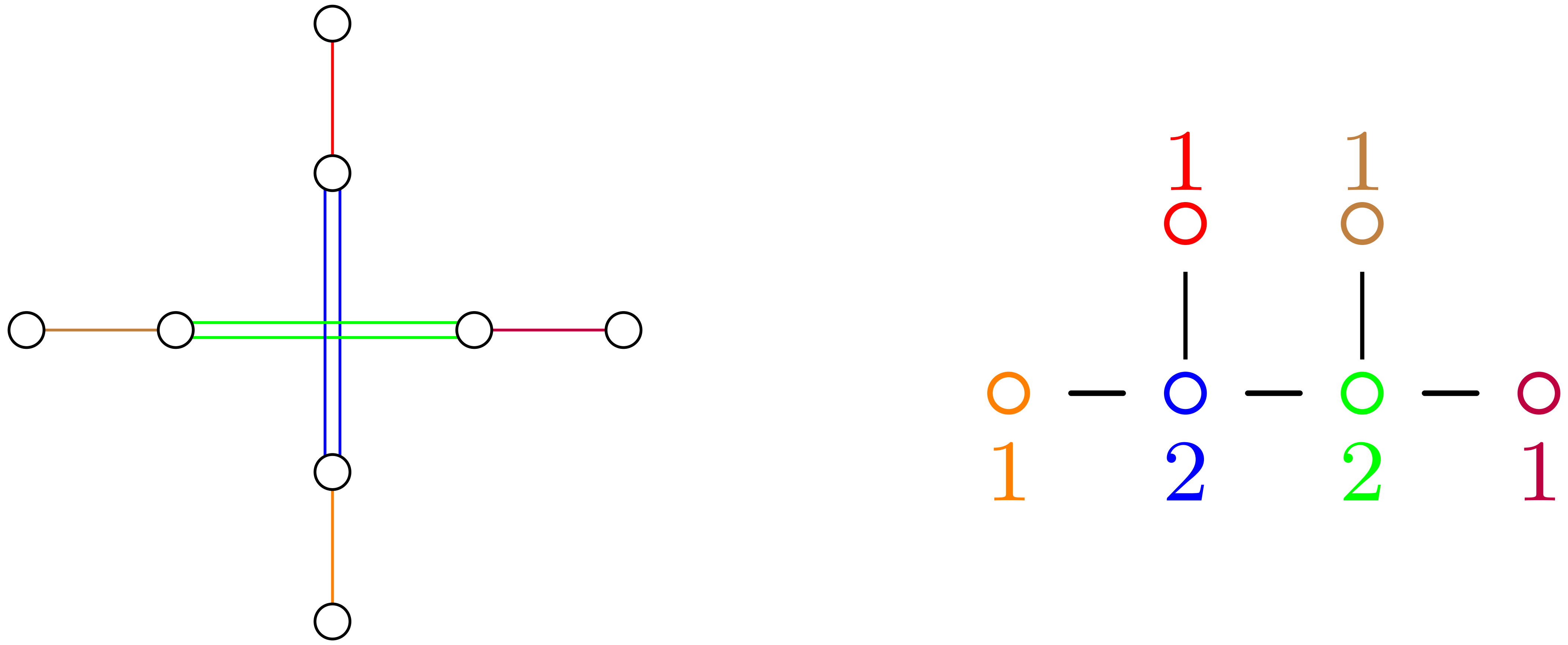


$E_5 = D_5$ Node multiplicity

SU(2) with 4 flavors



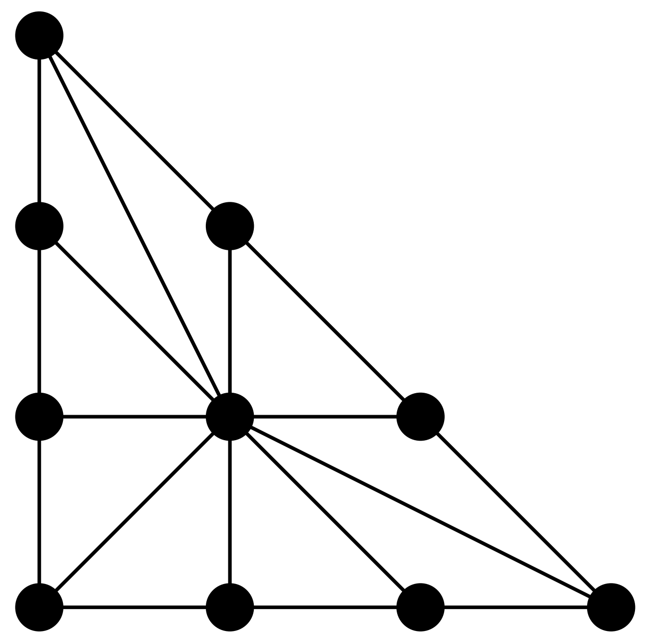


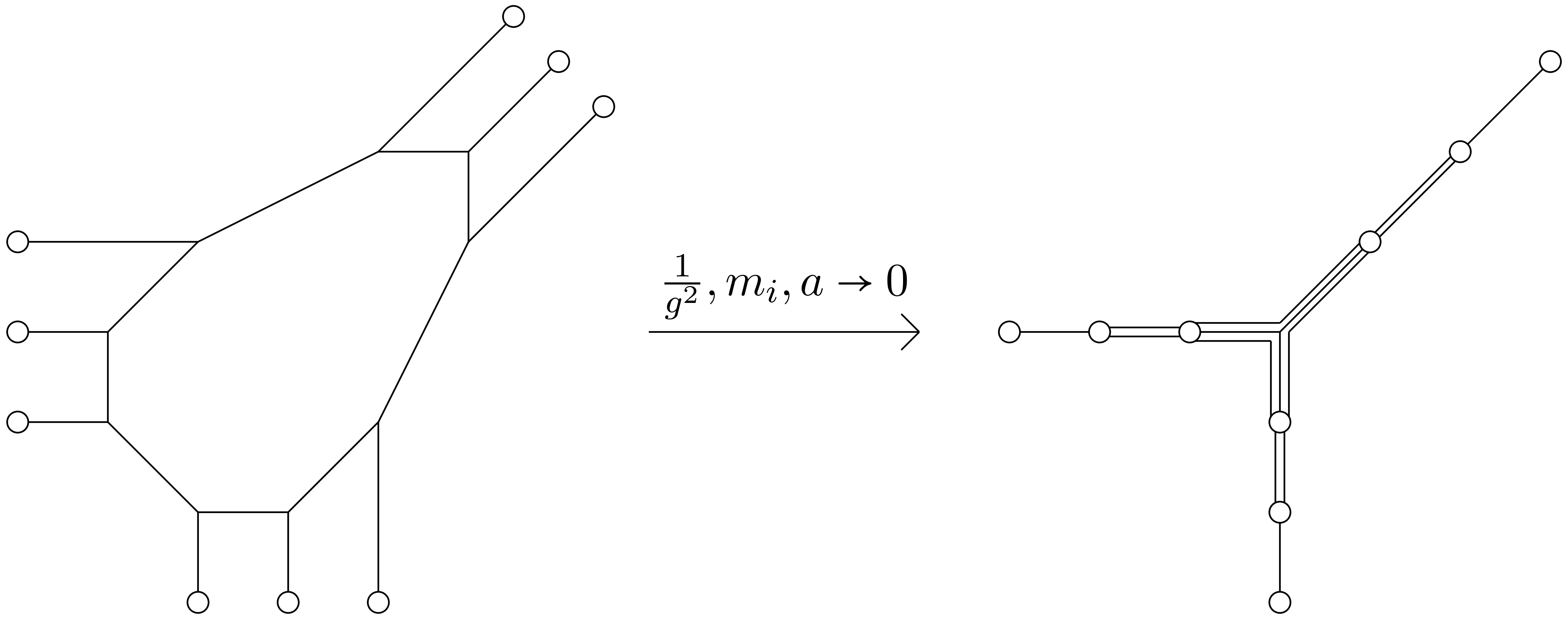


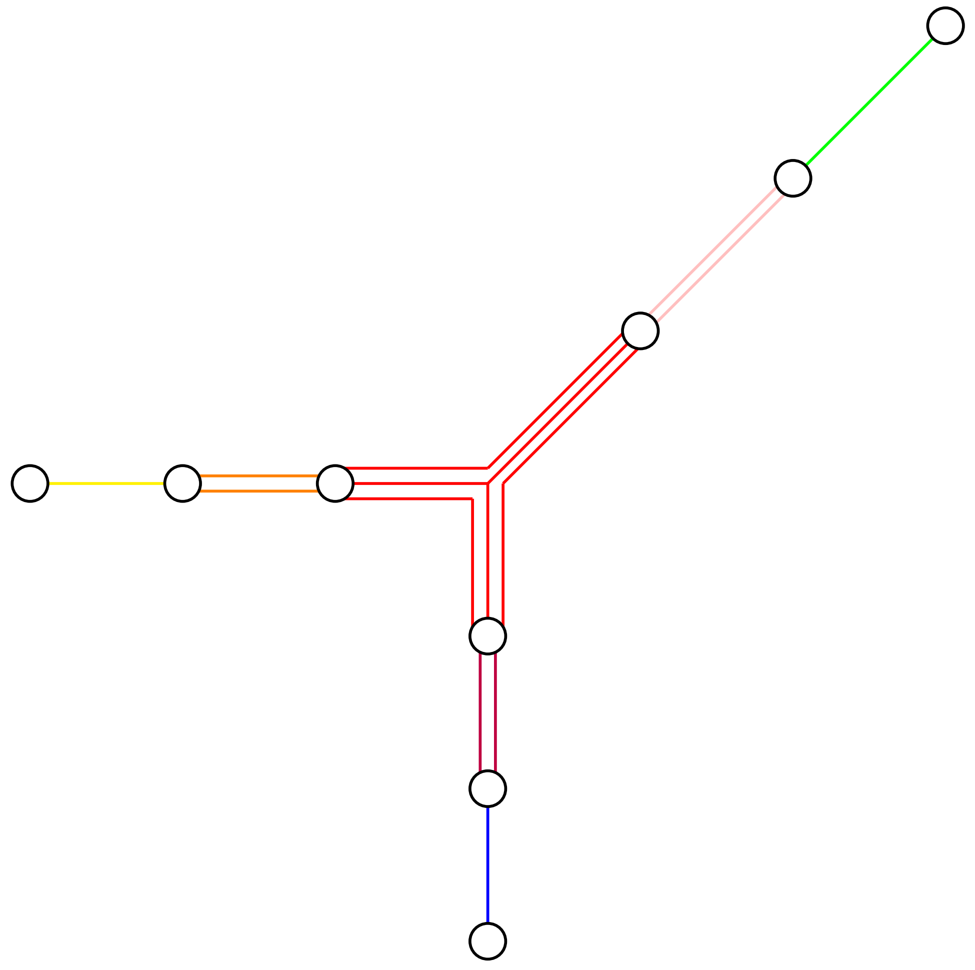
E_6

Exceptional algebra in brane physics

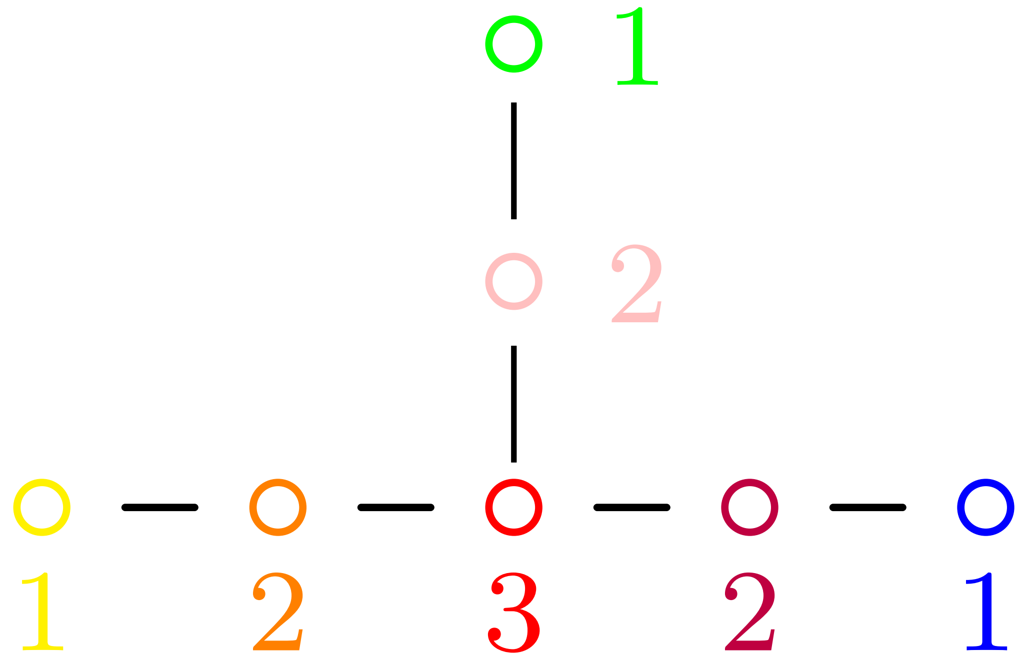
$$\mathcal{H}_\infty\left(\begin{array}{c} 5 \\ \square \\ | \\ \circ \\ SU(2) \end{array}\right) = \overline{\min_{E_6}}$$







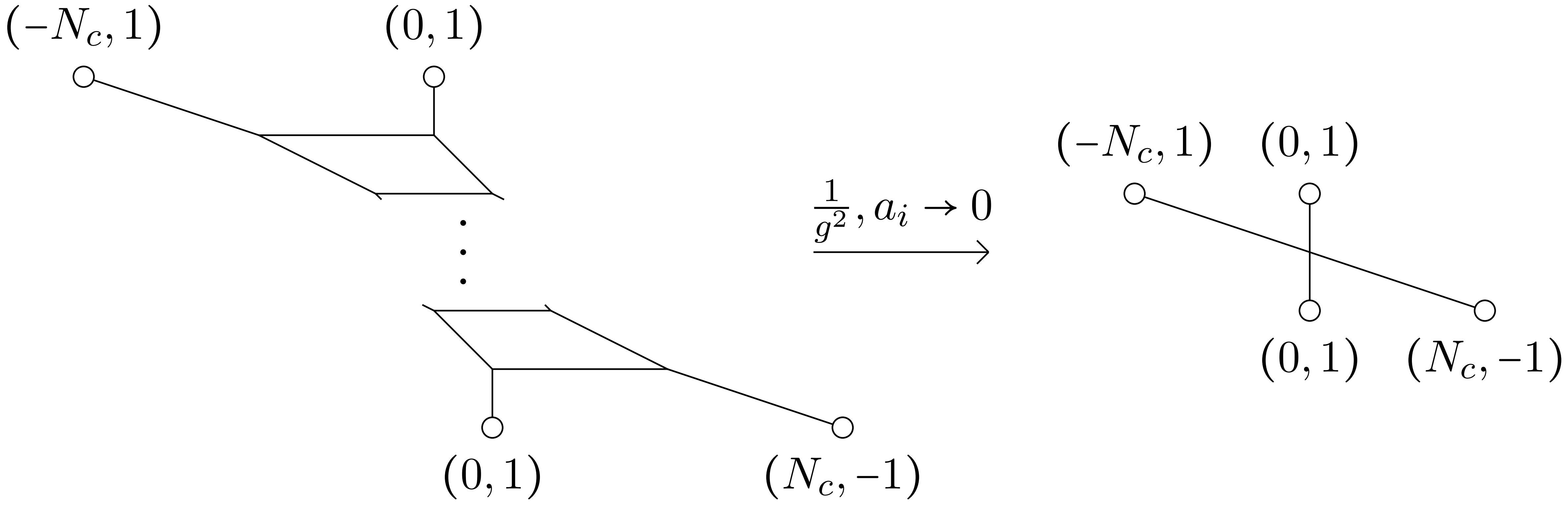
$$\overline{\min_{E_6}} = \mathcal{C}^{3d}\left(\begin{array}{ccccccccc} & & & & \circ\ 1 & & & & \\ & & & & | & & & & \\ & & & & \circ\ 2 & & & & \\ & & & & | & & & & \\ \circ & - & \circ & - & \circ & - & \circ & - & \circ \\ 1 & & 2 & & 3 & & 2 & & 1 \end{array}\right)$$

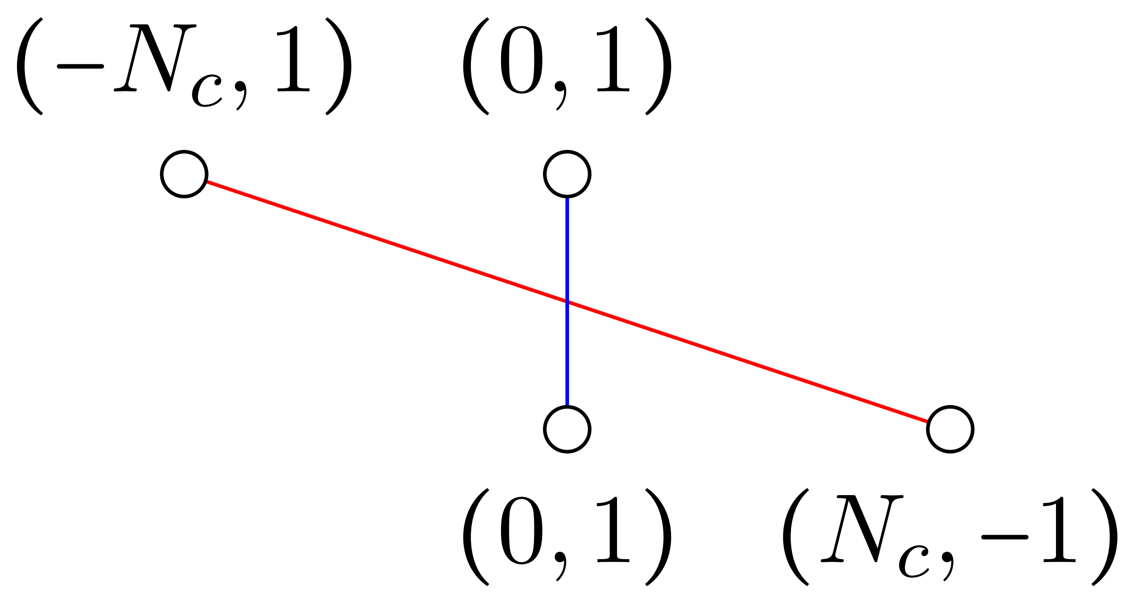


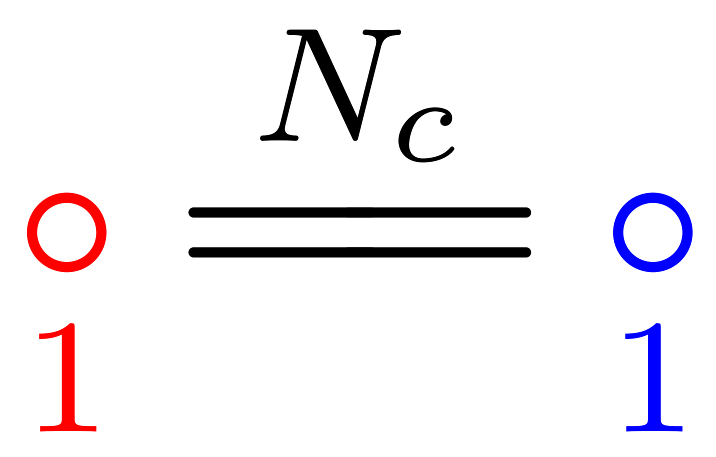
SYM — Edge Multiplicity

$SU(N)_0$

$$\mathcal{H}_\infty\left(\begin{array}{c} 0 \\ \square \\ | \\ \circ \\ SU(N_c)_0 \end{array}\right) = \mathbb{C}^2/\mathbb{Z}_{N_c}$$



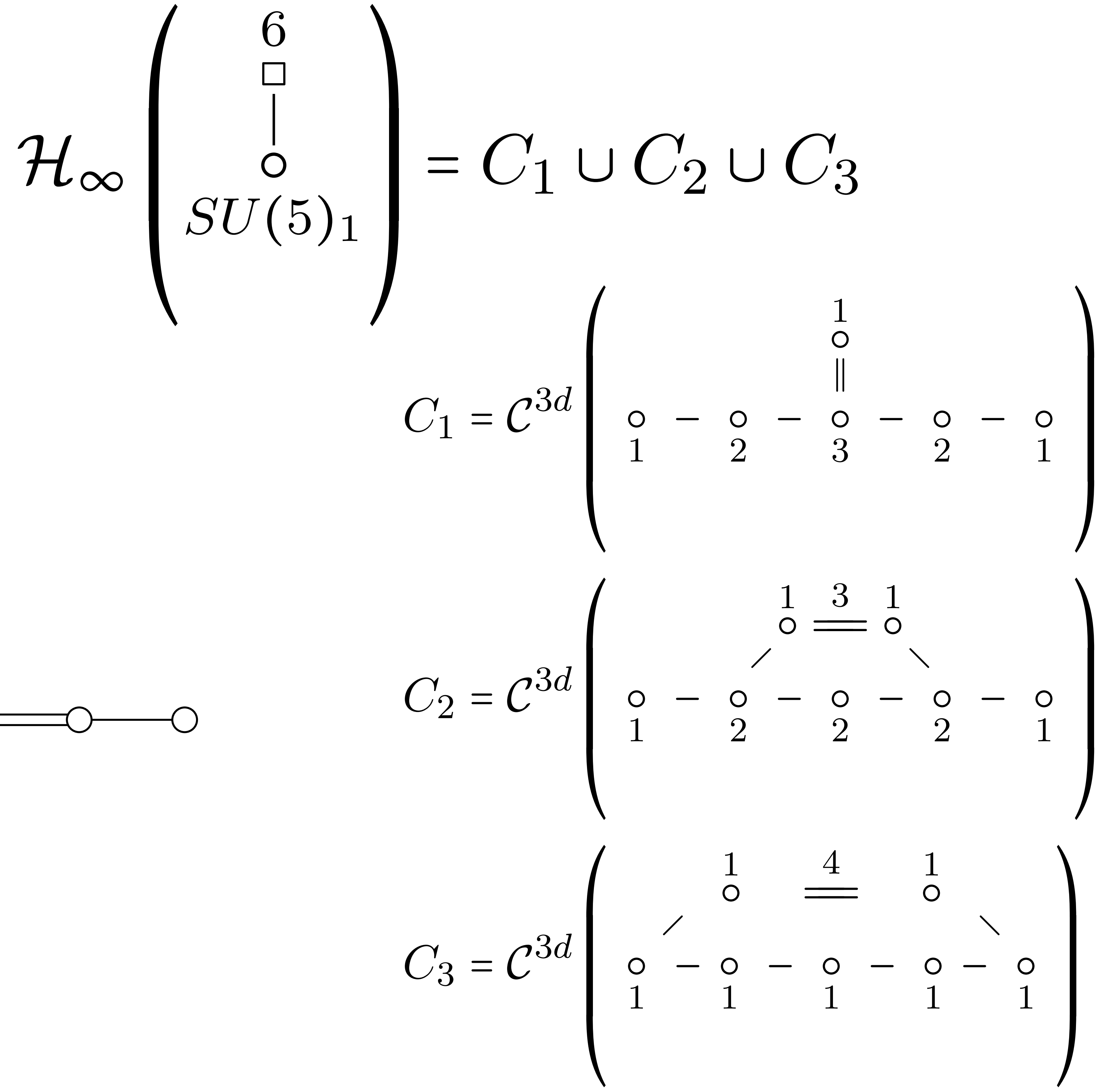


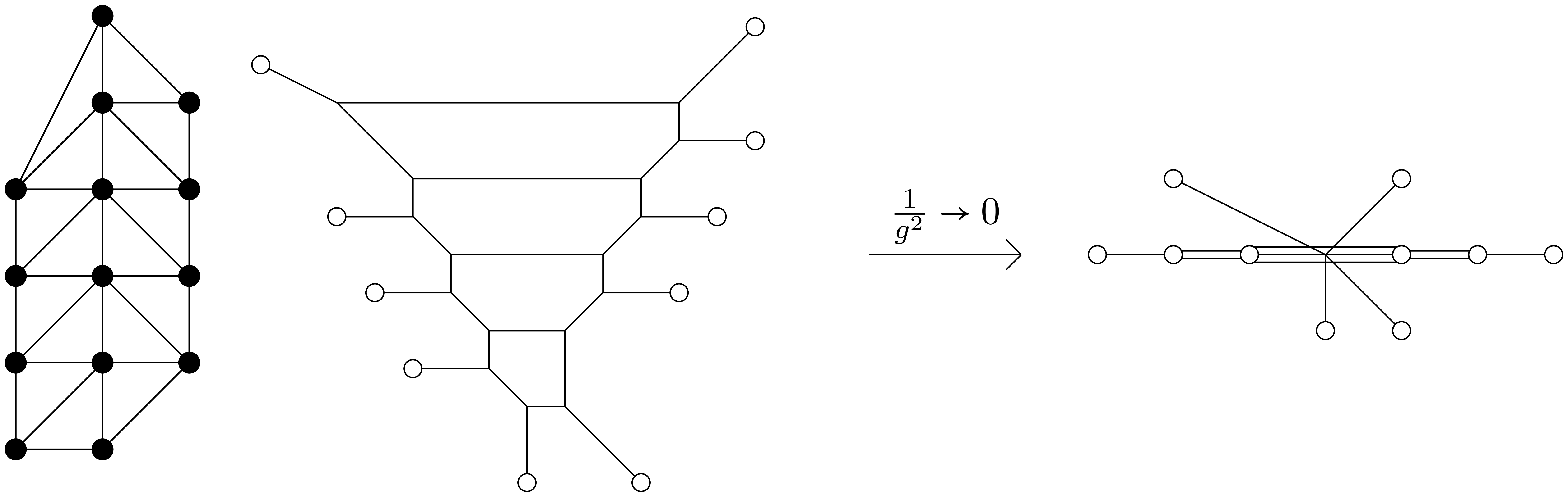


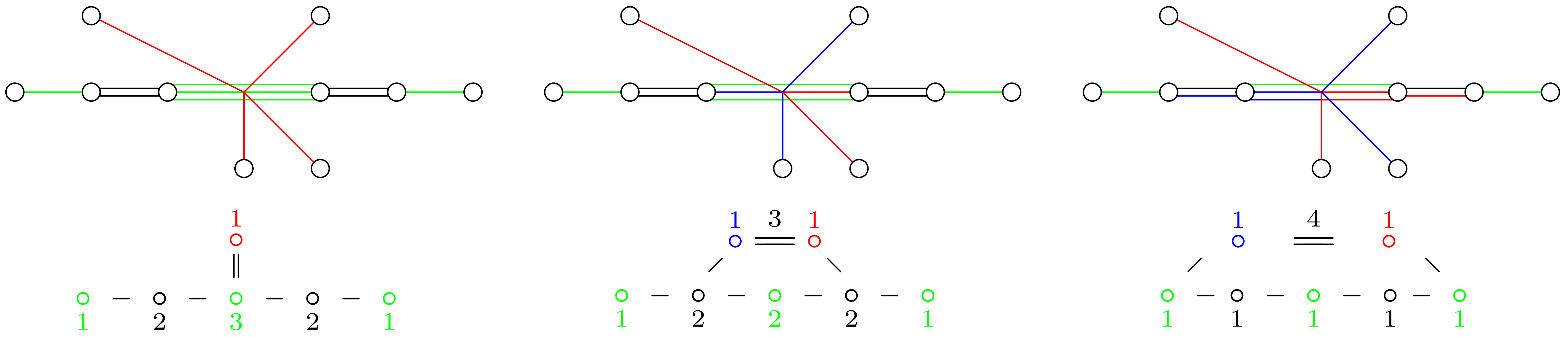
$$\mathbb{C}^2/\mathbb{Z}_{N_c} = \mathcal{C}^{3d}\left(\underset{1}{\circ} \overset{N_c}{=\!=\!=} \underset{1}{\circ}\right)$$

Union of 3 cones

new physics

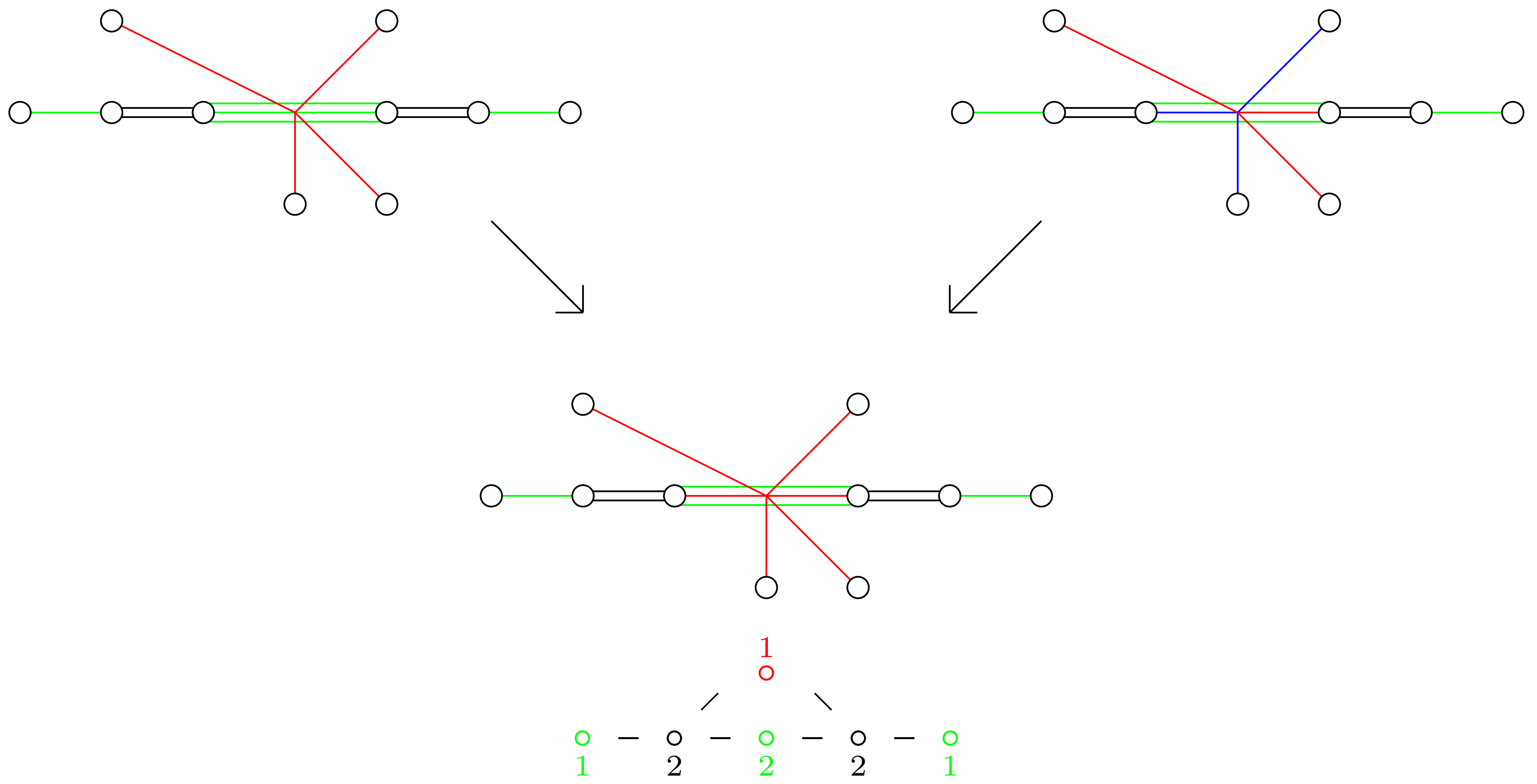


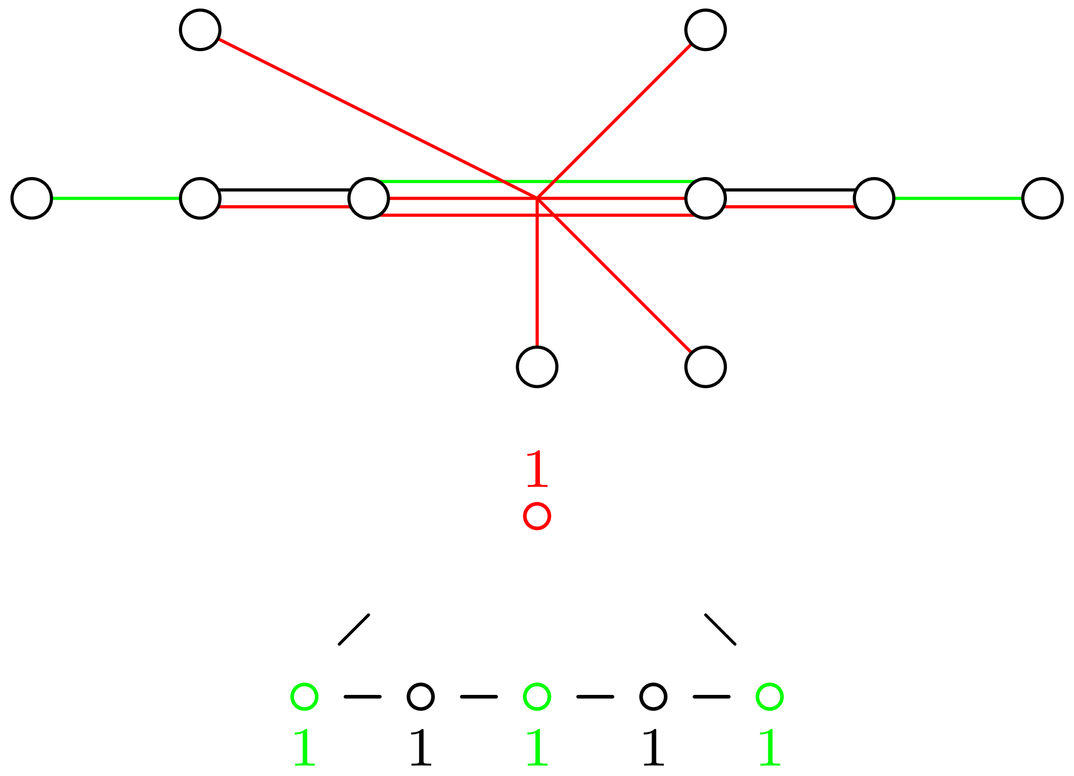


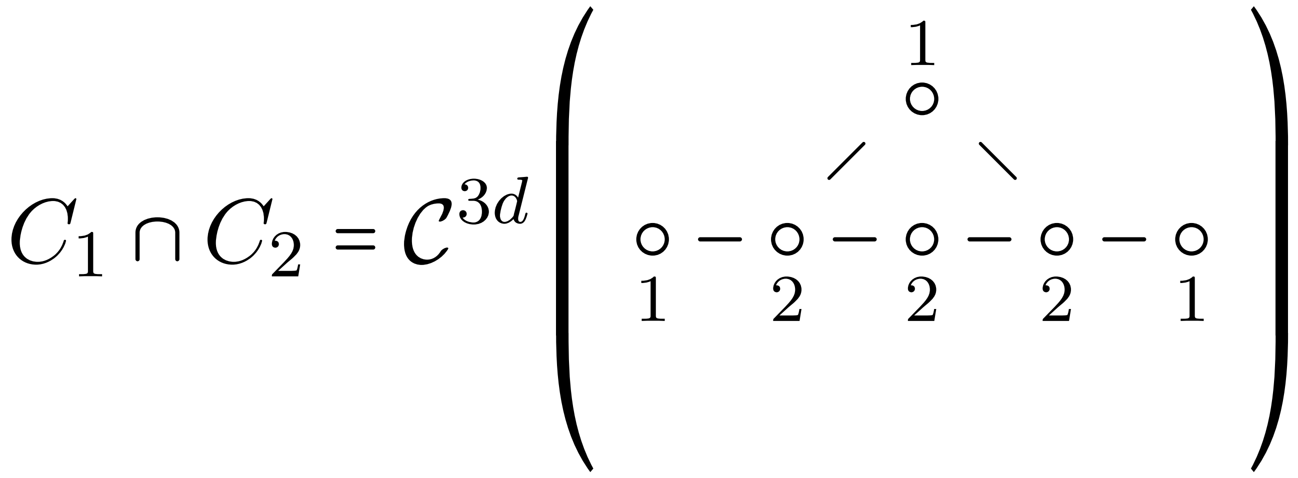


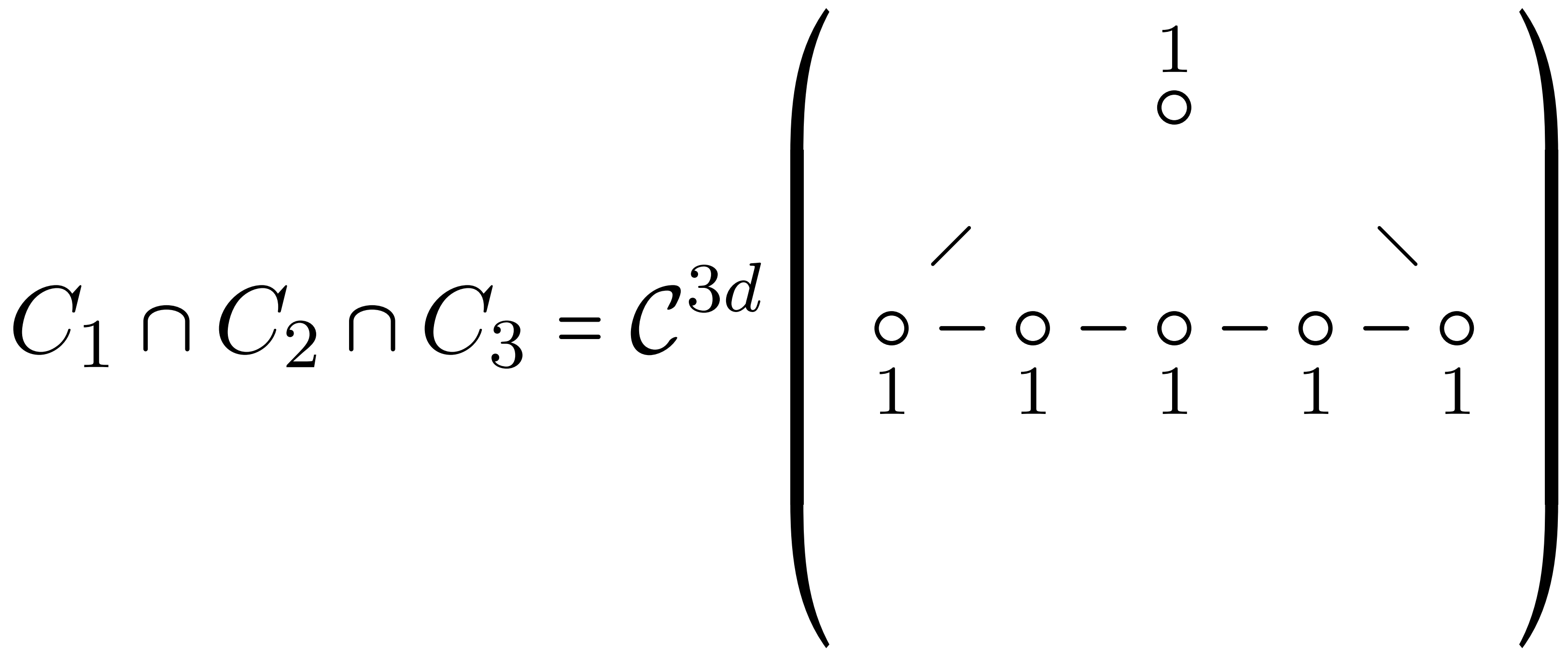
intersection of cones

of 2 cones and of 3 cones







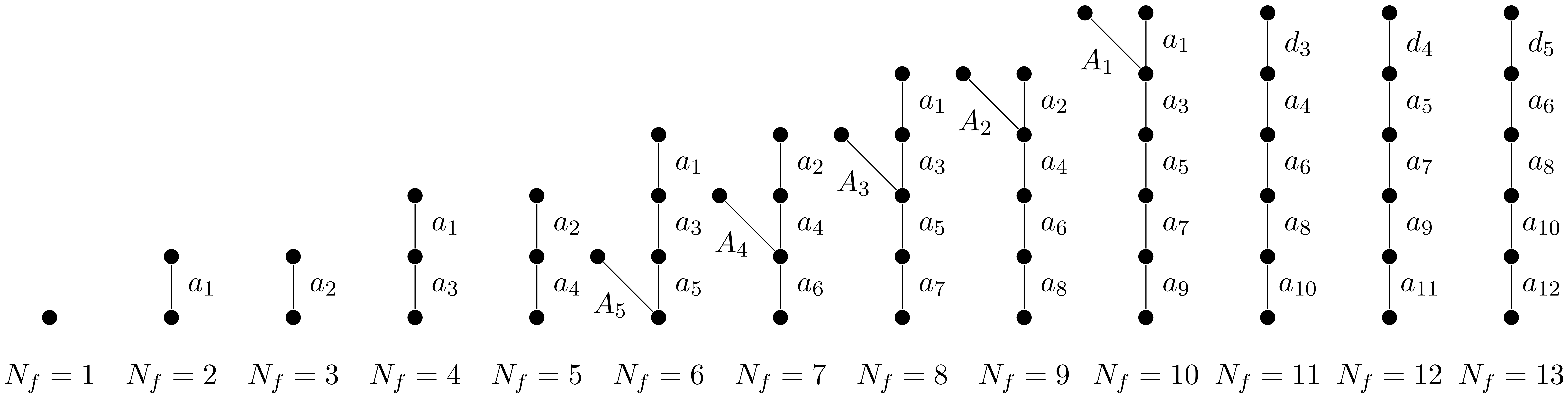


$$C_1 \cap C_2 = \overline{\mathrm{n.min}_{A_5}}$$

$$C_1 \cap C_2 \cap C_3 = \overline{\mathrm{min}_{A_5}}$$

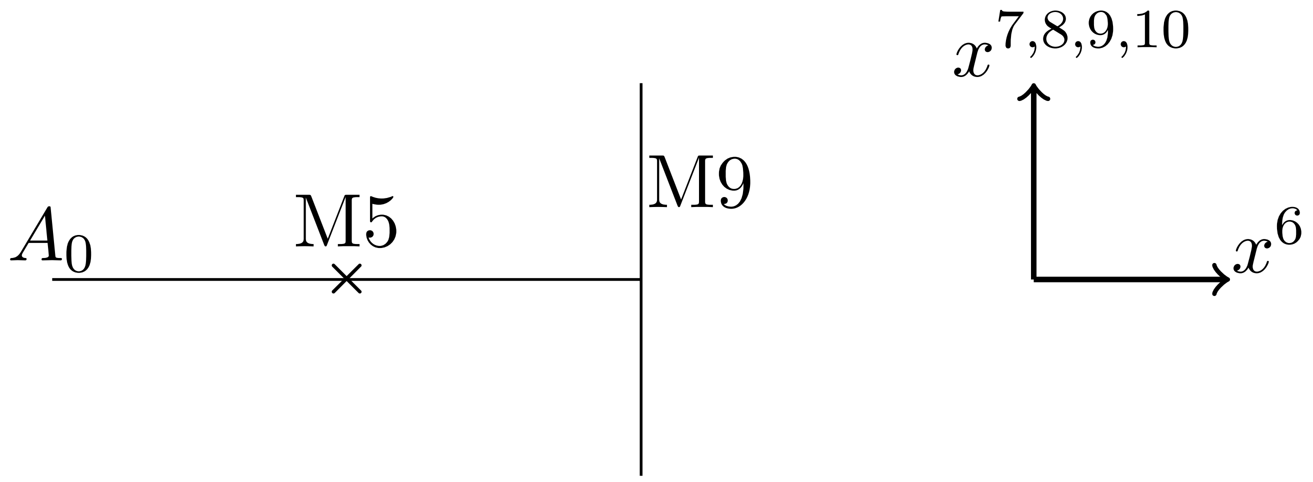
4d $\mathcal{N} = 2$ SU(6) with fundamental matter

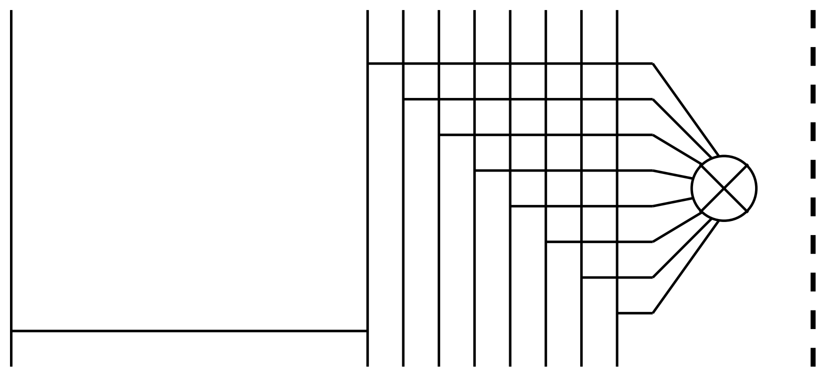
union of 2 cones

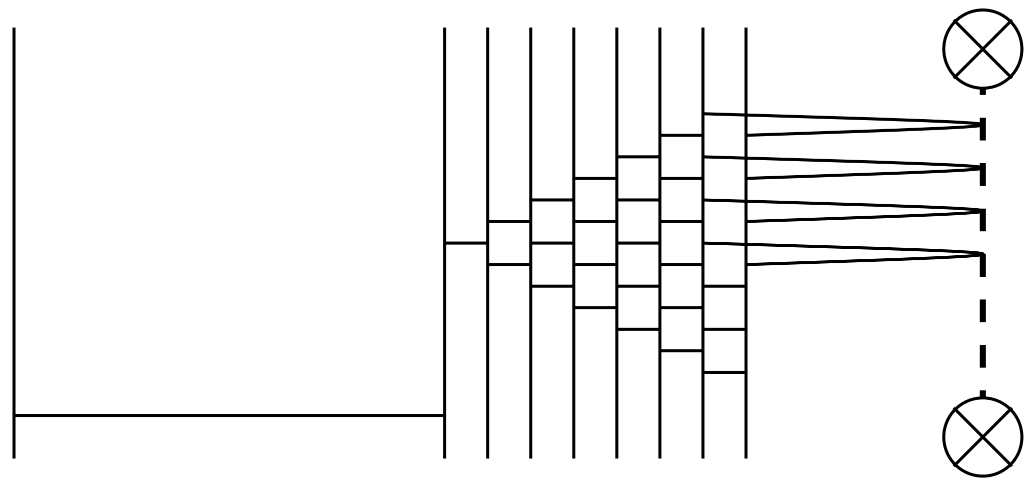


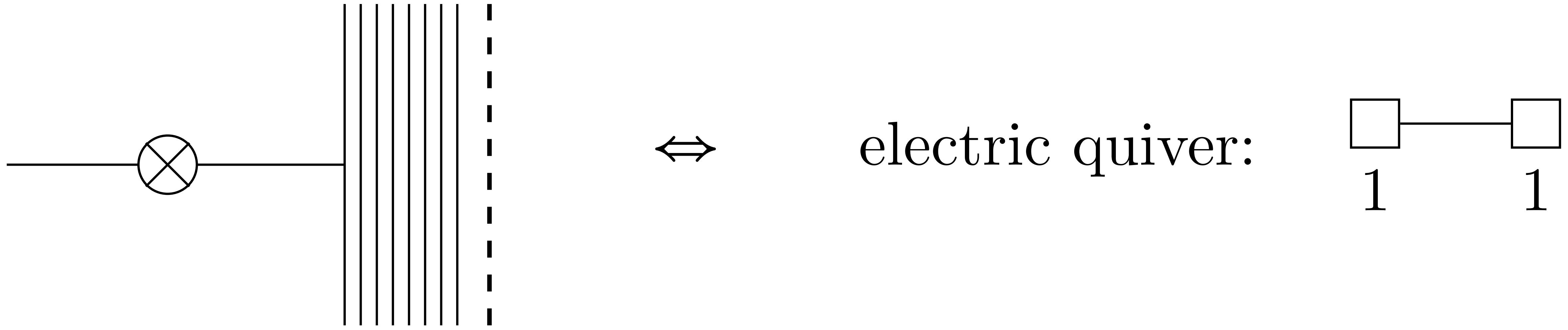
physical effects in 6d

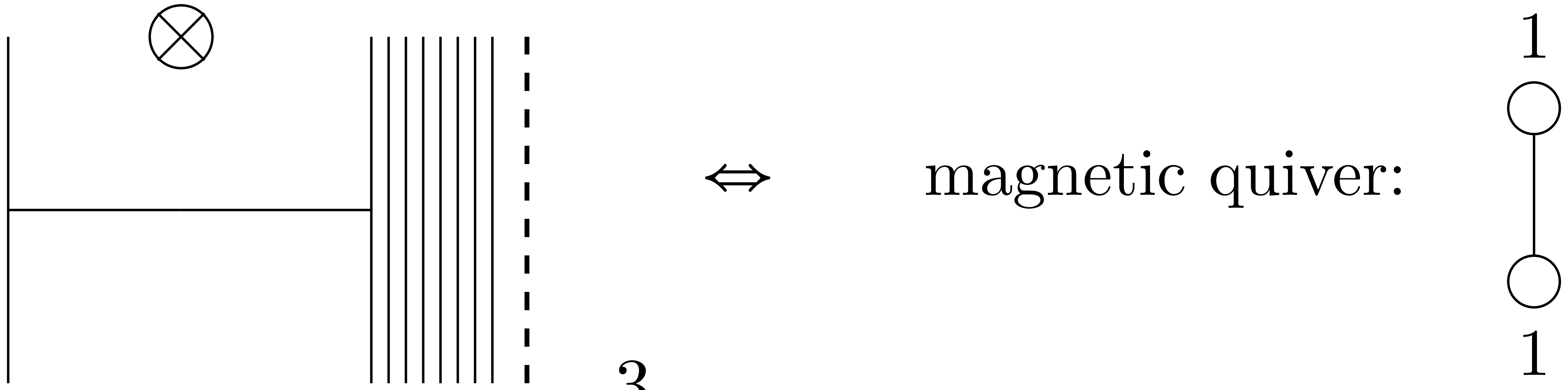
Small instanton transition: 1T <—> 29 H

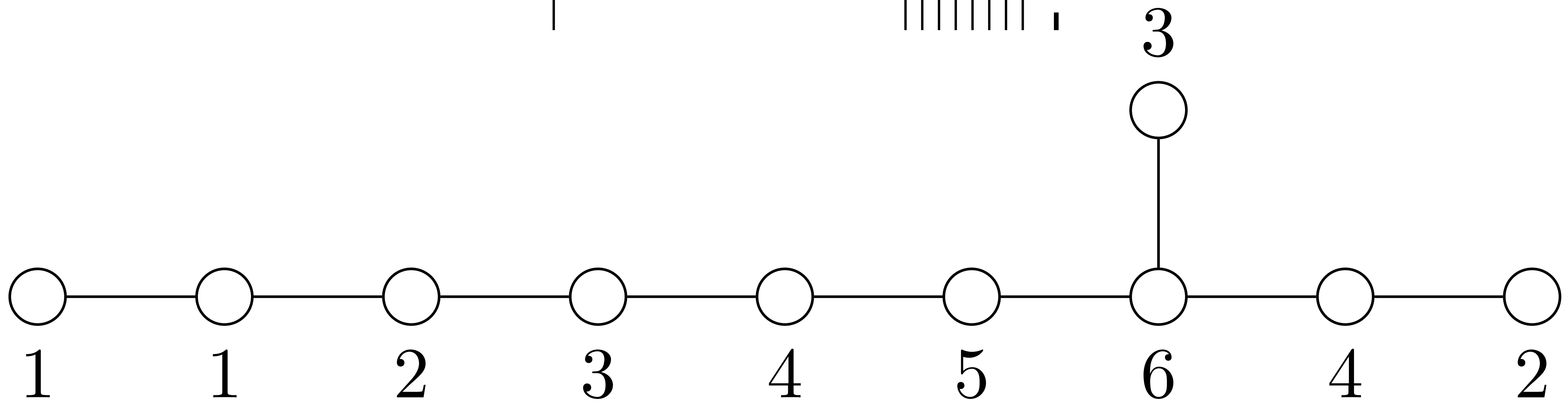








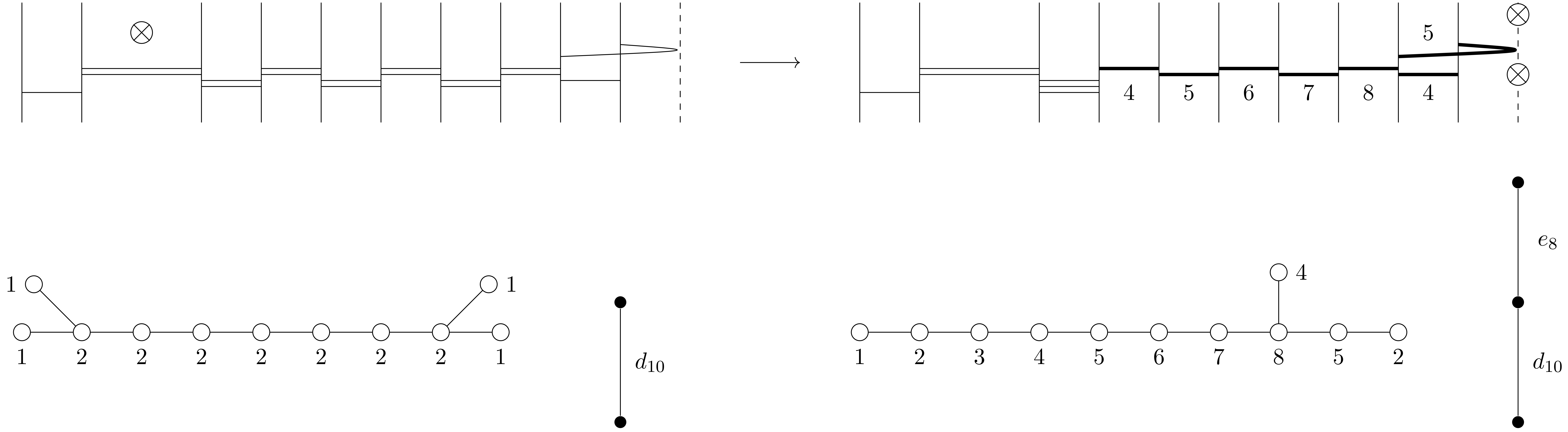




6d — small instanton transition

SU(2) with 10 flavors

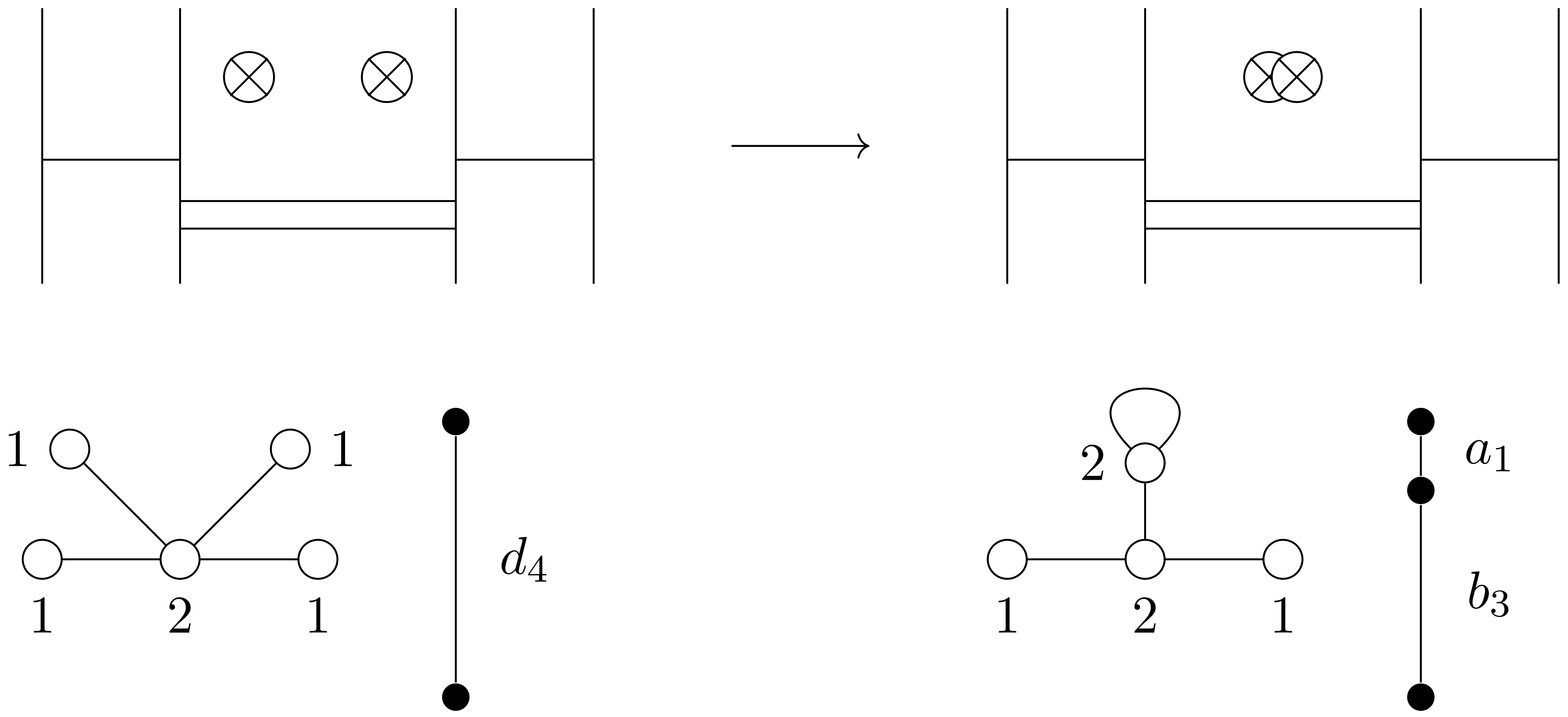
- The Classical Higgs branch — minimal nilpotent orbit of SO(20)
- The moduli space of 1 SO(20) instanton on $\mathbb{C}^2$



6d — tensionless strings and discrete gauging

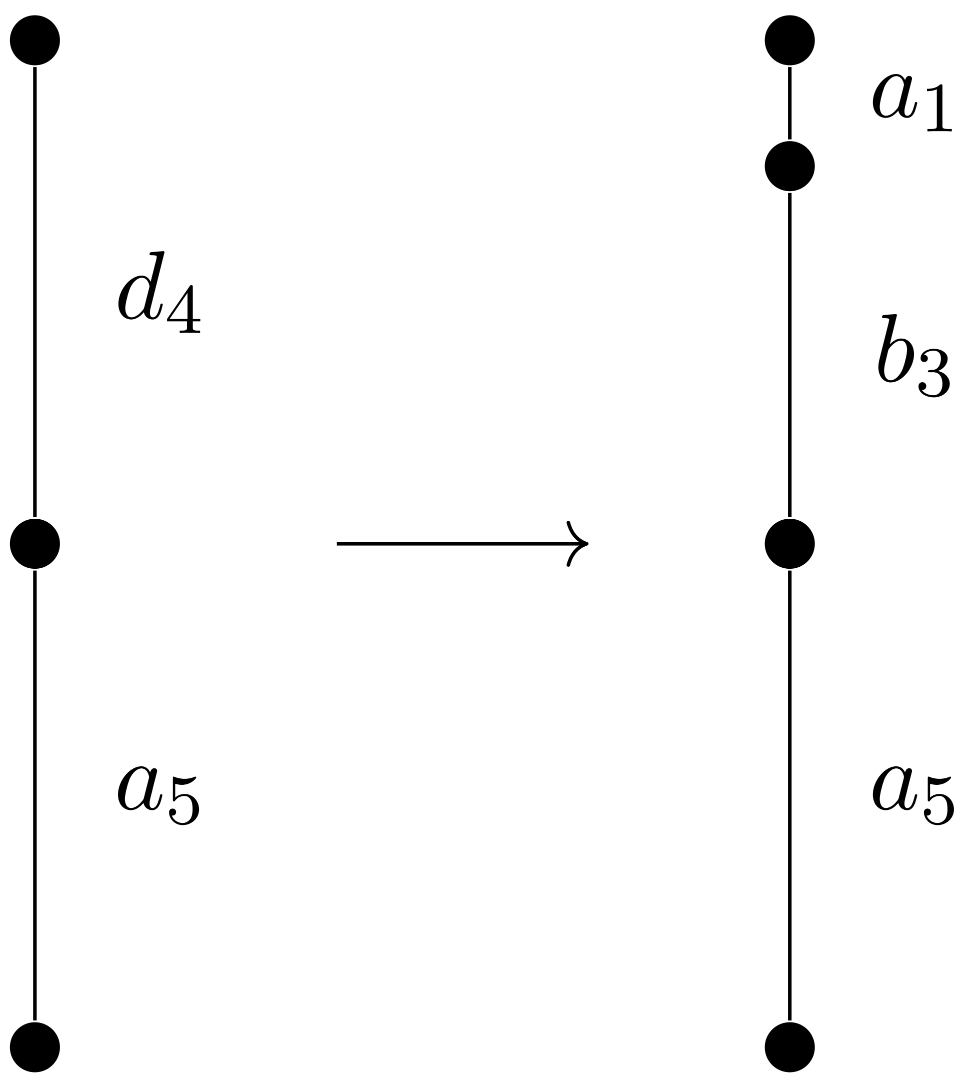
SU(2) with 4 flavors

- When n M5 branes coincide on an A-type singularity an S_n group is gauged
- There is symmetry reduction for the A_1, but not for higher values



6d — tensionless strings and S_2 gauging
SU(3) with 6 flavors

- Phase diagram — finite / infinite coupling



Summary

Changing the way we think

- Magnetic Quivers — encodes all data needed to understand strongly coupled moduli spaces
- Phase (Hasse) diagrams — changes the way we analyze symplectic singularities
- Brane systems — very instrumental in getting this progress
- Monopole formula — opened the window to all recent achievements

Thank you !