# Berezin-type and Fedosov-type quantizations on arbitrary smooth manifolds.

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# Introduction

- We wish to study systems which donot have symplectic structure (or group action) and study if they are a semi-classical limit of some quantum system as $\hbar$ goes to zero. A Poisson structure is needed but it is induced from another Poisson manifold where the manifold is embedded, usually a symplectic manifold (for Fedosov-type quantization) and $\mathbb{C}P^n$ or $\mathbb{C}^n$ (in case of Berezin-type of quantization).
- In [3] Berezin had focussed on Kähler manifolds and that too very special ones, and in [9] Fedosov had focussed on symplectic manifolds. We will talk of Berezin and Fedosov-type quantizations on smooth manifolds. The first one uses coherent states in a very essential way. The literature on coherent states is vast, see for instance a review by Ali, Gazeau, Antoine and Mueller [1]. After Berezin's orginal work [3], Berezin quantization has been generalized to many domains and manifolds, see Englis [8] for an example.
- We embed a compact smooth manifold into $\mathbb{C}P^n$ (using Whitney embedding or any other embedding) and pull back coherent states from $\mathbb{C}P^n$ [5]. Role of $\mathbb{C}P^n$ or $\mathbb{C}^n$

can be taken by other appropriate manifolds too. For instance, we can embed the compact smooth manifold in a symplectic manifold of twice the dimension which has a reproducing kernel Hilbert space (namely coherent states).

- ✦ In [3] Berezin gave a way of defining a star product on the symbol of bounded linear operators acting on a Hilbert space (with a reproducing kernel) on a Kähler manifold under certain conditions. There is a parameter in the theory (namely $\hbar$) such that in the limit $\hbar \to 0$ the star product tends to usual product and the commutator of the star product is proportional to the Poisson bracket upto first order. This is called the correspondence principle.

- ✦ We embed a compact smooth manifold into $\mathbb{C}P^n$ and pull back the reproducing kernel Hilbert space. Pullback coherent states give symbols of bounded linear operators induced from those corresponding to $\mathbb{C}P^n$ and it is easy to see that they satisfy the correspondence principle.

- ✦ In this context we recall that in [5], R.D. and Ghosh had considered pull back coherent states and totally real submanifolds of $\mathbb{C}P^n$ and defined pull back operators and their $\mathbb{C}P^n$-symbols and showed that they satisfied the correspondence principle. This was part of Ghosh's thesis [11]. **Our present work is a generalization**

**of this, as we donot need the condition totally real submanifolds. The condition of totally real submanifolds can give topological obstructions, so we circumvent that**.

- R.D. and Ghosh had also defined in [6] a Berezin-type quantization on even dimensional compact manifolds (of real dimension $2d$) by removing a set of measure zero and embedding it into $\mathbb{C}P^d$. This was part of Ghosh's thesis [11]. In this work our approach is slightly different. We embed a $d$ real dimensional manifold into $\mathbb{C}P^d$.
- In [9] Fedosov constructs a deformation quantization on symplectic manifolds. We embed a smooth compact manifold of real dimension $d$ into any symplectic manifold of real dimension $2d$ by the Whitney embedding and induce the Fedosov quantization on the submanifold.
- All the quantizations depend on the embedding.

# 1 Review of Geometric Quantization and Coherent states on $\mathbb{C}P^n$

✦ Berezin considers $\mathbb{C}P^n$ as a homogenous Kähler manifold and obtains the quantization. We give a local picture.

✦ Let $U_0 \subset \mathbb{C}P^n$ given by $U_0 = \{\tau_0 \neq 0\}$ where $[\tau_0, ...., \tau_n]$ are homogeneous coordinates on $\mathbb{C}P^n$. Let $(\mu_1, \mu_2, ...\mu_n)$ be inhomogenous coordinates on $U_0 \equiv \mathbb{C}^n$ such that $[1, \mu_1, \mu_2, ..., \mu_n] \in U_0$.

Here $\mathbb{C}P^n = \cup_{i=0}^{n} U_i$, where $U_i$ is the set of $(\mu_0, ...\mu_{i-1}, 1, \mu_{i+1}, ..., \mu_n)$, $U_i$ are the inhomogenous coordinate neighborhoods.

✦ The Fubini-Study form is given by $\Omega_{FS} = \sum_{i,j=1}^{n} \Omega_{ij}^{FS} d\mu_i \wedge d\bar{\mu}_j$, where the Kähler metric $G$ and the Kähler form $\Omega_{FS}$ are related by $\Omega_{FS}(X,Y) = G(IX,Y)$.

✦ The Poisson bracket of two functions $t$ and $s$:

$\{t,s\}_{FS} = \sum_{i,j=1}^{n} \Omega_{FS}^{ij} \left( \frac{\partial t}{\partial \bar{\mu}_j} \frac{\partial s}{\partial \mu_i} - \frac{\partial s}{\partial \bar{\mu}_i} \frac{\partial t}{\partial \mu_j} \right)$ where $(\Omega_{FS}^{ij})$ are the matrix coefficients of the inverse of the matrix $(\Omega^{FS}_{ij})$ of the Fubini-Study form.

✦ Let $H^{\otimes m}$ be the $m$-th tensor product of the hyperplane section bundle $H$ on $CP^n$. Recall that $m\Omega_{FS}$ is its curvature form and $m\Phi_{FS}$ is a local Kähler potential where $e^{m\Phi_{FS}(\mu,\bar{\mu})} = (1+|\mu|^2)^m$. Let $\{\phi_i\}_{i=1}^N$ be an orthonormal basis for the space of holomorphic sections.

✦ On $U_0$ the sections of $H^{\otimes m}$ are functions since the bundle is trivial when restricted to $U_0$. They can be identified with polynomials in $\{\mu_i\}_{i=1}^n$ of degree at most $m$.

✦ Let $\hbar = \frac{1}{m}$ be a parameter. Then $\{\phi_i\}$ depend on $\hbar$.

✦ We define

$dV(\mu) = |\Omega^n_{FS}(\mu)|_{U_0}| = \mathcal{G}(\mu)\Pi^n_{i=1}|d\mu_i \wedge d\bar{\mu}_i| = \mathcal{G}(\mu)|d\mu \wedge d\bar{\mu}| = \frac{|d\mu\wedge d\bar{\mu}|}{(1+|\mu|^2)^{n+1}}$ to be a volume form on $U_0$ which is identified with $\mathbb{C}^n$, where $\mathcal{G} = \det[g^{ij}|_{U_0}]$.

Then $V = \int_{\mathbb{C}^n} dV = \int_{\mathbb{C}^n} \frac{|d\mu\wedge d\bar{\mu}|}{(1+|\mu|^2)^{n+1}} < \infty$.

✦ Let $(c(m))^{-1} = \int_{U_0} \frac{1}{(1+|\nu|^2)^m} dV(\nu) = \int_{U_0} e^{-m\Phi_{FS}(\nu,\bar{\nu})} dV(\nu)$

Let an innerproduct on the space of functions on $U_0$ be defined as

$$\langle f, g\rangle = c(m) \int_{U_0} \frac{\overline{f(\nu)} g(\nu)}{(1+|\nu|^2)^m} dV(\nu) = c(m) \int_{U_0} \overline{f(\nu)} g(\nu) e^{-m\Phi_{FS}(\nu,\bar{\nu})} dV(\nu).$$

✦ Also, $D_{(q_1,q_2,\ldots q_n;q)} = c(m) \int_{U_0} \frac{|\nu_1|^{2q_1} \ldots |\nu_n|^{2q_n}}{(1+|\nu|^2)^m} dV(\nu)$, where $q_i's$ are all possible positive integers such that $q_1 + \ldots + q_n = q; q = 0, \ldots, m$.

Let $\Phi_{(q_1,q_2,\ldots,q_n;q)}(\mu) = \frac{1}{\sqrt{D_{(q_1,\ldots,q_n;q)}}} \mu_1^{q_1} \ldots \mu_n^{q_n}$ where $q_1 + \ldots + q_n = q; q = 0, \ldots, m$.

It is easy to check that $\{\Phi_{(q_1,\ldots,q_n;q)}\}$ are orthonormal in $\mathbb{C}^n$ with respect to the inner product defined as above and are restriction of a basis for sections of $H^{\otimes m}$ to $U_0$. The span of these form a Hilbert space with the above inner product.

✦ **Definition:** The Rawnsley-type coherent states [15], [16] are given on $U_0$ by $\psi_\mu$ reading as follows:

$\psi_\mu(\nu) := \sum_{q_1+q_2+\ldots+q_n=q;q=0,1,\ldots,m} \overline{\Phi_{(q_1,q_2,\ldots,q_n;q)}(\mu)} \Phi_{(q_1,q_2,\ldots,q_n;q)}(\nu)$.

In short hand notation $\psi_\mu := \sum_I \overline{\Phi_I(\mu)} \Phi_I$ where the multi-index $I = (q_1, \ldots, q_n; q)$ runs over the set $q_1 + \ldots + q_n = q; q = 0, \ldots, m$.

✦ This is a reproducing kernel in the sense below.

✣ Reproducing kernel property. If $\Psi$ is any other section, then $\langle \psi_\mu, \Psi\rangle = \Psi(\mu)$. In particular, $\langle\psi_\mu, \psi_\nu\rangle = \psi_\nu(\mu)$.

- Resolution of identity property:

$$c(m)\int_{U_0} \langle\Psi_1, \psi_\mu\rangle \langle\psi_\mu, \Psi_2\rangle\, e^{-m\Phi_{FS}(\mu,\bar{\mu})} dV(\mu) = \langle\Psi_1, \Psi_2\rangle\,.$$

  In particular,

$$c(m)\int_{U_0} \langle\psi_\nu, \psi_\mu\rangle \langle\psi_\mu, \psi_\nu\rangle\, e^{-m\Phi_{FS}(\mu,\bar{\mu})} dV(\mu) = \langle\psi_\nu, \psi_\nu\rangle\,.$$

- Overcompleteness property holds.

The proofs of these are easy and can be found for instance in [6]. It is in general true of Rawnsley type coherent states.

# 2 A reproducing kernel Hilbert space on a compact smooth manifolds and coherent states

- In this section we construct a reproducing kernel Hilbert space and coherent states on a compact smooth manifolds by emedding it into $\mathbb{C}P^d$. This generalizes a result of R.D.and Ghosh [5]. We proceed similar to [5], but we donot need the "totally real" condition. We use the Whitney embedding of any compact smooth manifold. Any other smooth embedding will also work.
- Let $M^d$ be a compact smooth manifold of real dimension $d$. Let $\epsilon : M \to \mathbb{R}^{2d}$ be any embedding (for instance Whitney embedding). Let $i : \mathbb{R}^{2d} \to \mathbb{C}P^d$ be the inclusion such that $R^{2d}$ is identified with $U_0 \subset \mathbb{C}P^d$ and $\chi = i \circ \epsilon$. It is clear that $\chi : M \to \mathbb{C}P^d$ is an embedding and that $\epsilon : M \to \epsilon(M)$ is a diffeomorphism. Let $\Sigma = \epsilon(M)$.
- Let $\mathcal{H}_m$ be the sections of $H^{\otimes m}$ with norm denoted for short as $||s||_{\mathbb{C}P^d}$.

Let $\Psi_{(q_1,q_2,...,q_n;q)}(\mu) = \frac{1}{\sqrt{D_{(q_1,...,q_n;q)}}}\mu_1^{q_1}...\mu_n^{q_n}$ where $q_1 + ... + q_n = q; q = 0, ..., m$ be an orthonormal basis for it as mentioned in the previous section.

- ✦ Let $\mathcal{H}_{1m} = i^*(\mathcal{H}_m)$ be the pullback Hilbert space on $\Sigma = \epsilon(M)$. The norm on $\mathcal{H}_{1m}$ is given by $||\tilde{s}||_\Sigma = \min_{\mathrm{s}\in\mathcal{H}_\mathrm{m}}\{||\mathrm{s}||_{\mathbb{C}\mathrm{P}^\mathrm{d}} : \tilde{\mathrm{s}} = \mathrm{s}\circ\mathrm{i}\}$.

  Let $\mathcal{H}_{2m} = \chi^*(\mathcal{H}_m)$ be the pullback Hilbert space on $M$. Thus if $\tilde{s} \in \mathcal{H}_{2m}$, it is of the form $\tilde{s} = s \circ \chi$. The norm on $\mathcal{H}_{2m}$ is given by $||\tilde{s}||_M = \min_{\mathrm{s}\in\mathcal{H}_\mathrm{m}}\{||\mathrm{s}||_{\mathbb{C}\mathrm{P}^\mathrm{d}} : \tilde{\mathrm{s}} = \mathrm{s}\circ\chi\}$.

- ✦ Let $\epsilon^{-1} : \Sigma \to M$ be the inverse of $\epsilon$ on $\Sigma$.

  - ✦ $\mathcal{H}_{1m}$ and $\mathcal{H}_{2m}$ are Hilbert spaces in the respective norms.

    For proof see for instance [13].

  - ✦ Let $\eta_I$ be an orthonormal basis for $\mathcal{H}_{2m}$.

    **Definition:**

    The Rawnsley-type coherent states on $M$ are defined locally as $\psi_p = \sum_{k=1}^{l} \overline{\eta_{I_k}(p)}\eta_{I_k}$ where $p \in M$.
    As before they are overcomplete, have reproducing kernel property, resolution of identity property.

# 3 Local induced Poisson structure on a smooth manifold embedded in a symplectic manifold

✦ Let $(X, \omega)$ be a symplectic manifold of real dimension $2n$. By cell decomposition, there is a top dimensional open cell, $U \subset X$ such that $U$ is homeomorphic to $\mathbb{R}^{2n}$ and $X \setminus U$ is of dimension lower than $2n$. Let $M$ be a compact smooth manifold of dimension $d$ (not necessarily even). We can embed it by Whitney embedding theorem in $U$ since the latter is homeomorphic to $\mathbb{R}^{2n}$.

(There could be other embeddings too from $M^d$ to $X^{2n}$, what follows holds for all of them).

- ✦ Let $\{V, (x_1, y_1, ...x_n, y_n)\}$ be an open neighbourhood of $X^{2n}$ special in the sense that the embedding $\epsilon$ looks simply,
  $U = V \cap \epsilon(M) = \{(x_1, x_2, ...x_{2n-1}, x_{2n}) | x_{d+1}, ..., x_{d+1+s} = 0\}$.
  On $U_M = \epsilon^{-1}(U)$ to be $\{(m_1, ..., m_d) = (x_1, x_2, ...., x_d)\}$, where $d$ is the dimension of $M$. Let $\epsilon^{-1} : \epsilon(M) \to M$ and $\frac{\partial}{\partial m_i} = \epsilon_*^{-1}(\frac{\partial}{\partial x_i})$. Similarly $dm_i = \epsilon^*(dx_i)$, $i = 1, ..., d$ .

## 3.1 Local Poisson structure from embedding a compact smooth manifold in a compact symplectic manifold

- Let $\iota$ be the inclusion map such that $\iota : \epsilon(M) \rightarrow X$. and $f_1, f_2 \in C^{\infty}(M)$. We know $\epsilon^{-1} : \epsilon(M) \rightarrow M$ pulls back $f_1, f_2$ to $\epsilon(M)$.and $f_1 = \epsilon^* \circ \iota^*(F_1)$ and $f_2 = \epsilon^* \circ \iota^*(F_2)$, where $\iota^*(F_1), \iota^*(F_2)$ are any extensions of $\epsilon^{-1*}(f_1)$ and $\epsilon^{-1*}(f_2)$. Let $i, j = 1...d$ such that We can define the Poisson structure as

$$\{f, g\}_M := \epsilon^* \left( \sum_{i,j=1}^{d} \omega^{ij} \frac{\partial \iota^*(F_1)}{\partial x_i} \frac{\partial \iota^*(F_2)}{\partial x_j} \right).$$

  where $\frac{\partial}{\partial x_i}$ does not occur in this sum if $x_i$ it is one of $\{x_{d+1}, ..., x_{d+1+s}\}$ which are zero on $\epsilon(M)$.

- The induced Poisson bracket $\{f, g\}_M$ is independent of the choice of the extension $\iota^*(F_1)$ and $\iota^*(F_2)$.

From now on we will not use $\iota$ anymore when it is obvious that we are referring to the restriction of $F \in C^{\infty}(X)$ to $\epsilon(M)$.

✦ The ambient space being symplectic the bracket induced by the symplectic form is Poisson i.e. satisfies the Jacobi identity. This can be proved by the fact that in Darboux coordinates the non-degenerate Poisson bracket looks like that in Euclidean space. On $\epsilon(M)$ (and hence on $M$) the induced Poisson structure, even though degenerate, also satisfy Jacobi identity (which can be proved by restricting the structure of the Poisson bracket on the ambient manifold in Darboux coordinates).

# 4 Induced Berezin-type quantization on compact smooth manifolds

(This is work done for a conference in Cotonou, Benin and published in [4]).

✦ Let $M$ be a compact smooth manifold.
Let $\chi : M \mapsto \mathbb{C}P^n$ as previously explained.

✦ Let us continue on $\mathbb{C}P^n$ and recall the Berezin quantization on it.

✦ **Notation:** As in [3], we denote by $\mathcal{L}_m(\mu, \bar{\mu}) = \langle \psi_\mu, \psi_\mu \rangle = \psi_\mu(\mu)$, $\mathcal{L}_m(\mu, \bar{\nu}) = \langle \psi_\mu, \psi_\nu \rangle = \psi_\nu(\mu)$.
Let $\hat{A}$ be a bounded linear operator acting on $\mathcal{H}$. Then, as in [3], one can define a symbol of the operator as

$$A(\nu, \bar{\mu}) = \frac{\left\langle \psi_\nu, \hat{A}\psi_\mu, \right\rangle}{\langle \psi_\nu, \psi_\mu \rangle}.$$

One can show that one can recover the operator from the symbol by a formula

[3].

✦ Let $\hat{A}_1, \hat{A}_2$ be two such operators and let $\hat{A}_1 \circ \hat{A}_2$ be their composition.
Then the symbol of $\hat{A}_1 \circ \hat{A}_2$ will be given by the star product defined as in [3]:

$$\begin{aligned} & (A_1 * A_2)(\mu, \bar{\mu}) \\ = \quad & c(m) \int_{U_0} A_1(\mu, \bar{\nu}) A_2(\nu, \bar{\mu}) \frac{\mathcal{L}_m(\mu, \bar{\nu}) \mathcal{L}_m(\nu, \bar{\mu})}{\mathcal{L}_m(\mu, \bar{\mu}) \mathcal{L}_m(\nu, \bar{\nu})} \mathcal{L}_m(\nu, \bar{\nu}) e^{-m\tilde{\Phi}(\nu, \bar{\nu})} dV(\nu), \end{aligned}$$

where recall $\frac{1}{c(m)} = \int_{U_0} e^{-m\Phi_{FS}(\nu, \bar{\nu})} dV(\nu)$.
This is the symbol of $\hat{A}_1 \circ \hat{A}_2$.

✦ **Theorem** [Berezin]
Let $\mu \in \mathbb{C}^n$.
The star product satisfies the correspondence principle:
1. $\lim_{m \to \infty} (A_1 \star A_2)(\mu, \bar{\mu}) = A_1(\mu, \bar{\mu}) A_2(\mu, \bar{\mu})$,
2. $\lim_{m \to \infty} m(A_1 \star A_2 - A_2 \star A_1)(\mu, \bar{\mu}) = i\{A_1, A_2\}_{FS}(\mu, \bar{\mu})$.

✦ See [3], [6] for proof.

## 4.1 Induced operators and correspondence principle

✦ Let $\hat{A}$ be a bounded linear operator from $\mathcal{H}_m$ to itself. Now we turn to $M$. Let $\hat{B} = \chi^*(\hat{A})$ be a bounded linear operator from $\mathcal{H}_{2m} = \chi^*(\mathcal{H}_m)$ to itself. It is defined by $\chi^*(\hat{A})(\chi^*(s)) = \chi^*(\hat{A}s)$. Given $\hat{B}$, $\hat{A}$ is not unique. Suppose we have $\hat{B} = \chi^*(\hat{A}_1) = \chi^*(\hat{A}_2)$. Let $\hat{A}$ be the one of lowest norm i.e. if $\hat{B} = \chi^*(\hat{A}) = \chi^*(\hat{A}_1)$, then $||\hat{A}_1|| \geq ||\hat{A}||$.

✦ **Definition:** Let $B : M \times M \to \mathbb{C}$ be the symbol of $\hat{B}$ in the coherent states $\psi_p$, i.e. $B(p,p) = \frac{\langle \psi_p, \hat{B}(\psi_p) \rangle_M}{\langle \psi_p, \psi_p \rangle_M}$ and $B(p,q) = \frac{\langle \psi_p, \hat{B}(\psi_q) \rangle_M}{\langle \psi_p, \psi_q \rangle_M}$.

Here the norm is defined as before.

✦ $B(p,q) = \frac{\langle s_{\chi(p)}, \hat{A}(s_{\chi(q)}) \rangle_{\mathbb{C}P^d}}{\langle s_{\chi(p)}, s_{\chi(q)} \rangle_{\mathbb{C}P^d}} = A(\chi(p), \overline{\chi(q)})$, where $\hat{B} = \chi^*(\hat{A})$ and $\hat{A}$ is the one of lowest norm over all $\hat{C}$ such that $\hat{B} = \chi^*(\hat{C})$.

✦ **Proof**

$\psi_p = \chi^*(s_{\chi(p)})$, where $s_{\chi(p)} = \sum_{k=1}^{l} \overline{\Phi_{I_k}(\chi(p))} \Phi_{I_k}$ is the coherent state in $\mathcal{H}_m$. This is because $\eta_{I_k} = \chi^*(\Phi_{I_k})$.

$B(p,q) = \frac{\hat{B}(\psi_q)(p)}{\psi_q(p)} = \frac{\hat{A}(s_{\chi(q)})(\chi(p))}{s_{\chi(q)}(\chi(p))} = \frac{\langle s_{\chi(p)}, \hat{A}(s_{\chi(q)}) \rangle_{\mathbb{C}P^d}}{\langle s_{\chi(p)}, s_{\chi(q)} \rangle_{\mathbb{C}P^d}}$.

✦ **Definition:**

Let $B_1$ and $B_2$ be symbols of $\hat{B}_1$ and $\hat{B}_2$ bounded linear operators on $\mathcal{H}_{2m}$. Then $B_1 * B_2$ is the symbol of $\hat{B}_1 \circ \hat{B}_2$, [3].

One sees that $\hat{B}_1 \circ \hat{B}_2 = \chi^*(\hat{A}_1 \circ \hat{A}_2)$, where we can take $\hat{A}_1 \circ \hat{A}_2$ is the one of the least norm.

Then, $B_1 * B_2 = \chi^*(A_1 * A_2)$ where the second one is $\mathbb{C}P^d$ star product. Let $\{B_1, B_2\}_M(p,p)$ be the induced Poisson bracket.

✦ **Theorem** The star product on the symbol of bounded linear operators on $\mathcal{H}_{2m}$ satisfies the correspondence principle:

(1) $\lim_{m\to\infty}(B_1 * B_1)(p,p) = B_1(p,p)B_2(p,p)$.

(2) $\lim_{m\to\infty} m(B_1 * B_2 - B_2 * B_1)(p,p) = i\{B_1, B_2\}_M(p,p)$.

✦ **Proof:** This follows from the fact that $A_1 * A_2$ satisfy the correspondence principle, see Theorem 4 (Berezin).

# 5 Fedosov star product on “algebra” of observables on compact smooth manifolds

✦ This is work in progress by Satyen Patel ( for his MSc thesis project) and R.D.

✦ The problem Fedosov answers for symplectic manifolds is as follows: (quoted from his seminal paper [9]):

“The problem is to define an associative multiplication operation *, depending on parameter h (Planck constant), of two functions so that the space $C^\infty(M)$ with usual linear operators and *-product would be a formal deformation of commutative algebra of functions with a Poisson bracket. More exactly it means the following. Let $Z$ be the linear space, the elements of which are formal series

$$a = a(x,h) = \sum_k a_k(x) h^k$$

where $a_k(x) \in C^\infty(M)$. Further for any $a, b \in Z$ let an associative product operation

$$a * b = c = \sum c_k(x) h^k$$

be defined with the following properties:

(i) $c_k$ are polynomials in $a_k$, $b_k$ and their derivatives;

(ii) $c_0(x) = a_0(x) b_0(x)$

(iii) $[a,b] = a * b - b * a = -ih\{a_0, b_0\} + ...$ , where dots mean the terms of higher orders in $h$.

The algebra $Z$ is called the algebra of quantum observables. Property (i) means the locality of *-product, property (ii) means that algebra Z is a deformation of the commutative algebra of $C^\infty$ functions, property (iii) is the so-called correspondence principle."

- ✦ Fedosov constructs this $*$ product for a symplectic manifold $X$. which has a non-degenerate Possion structure induced by the symplectic form. Our aim is to define this on a smooth manifold $M$ embedded in $X$, which has a possibly degenerate Poisson structure locally, induced from the one on $X$.
- ✦ Let $(X, \omega)$ be a symplectic manifold of real dimension $2n$. By cell decomposition, there is a top dimensional open cell, $U \subset X$ such that $U$ is homeomorphic to $\mathbb{R}^{2n}$ and $X \setminus U$ is of dimension lower than $2n$. Let $M$ be a compact smooth manifold of dimension $d$ (not necessarily even). We can embed it by Whitney embedding theorem in $U$ since the latter is homeomorphic to $\mathbb{R}^{2n}$.

(There could be other embeddings too from $M^d$ to $X^{2n}$, what follows holds for all of them).

- ✦ Let $\{V, (x_1, y_1, ...x_n, y_n)\}$ be an open neighbourhood of $X^{2n}$ special in the sense that the embedding $\epsilon$ looks simply, $U = V \cap \epsilon(M) = \{(x_1, x_2, ...x_{2n-1}, x_{2n}) | x_{d+1}, ...,$ $0\}$. On $U_M = \epsilon^{-1}(U)$ to be $\{(m_1, ..., m_d) = (x_1, x_2, ...., x_d)\}$, where $d$ is the dimension of $M$. Let $\epsilon^{-1} : \epsilon(M) \to M$ and $\frac{\partial}{\partial m_i} = \epsilon_*^{-1}(\frac{\partial}{\partial x_i})$. Similarly $dm_i = \epsilon^*(dx_i), i = 1, ..., d$ .

## 5.1 Local Poisson structure from embedding a compact smooth manifold in a compact symplectic manifold

- Let $\iota$ be the inclusion map such that $\iota : \epsilon(M) \rightarrow X.$ and $f_1, f_2 \in C^{\infty}(M)$. We know $\epsilon^{-1} : \epsilon(M) \rightarrow M$ pulls back $f_1, f_2$ to $\epsilon(M)$.and $f_1 = \epsilon^* \circ \iota^*(F_1)$ and $f_2 = \epsilon^* \circ \iota^*(F_2)$, where $\iota^*(F_1), \iota^*(F_2)$ are any extensions of $\epsilon^{-1*}(f_1)$ and $\epsilon^{-1*}(f_2)$.

Let $i, j = 1...d$ such that We can define the Poisson structure as

$$\{f, g\}_M := \epsilon^* \left( \sum_{i,j=1}^{d} \omega^{ij} \frac{\partial \iota^*(F_1)}{\partial x_i} \frac{\partial \iota^*(F_2)}{\partial x_j} \right).$$

where $x_i$ does not occur in this sum if it is one of $\{x_{d+1}, ..., x_{d+1+s}\}$ which are zero on $\epsilon(M)$.

✦ The induced Poisson bracket $\{f, g\}_M$ is independent of the choice of the extension $\iota^*(F_1)$ and $\iota^*(F_2)$.

From now on we will not use $\iota$ anymore when it is obvious that we are referring to the restriction of $F \in C^\infty(X)$ to $\epsilon(M)$.

✦ The ambient space being symplectic the bracket induced by the symplectic form is Poisson i.e. satisfies the Jacobi identity. This can be proved by the fact that in Darboux coordinates the non-degenerate Poisson bracket looks like that in Euclidean space. On $\epsilon(M)$ (and hence on $M$) the induced Poisson structure, even though degenerate, also satisfy Jacobi identity (which can be proved by restricting the structure of the Possion bracket on the ambient manifold in Darboux coordinates).

## 5.2 Fedosov star product on $Z(M)$ and the correspondence principle

### 5.2.1 The star product

a) Review of Fedosov star product on the symplectic manifold $(X, \omega)$ and its induction to $M$.

✦ Note that we can choose a local coordinate $(x_1, ...x_{2n}, y_1, ..., y_{2n})$ on $TX$ such that locally on a special neighbourhood $O_{\epsilon(M)}$ of $T(\epsilon(M))$ has coordinates

$(x_1, x_2, ...., x_d, 0, 0, ..., 0, y_1, ..., y_d, 0, 0, ..0)$.

- Sections of the bundle of Weyl algebras $W$ on $X$ locally look like $a(x, y, h) = \sum_{2k+l \geq 0} h^k a_{k,i_1...i_l}(x) y^{i_1}...y^{i_l}$ where $a_{k,i_1...i_l}(x)$ are symmetric covariant tensor fields on $X$.
- Let us consider $a$ and $b$ two elements of the Weyl algebra $W_x$ for $x \in M$. We know from Fedosov [9] that
  $a \circ b := \sum_{k=0}^{\infty} -(\frac{ih}{2})^k \frac{1}{k!} \omega^{i_1 j_1}...\omega^{i_k j_k} \frac{\partial^k a}{\partial y^{i_1} \partial y^{i_2}...\partial y^{i_k j_k}} \frac{\partial^k b}{\partial y^{i_1} \partial y^{i_2}...\partial y^{i_k j_k}}$.
- This can be defined on $\epsilon(M)$ by setting all the $\frac{\partial}{\partial y_i} = 0$ if $i = d+1, ...., n$, i.e. the $y_i$ donot appear if they are identically zero locally on the special neighbourhood $O_{\epsilon(M)}$ on $T(\epsilon(M))$.
- Then, to define on $M$, we push forward the contravariant tensors by $\epsilon_*^{-1}$ and pull back covariant tensors by $\epsilon^*$.
- For sections of the Weyl bundle W, we first restrict to $\epsilon(M)$ and then on $M$ one can define
  $\tilde{a}(x, y, h) = \sum_{2k+l \geq 0} h^k \epsilon^*(a_{k,i_1...i_l}(x))(\epsilon_*^{-1}(y))^{i_1}...(\epsilon_*^{-1}(y))^{i_l})$ where we omit those indices which donot appear in the special neighbourhood $O_{\epsilon(M)}$ on $T(\epsilon(M))$.
- In [9] Fedosov also introduces differential forms on $X$ which take values in $W$.

$a(x,y,h) = \sum_{2k+l\geq 0} h^k a_{k,i_1...i_p,j_1...j_p}(x) y^{i_1}...y^{i_p} dx^{j_1} \wedge ... \wedge dx^{j_q}$,
such that $a_{k,i_1...i_p,j_1...j_p}(x)$ are covariant tensor fields which are symmetric w.r.t. indices $i_1, ..., i_p$ and antisymmetric w.r.t. indices $j_1, ..., j_q$.

- In order to define it on $M$, we do as before, namely, omit the indices which are identically zero in $O_{\epsilon(M)}$ and pull back the covariant tensors by $\epsilon$ and push back contravariant tensors by $\epsilon^{-1}$. Once again we restrict to $\epsilon(M)$ and then define for $M$.

  Namely, on $M$,
  $\tilde{a}(x,y,h) := \sum_{2k+l\geq 0} h^k \epsilon^*(\tilde{a}_{k,i_1...i_p,j_1...j_p}(x)) \epsilon_*^{-1}(y)^{i_1}...\epsilon_*^{-1}(y)^{i_p} \epsilon_*(dx^{j_1} \wedge ... \wedge dx^{j_q})$
  where $x \in \epsilon(M)$. Here too, notice, we omit those indices which are identically zero in the special neighbourhood $O_{\epsilon(M)}$ on $T(\epsilon(M))$.

## Definition of the star product

- Let $Z(X)$ be the algebra of quantum observables for $X$ and $c_0, d_0$ be in $Z(X)$. Let $W_D$ be a flat Weyl bundle (w.r.t. a flat connection) on $W$. Then for each $c_0$ there is a (unique) lift $c = \sigma^{-1}(c_0)$ to $W_D$ such that the symbol $\sigma(c) = c_0$.
- One can recover $c$ from its symbol $c_0$ as follows, [9] pg. 222: $c = c_0 + \partial_i c_0 y^i + \frac{1}{2}\partial_i\partial_j c_0 y^i y^j + .... - \frac{1}{24} R_{ijkl}\omega^{lm}\partial_m c_0 y^i y^j y^k + ....$
- By (3.14) [9]

$$c_0 *_{Fed} d_0 := \sigma(\sigma^{-1}(c_0) \circ \sigma^{-1}(d_0)).$$

where recall fiberwise on $W_x$ the $\circ$ is defined to be $c_0 \circ d_0 := \sum_{k=0}^{\infty} -(\frac{ih}{2})^k \frac{1}{k!}\omega^{i_1 j_1}...\omega^{i_k j_k} \frac{\partial^k c_0}{\partial y^{i_1}\partial y^{i_2}...\partial y^{i_k j_k}} \frac{\partial^k d_0}{\partial y^{i_1}\partial y^{i_2}...\partial y^{i_k j_k}}$.

- Let $Z(X)$ and $Z(M)$ denote the algebra of quantum observables for $X$ and $M$ respectively, $W_D$ be the flat Weyl bundle (w.r.t. a fixed flat connection ) on $X$ and $W_{\epsilon(M)}$ be the bundle on $\epsilon(M)$ which is obtained by restricting the bundle $W_D(X)$ on $\epsilon(M)$ and omitting all convariant and contravariant tensors which involve indices $i = d+1, ..., n$.

✦ Let $c_0, d_0 \in Z(X)$ be arbitrary extensions of $a_0, b_0 \in Z(M)$. We know, page 232, [9] that $c_0, d_0$ are symbols of $c, d$ (flat sections of Weyl algebra $W_D(X)$).

What follows is independent of choice of the extension $c, d$.

Let $\tilde{a}, \tilde{b}$ be $\tilde{a} = \phi(c)$ and $\tilde{b} = \phi(d)$ be in $\phi(W_D)$. Then there is a formula to go from the symbols $c_0, d_0$ to $c, d$ and vice versa. It is easy to see that that if $\tilde{a} = \phi(c), \tilde{b} = \phi(d)$ then the symbol of these is exactly $\tilde{a}_0, \tilde{b}_0$ respectively.

✦ Thus $\tilde{a}_0$ and $\tilde{b}_0$ be symbols of $\tilde{a}, \tilde{b} \in \tilde{W} = \phi(W_D|_{\epsilon(M)})$ where $\phi$ is the push-forward and pull-back map, i.e. $\phi$ denotes push forward of covariant tensors by $\epsilon_*$ and pull-back of contravariant tensors by $\epsilon_*^{-1}$ and we omit from $W(X)$ indices which involve $i = d+1, ..., n$.

✦ Since $\epsilon : M \to \epsilon(M)$ is a diffeomorphism, we have given $\tilde{a}_0$, there is a unique $a_0$ such that $\tilde{a}_0 = \epsilon^*(a_0)$.

✦ Define the star product for $\tilde{a}_0$ and $\tilde{b}_0$ as:

$$\tilde{a}_0 * \tilde{b}_0 = \epsilon^*(a_0) * \epsilon^*(b_0) := \epsilon^*(a_0 *_F b_0)$$

where $*_F$ denotes star product which is the Fedosov star product $*_{Fed}$ on $\epsilon(M)$ with no derivatives $\frac{\partial}{\partial x_i}$ or forms $dx_i$ appear for $i = d+1, ..., n$.

- It is obvious that $(\tilde{a}_0 + \tilde{b}_0) * \tilde{c}_0 := \tilde{a}_0 * \tilde{c}_0 + \tilde{b}_0 * \tilde{c}_0$
- In fact, $a_0 *_F b_0 = 0$ if $a_0|_{\epsilon(M)} = 0$ This holds since $*_F$ doesnot involve any derivative or forms with indices from $d+1$ and above.
- Associativity follows from this.

- One can show that the correspondence principle also follows because it holds for $*_{Fed}$.
  Namely if $\tilde{a}_0, \tilde{b}_0 \in Z(M)$. Then
  1. $\lim_{h \to 0} \tilde{a}_0 * \tilde{b}_0 = \tilde{a}_0 \tilde{b}_0$
  2. $\lim_{h \to 0} \frac{1}{h}(\tilde{a}_0 * \tilde{b}_0 - \tilde{b}_0 * \tilde{a}_0) = \{a_0, b_0\}_M$.

**Further directions:**

**Some questions:**

- **How important is the role of $\mathbb{C}P^n$ and $\mathbb{C}^n$ for Berezin-type of quantization** We can replace them with some other symplectic/Poisson manifold $N$ where $N$ has reproducing kernel Hilbert space (coherent states), Berezin/Odzijewicz/Fedosov quantization.
- **Can this be generalized to other methods of deformation quantization.**
- **Physical Examples in which this method could be useful**
  Is it useful to study integral and fractional Quantum Hall effect, Berry's phase and some other phenomenon like spin, magnetic monopole, Aharonov-Bohm Effect with the view of applying this method of quantization or where the Hilbert space is obtained from restriction of the Hilbert space of quantization of $\mathbb{C}P^n$ ( or $\mathbb{C}^n$), yet the parameter space is some other $n$-manifold.

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# THANK YOU