# Unfolding the Toric Code

## Emergent Qubits & Quantum Circuits

**Brijesh Kumar**
JNU, Delhi

10th Indian Statistical Physics Community Meeting, ICTS
(24 April 2025)

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**JPSJ 94, 034001 (2025)**

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# Toric Code Model

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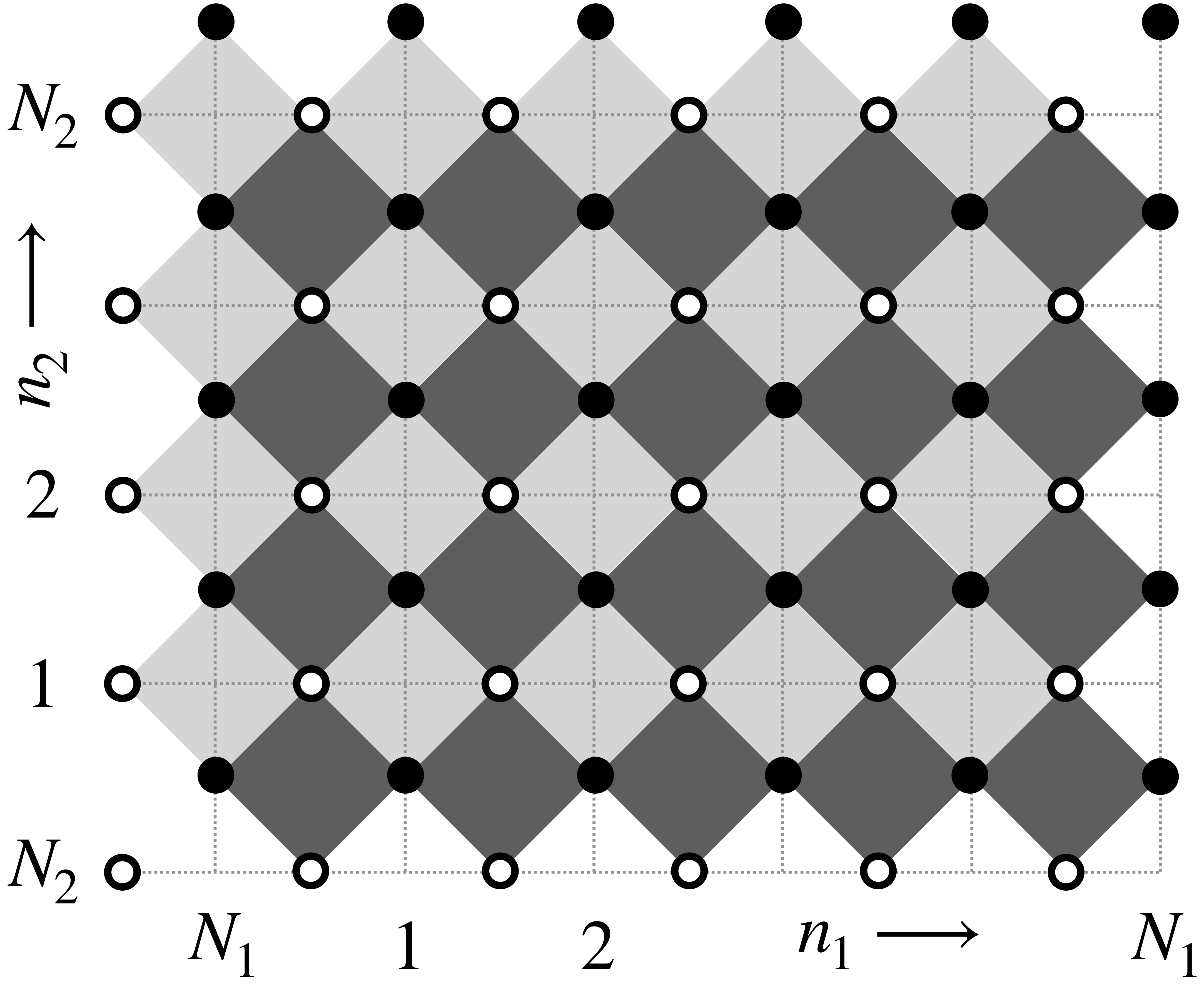


$N_1 \times N_2$ square lattice with period boundaries (torus)

$2N_1N_2$ spin-1/2's (qubits ○ , ●) sitting on the bonds

*Kitaev (2003)*

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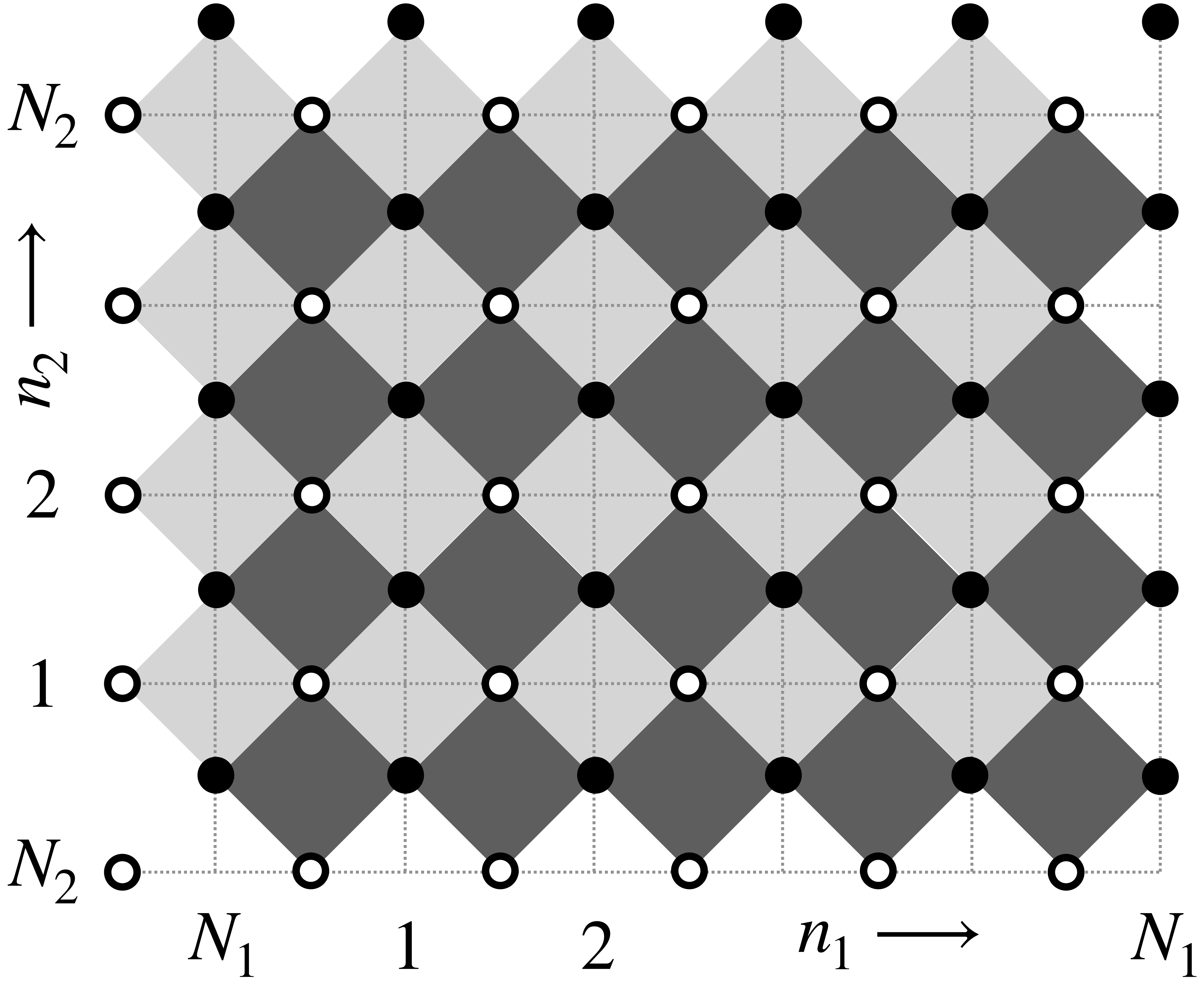
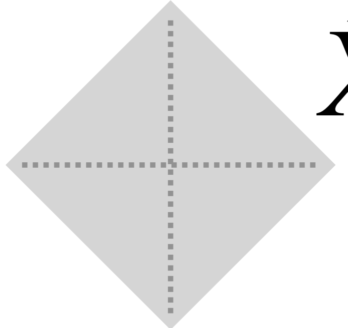


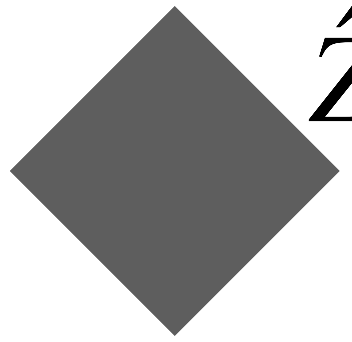

$N_1 \times N_2$ **square lattice with period boundaries** (torus)

$2N_1N_2$ **spin-1/2's (qubits ○ , ●) sitting on the bonds**

**four-qubit interactions**



$$\hat{X}_{n_1,n_2} = \hat{\sigma}^x_{n_1-\frac{1}{2},n_2}\hat{\sigma}^x_{n_1+\frac{1}{2},n_2}\hat{\sigma}^x_{n_1,n_2+\frac{1}{2}}\hat{\sigma}^x_{n_1,n_2-\frac{1}{2}}$$



$$\hat{Z}_{n_1+\frac{1}{2},n_2-\frac{1}{2}} = \hat{\sigma}^z_{n_1+\frac{1}{2},n_2}\hat{\sigma}^z_{n_1+\frac{1}{2},n_2-1}\hat{\sigma}^z_{n_1,n_2-\frac{1}{2},n_2}\hat{\sigma}^z_{n_1+1,n_2-\frac{1}{2}}$$

mutually commuting; conserved

$$\left[\hat{X}_{n_1,n_2}, \hat{X}_{n_1',n_2'}\right] = 0 \qquad \left[\hat{Z}_{n_1+\frac{1}{2},n_2-\frac{1}{2}}, \hat{Z}_{n_1'+\frac{1}{2},n_2'-\frac{1}{2}}\right] = 0 \qquad \left[\hat{X}_{n_1,n_2}, \hat{Z}_{n_1'+\frac{1}{2},n_2'-\frac{1}{2}}\right] = 0$$

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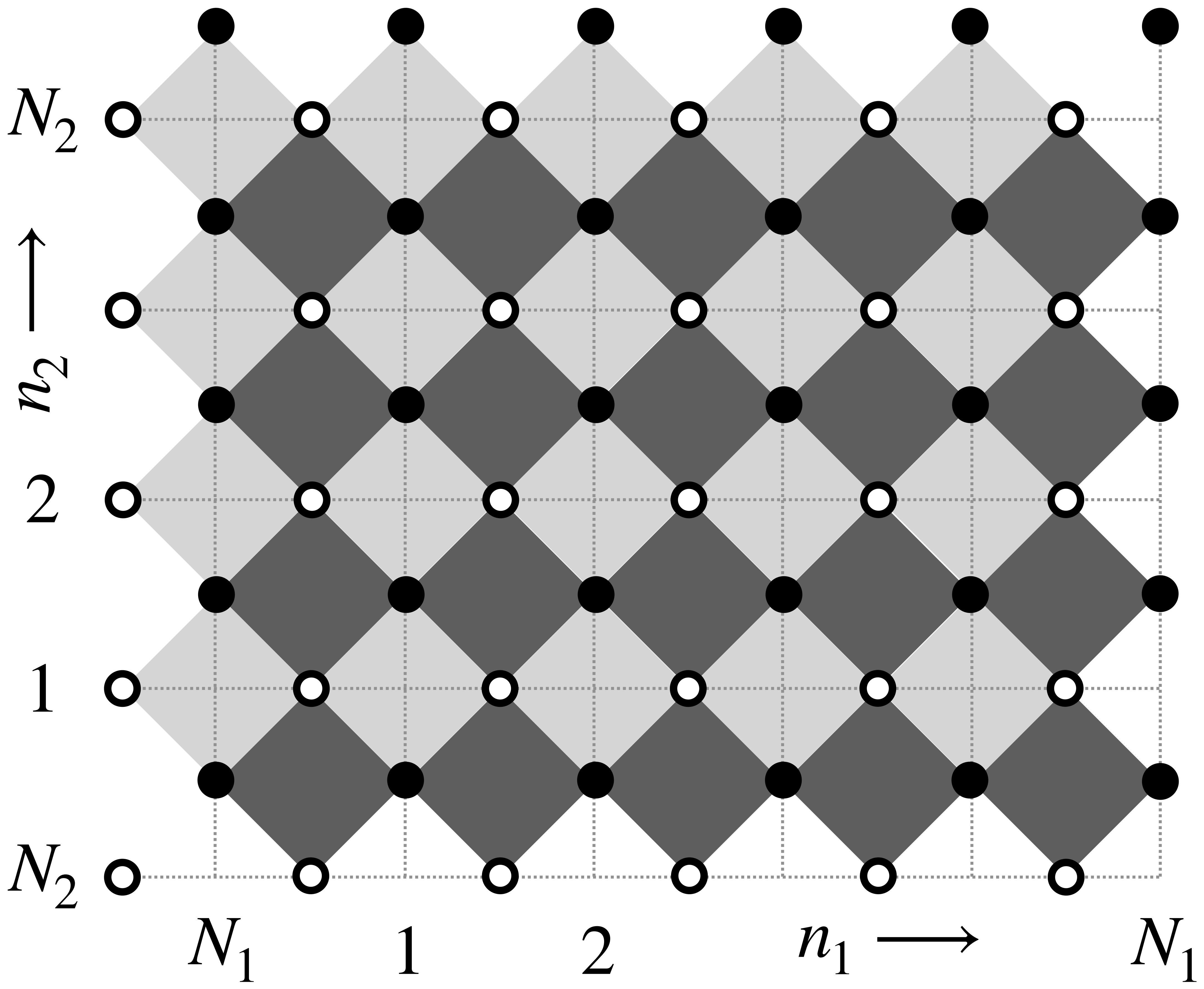


*Kitaev (2003)*

$$\hat{H} = -I_z \sum \hat{Z}_{n_1+\frac{1}{2},n_2-\frac{1}{2}} - I_x \sum \hat{X}_{n_1,n_2}$$

**two natural constraints on torus:**

$$\prod_{\text{all}} \hat{X}_{n_1,n_2} = \hat{1} \quad \textbf{and} \quad \prod_{\text{all}} \hat{Z}_{n_1+\frac{1}{2},n_2-\frac{1}{2}} = \hat{1}$$

**leading to fourfold topological degeneracy**

*a ground state wavefunction for* $I_z, I_x > 0$:

$$|\psi_g\rangle \sim \prod_{n_1,n_2} \left( \frac{\hat{1} + \hat{X}_{n_1,n_2}}{2} \right) \prod_{\text{all qubits}} |+\rangle$$

# Toric Code Model

$$|\psi_g\rangle \sim \prod_{n_1,n_2} \left( \frac{\hat{1} + \hat{X}_{n_1,n_2}}{2} \right) \prod_{\text{all qubits}} |+\rangle$$

*Kitaev (2003)*

**Experimental realisations**

**K. J. Satzinger et al, Science (2021)** planar

C. Liu et al, Optica (2019)

C. Song et al, Phys. Rev. Lett. (2018)

⋮

**D. Bluvstein et al, Nature (2022)** torus

implementation **on torus** remains ambiguous

# What We Do on Toric Code Model

B. Kumar, JPSJ 94, 034001 (2025)

# What We Do on Toric Code Model

B. Kumar, JPSJ 94, 034001 (2025)

1. Unfold the toric code model into independent emergent qubits

2. Derive complete set of exact toric code eigenstates

3. Devise exact quantum circuits

# Unfolding the Toric Code

BK, JPSJ (2025)

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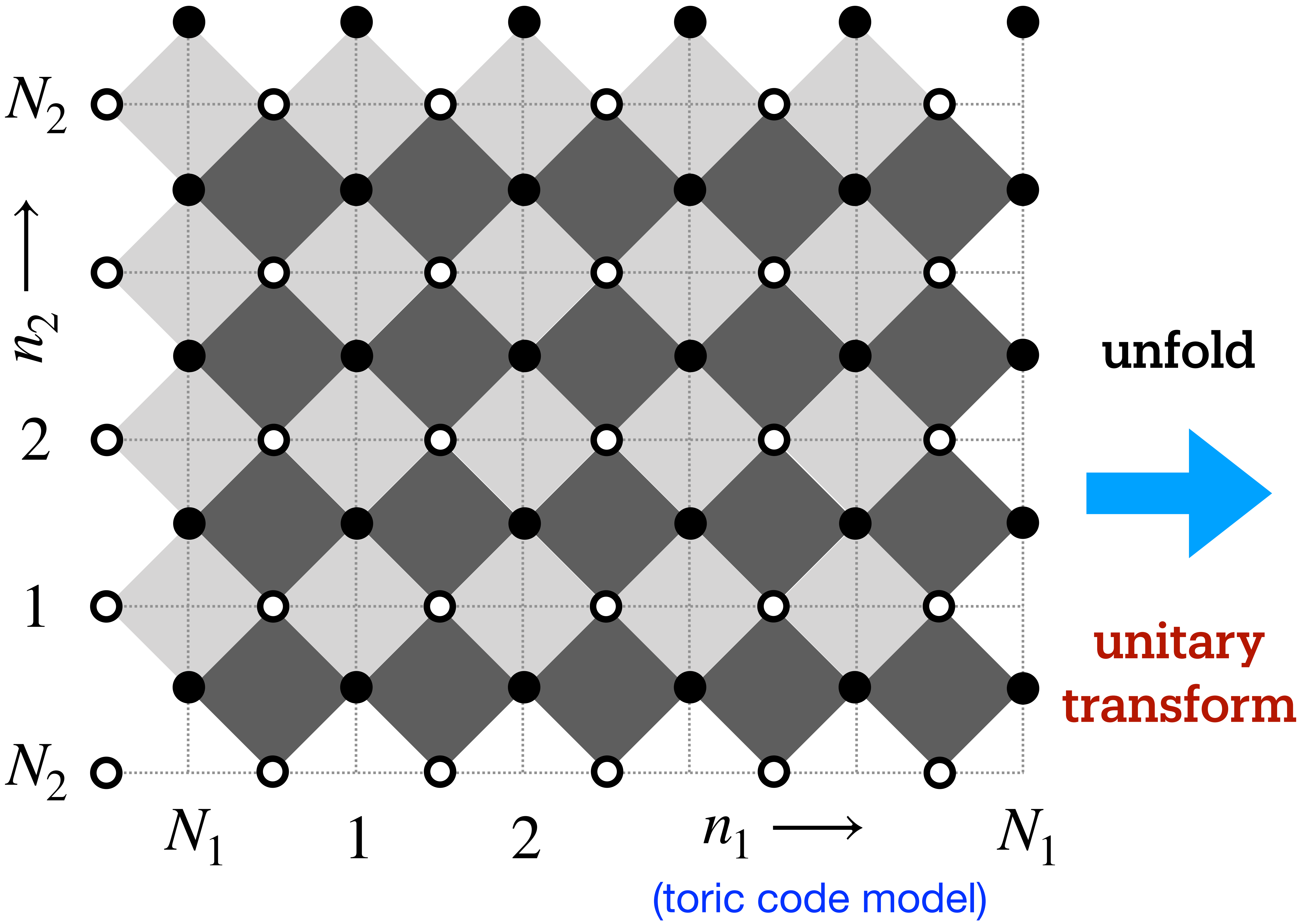

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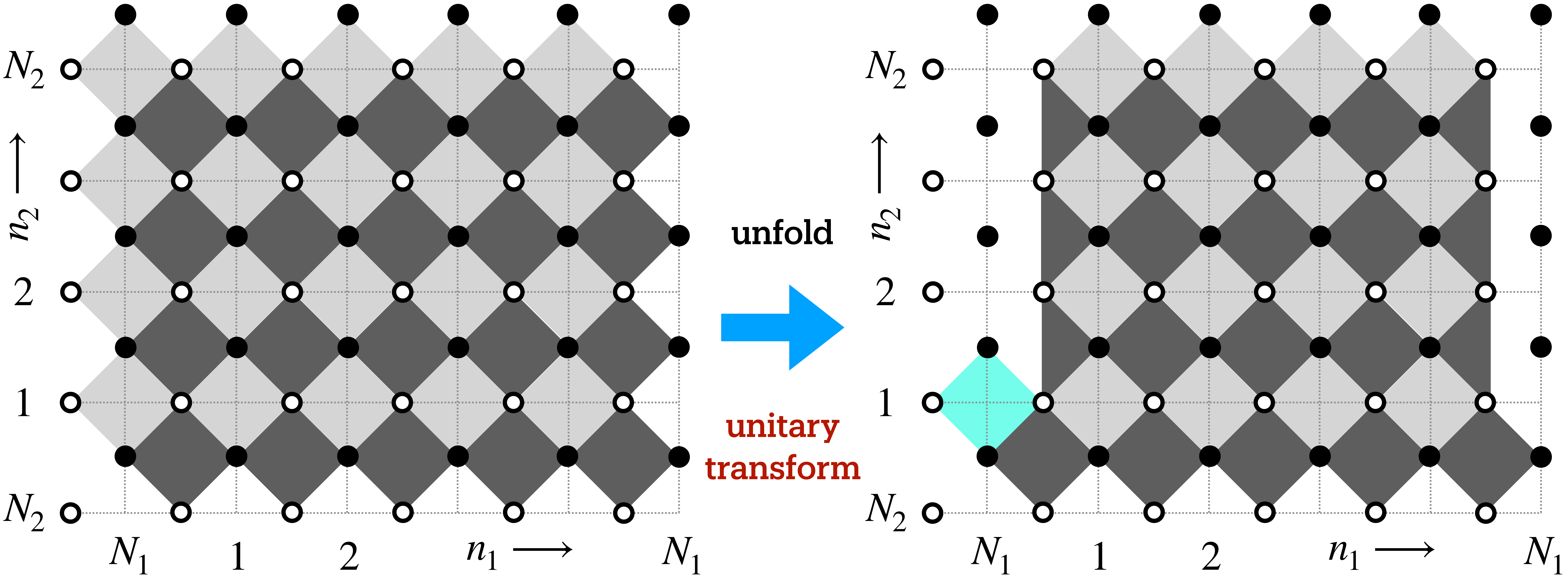


(toric code model)

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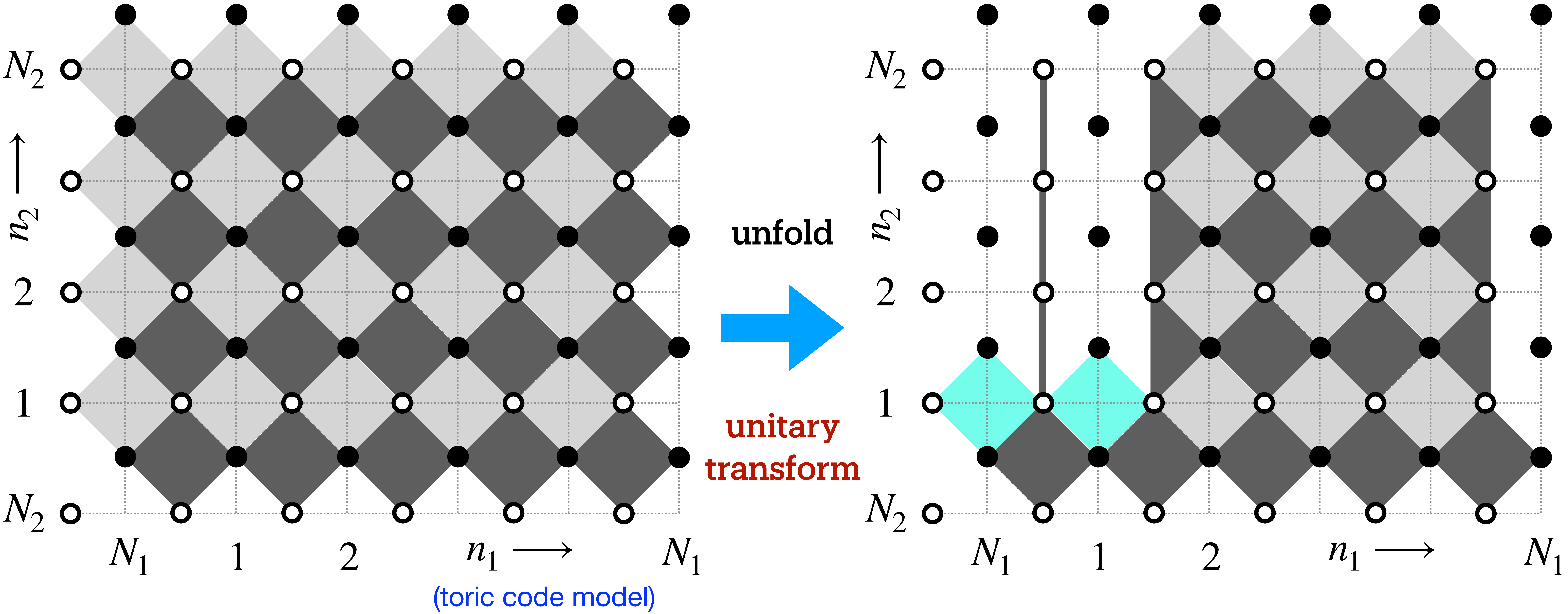

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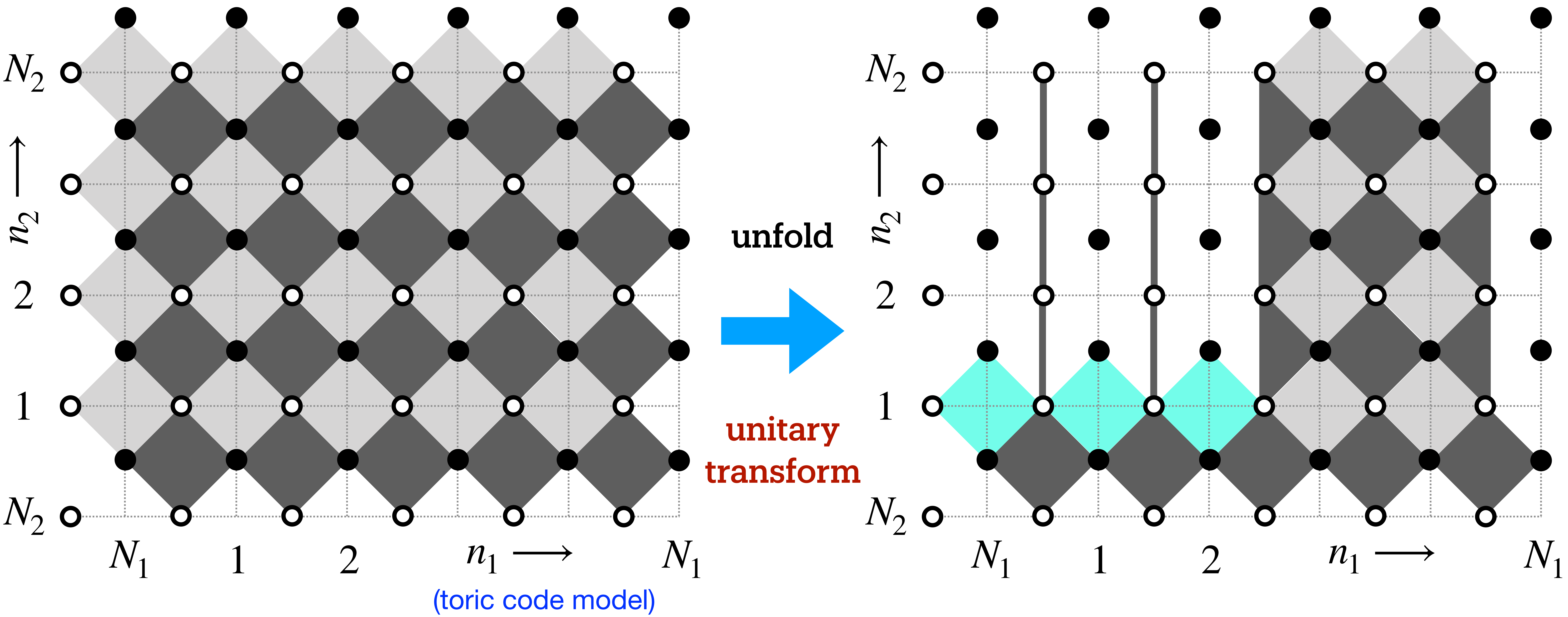

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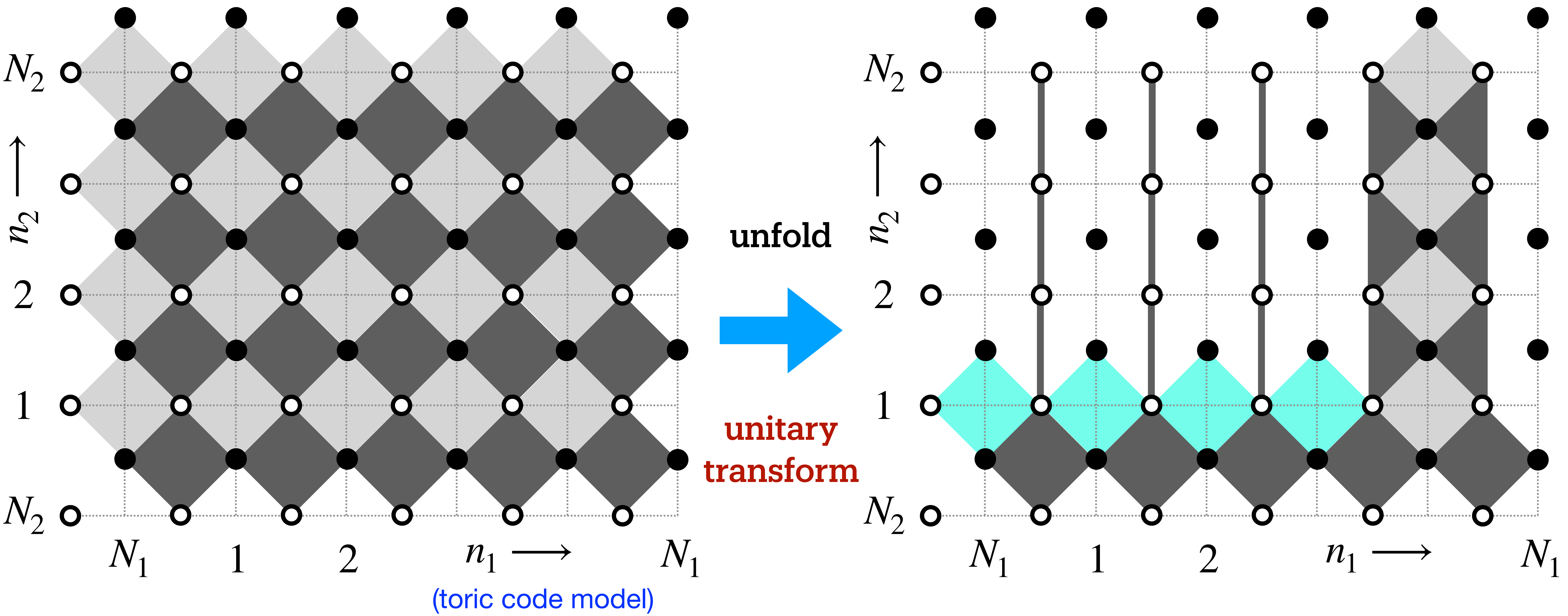

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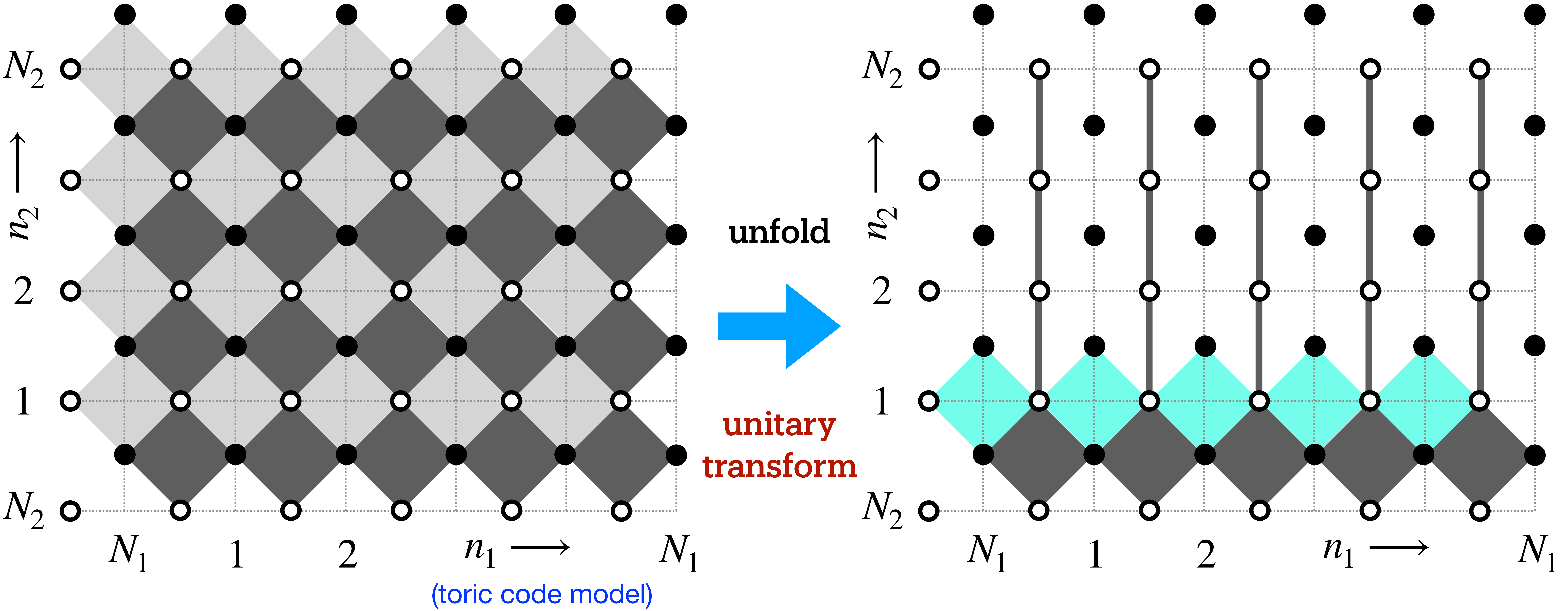

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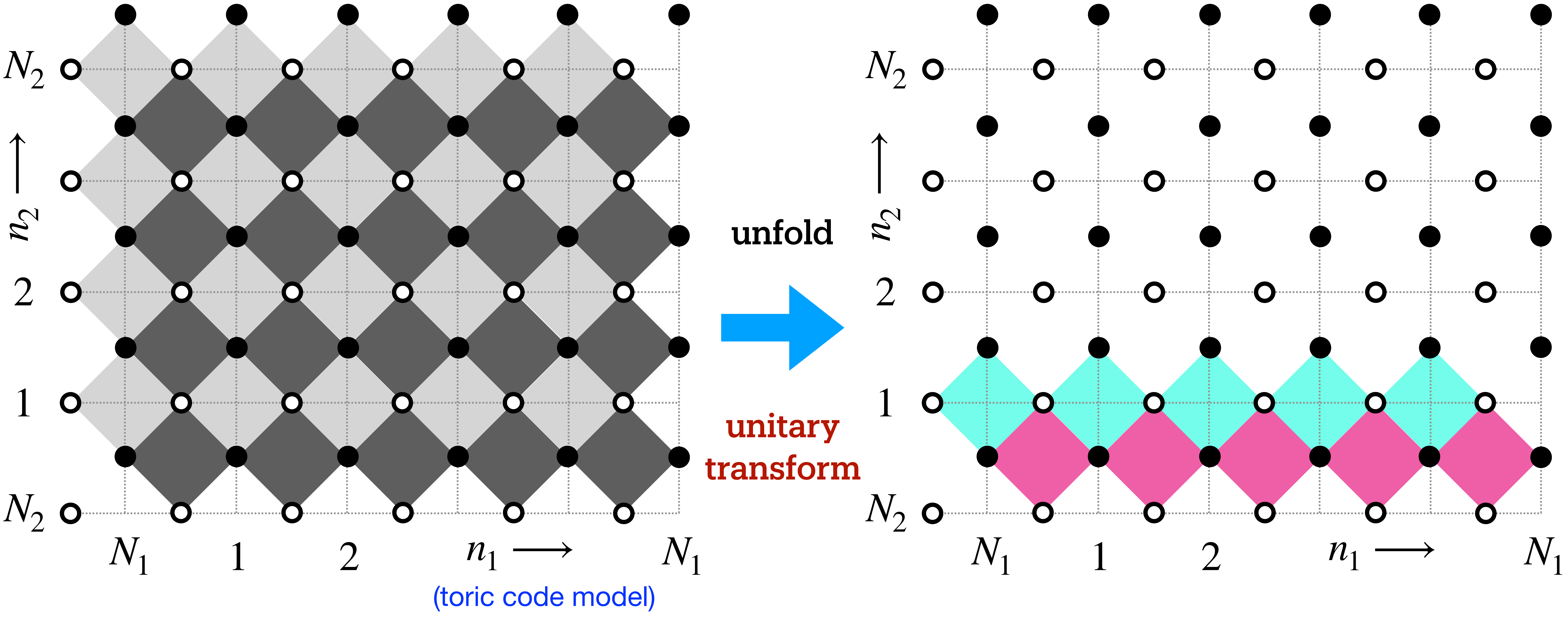

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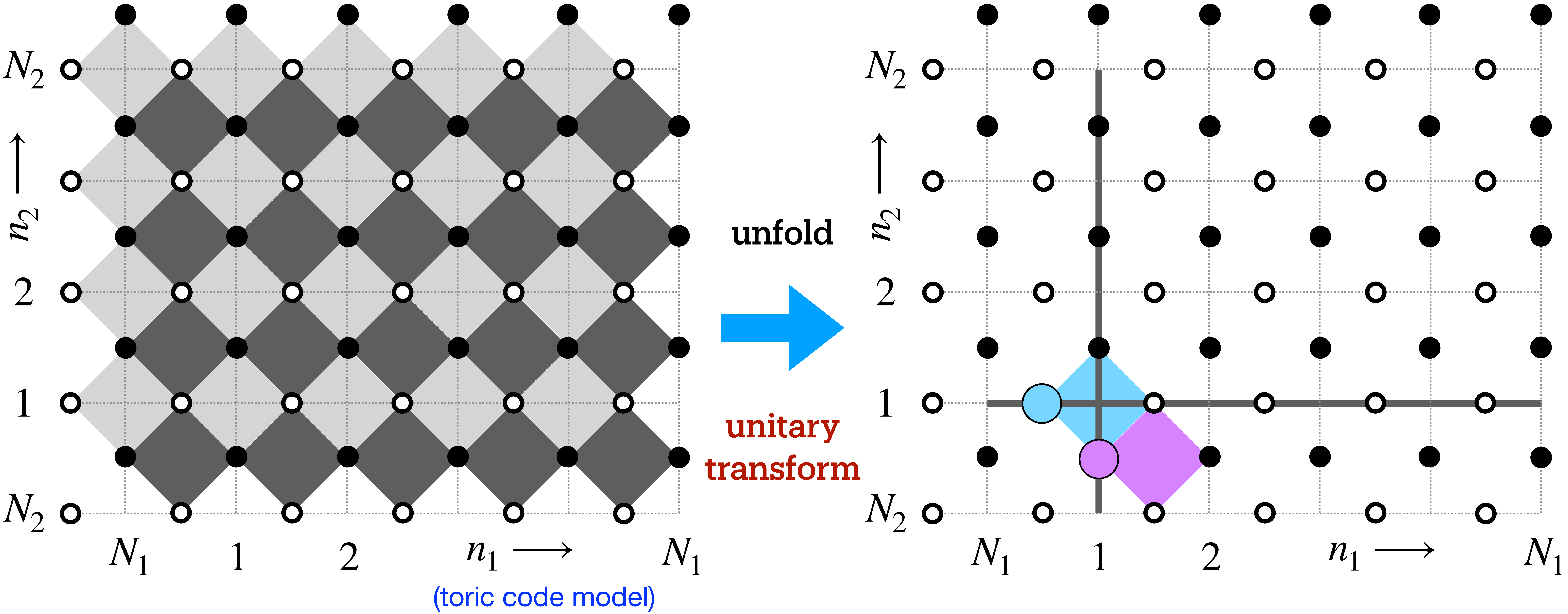

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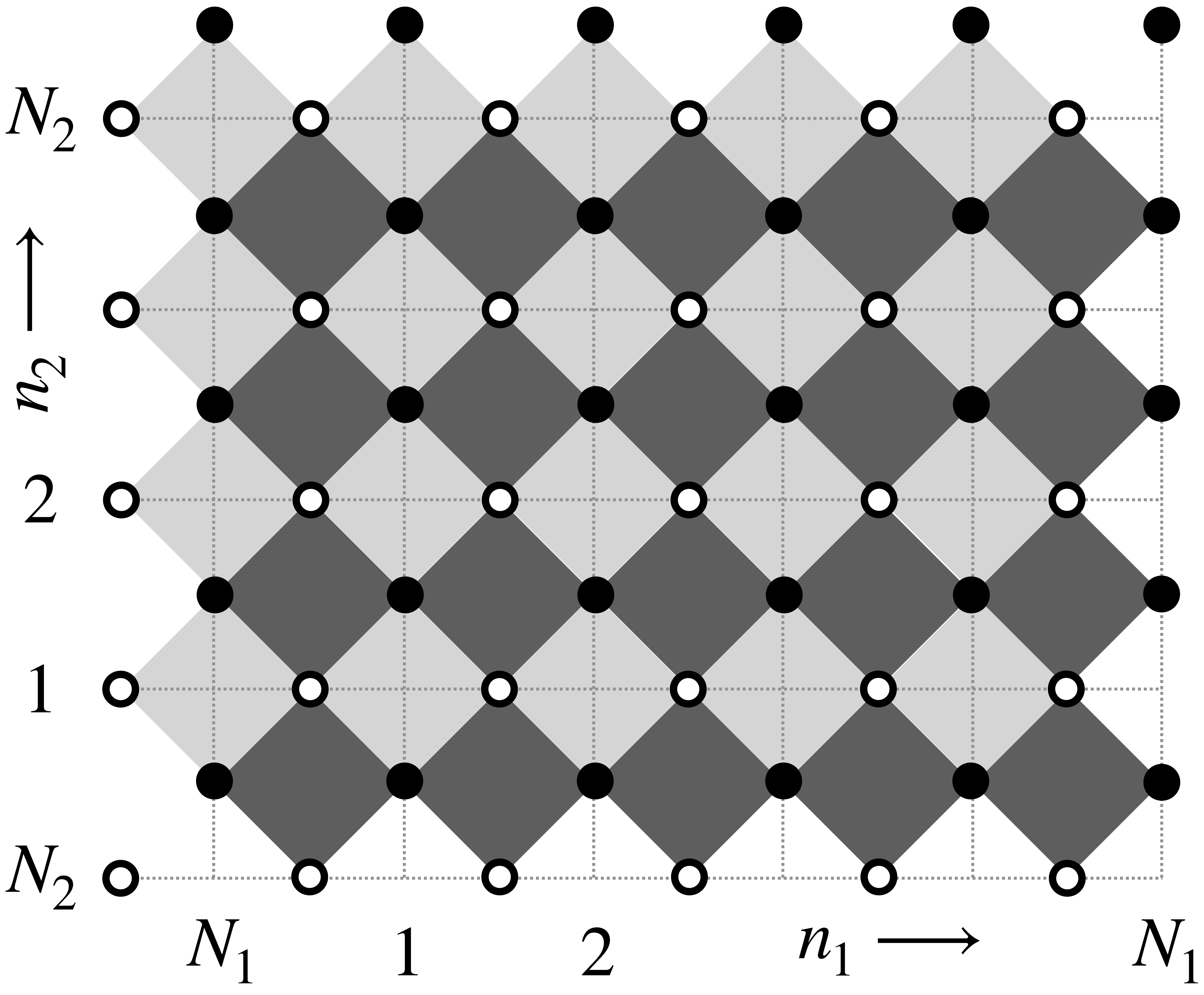
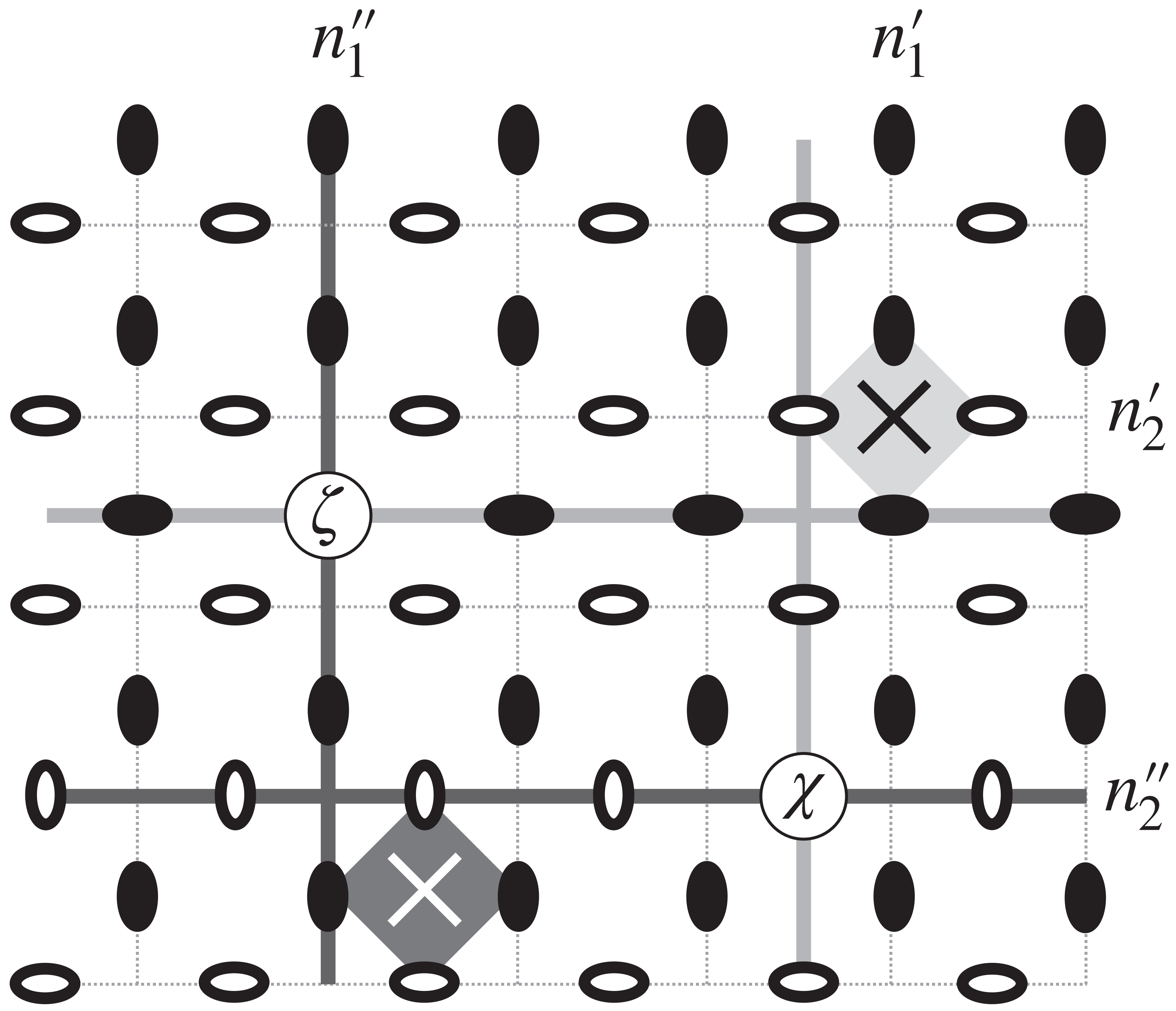



(toric code model)

unfold

**unitary transform**




independent "emergent" qubits

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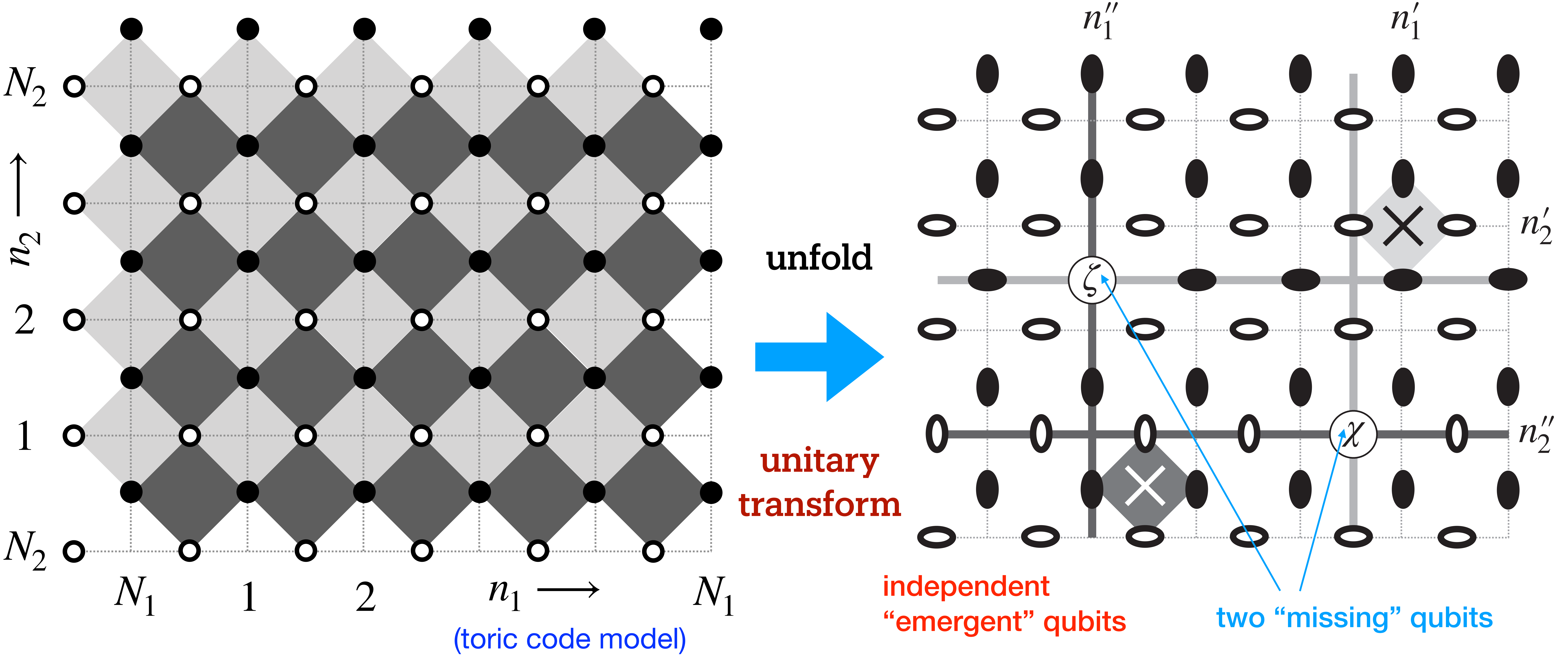

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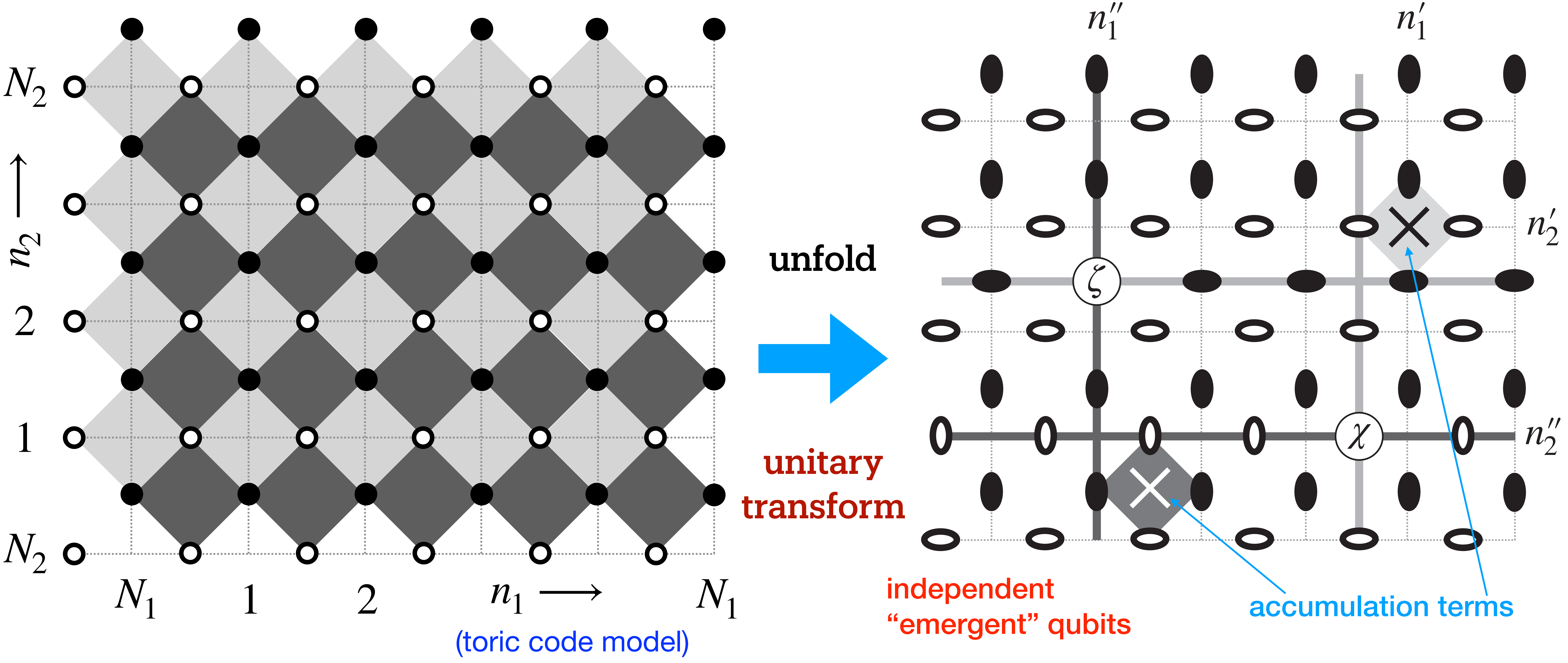

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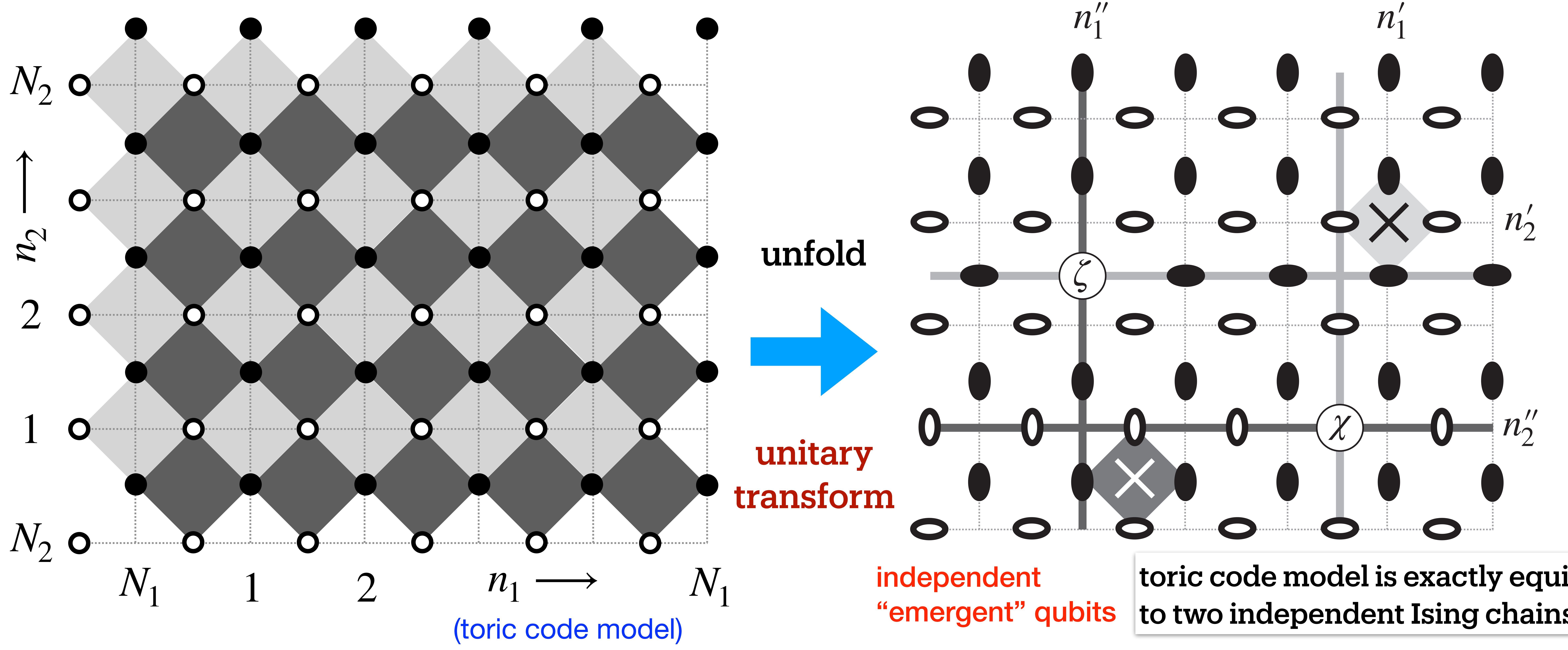


toric code model is exactly equivalent to two independent Ising chains

# Exact Eigenstates

BK, JPSJ (2025)

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BK, JPSJ (2025)

$$\hat{\rho}_{\text{toric-code}} = \prod_{(n_1,n_2)\neq(n'_1,n'_2)} \left( \frac{\hat{1} + x_{n_1,n_2}\hat{X}_{n_1,n_2}}{2} \right) \times \prod_{(n_1,n_2)\neq(n''_1,n''_2)} \left( \frac{\hat{1} + z_{n_1+\frac{1}{2},n_2-\frac{1}{2}}\hat{Z}_{n_1+\frac{1}{2},n_2-\frac{1}{2}}}{2} \right) \times \left[ \frac{1 + \chi \mathbf{u} \cdot \hat{\mathbf{A}}_{n'_1-\frac{1}{2},n''_2}}{2} \right] \times \left[ \frac{1 + \zeta \mathbf{v} \cdot \hat{\mathbf{B}}_{n''_1,n'_2-\frac{1}{2}}}{2} \right]$$

# Exact Eigenstates

BK, JPSJ (2025)

$2N_1N_2 - 2$ emergent qubit quantum numbers: $\{x = \pm 1\}$ and $\{z = \pm 1\}$ for conserved $\hat{X}$'s and $\hat{Z}$'s

$$\hat{\rho}_{\text{toric-code}} = \prod_{(n_1,n_2)\neq(n_1',n_2')} \left( \frac{\hat{1} + x_{n_1,n_2}\hat{X}_{n_1,n_2}}{2} \right) \times \prod_{(n_1,n_2)\neq(n_1'',n_2'')} \left( \frac{\hat{1} + z_{n_1+\frac{1}{2},n_2-\frac{1}{2}}\hat{Z}_{n_1+\frac{1}{2},n_2-\frac{1}{2}}}{2} \right) \times \left[ \frac{1 + \chi \mathbf{u} \cdot \hat{\mathbf{A}}_{n_1'-\frac{1}{2},n_2''}}{2} \right] \times \left[ \frac{1 + \zeta \mathbf{v} \cdot \hat{\mathbf{B}}_{n_1'',n_2'-\frac{1}{2}}}{2} \right]$$

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two "missing" qubits, having quantum numbers $\chi$ & $\zeta$, quantisation axes $\hat{u}$ & $\hat{v}$, and the "logical" qubit operators $\hat{A}$ & $\hat{B}$

$$\hat{\rho}_{\text{toric-code}} = \prod_{(n_1,n_2)\neq(n_1',n_2')} \left( \frac{\hat{1} + x_{n_1,n_2}\hat{X}_{n_1,n_2}}{2} \right) \times \prod_{(n_1,n_2)\neq(n_1'',n_2'')} \left( \frac{\hat{1} + z_{n_1+\frac{1}{2},n_2-\frac{1}{2}}\hat{Z}_{n_1+\frac{1}{2},n_2-\frac{1}{2}}}{2} \right) \times \left[ \frac{1 + \chi \mathbf{u} \cdot \hat{\mathbf{A}}_{n_1'-\frac{1}{2},n_2''}}{2} \right] \times \left[ \frac{1 + \zeta \mathbf{v} \cdot \hat{\mathbf{B}}_{n_1'',n_2'-\frac{1}{2}}}{2} \right]$$

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**very complete representation of the toric code eigenstates**

$$\hat{\rho}_{\text{toric-code}} = \prod_{(n_1,n_2)\neq(n_1',n_2')} \left(\frac{\hat{1} + x_{n_1,n_2}\hat{X}_{n_1,n_2}}{2}\right) \times \prod_{(n_1,n_2)\neq(n_1'',n_2'')} \left(\frac{\hat{1} + z_{n_1+\frac{1}{2},n_2-\frac{1}{2}}\hat{Z}_{n_1+\frac{1}{2},n_2-\frac{1}{2}}}{2}\right) \times \left[\frac{1 + \chi\mathbf{u}\cdot\hat{\mathbf{A}}_{n_1'-\frac{1}{2},n_2''}}{2}\right] \times \left[\frac{1 + \zeta\mathbf{v}\cdot\hat{\mathbf{B}}_{n_1'',n_2'-\frac{1}{2}}}{2}\right]$$

same algebraic closed form for all eigenstates; the excited states do not require be described any differently from the ground state.

# Exact Quantum Circuits

BK, JPSJ (2025)

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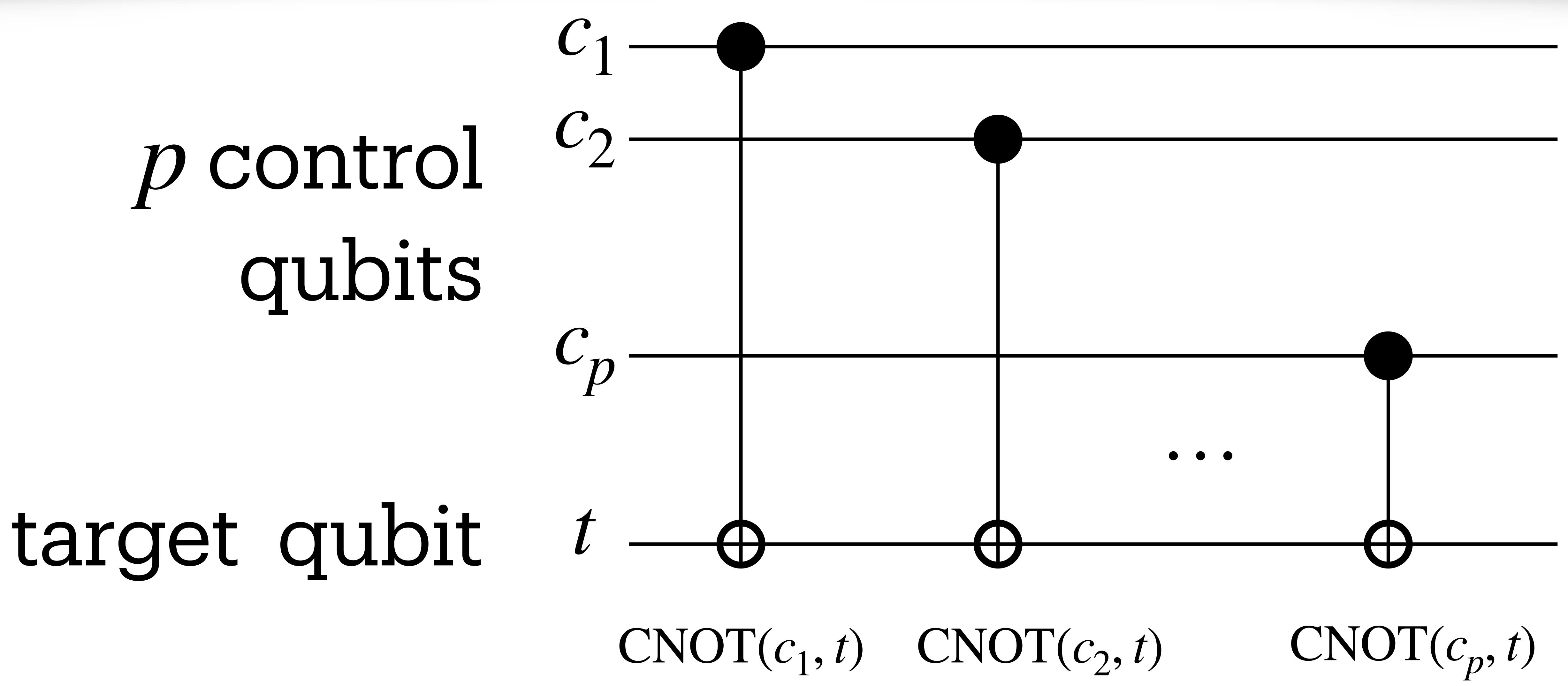
**Unitary transformation** that turns the toric code model into emergent qubits **reduce exactly into CNOT quantum gates**.

$$\left(\hat{\sigma}^z_{c_1}\hat{\sigma}^z_{c_2}\ldots\hat{\sigma}^z_{c_p}\right)^{\hat{Q}^x_t} = \mathrm{CNOT}(c_1,t)\cdot\mathrm{CNOT}(c_2,t)\cdots\cdot\mathrm{CNOT}(c_p,t)$$



# Exact Quantum Circuits

BK, JPSJ (2025)

unitary transformation for
the toric code model

$$\hat{U}_{4_1} = \prod_{n_1=1}^{N_1} \left[ \prod_{n_2=2}^{N_2} \left( \hat{\sigma}^x_{n_1-\frac{1}{2},n_2} \hat{\sigma}^x_{n_1,n_2+\frac{1}{2}} \hat{\sigma}^x_{n_1+\frac{1}{2},n_2} \right)^{\hat{Q}^z_{n_1,n_2-\frac{1}{2}}} \right]$$

$$\hat{U}_{4_2} = \prod_{n_1=1}^{N_1} \left[ \prod_{n_2=N_2}^{2} \left( \hat{\sigma}^z_{n_1+\frac{1}{2},n_2-1} \right)^{\hat{Q}^x_{n_1+\frac{1}{2},n_2}} \right]$$

$$\hat{U}_{4_3} = \prod_{n_1=2}^{N_1} \left[ \hat{\sigma}^x_{n_1+\frac{1}{2},1} \prod_{n_2\neq 1} \hat{\sigma}^x_{n_1,n_2-\frac{1}{2}} \right]^{\hat{Q}^z_{n_1-\frac{1}{2},1}}$$

$$\hat{U}_{4_4} = \prod_{n_1=2}^{N_1} \left[ \hat{\sigma}^z_{n_1+1,\frac{1}{2}} \prod_{n_2\neq 1} \hat{\sigma}^z_{n_1+\frac{1}{2},n_2} \right]^{\hat{Q}^x_{n_1,\frac{1}{2}}}$$

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$$\hat{U}_{4_2} = \prod_{n_1=1}^{N_1}\left[\prod_{n_2=N_2}^{2}\left(\hat{\sigma}^z_{n_1+\frac{1}{2},n_2-1}\right)^{\hat{Q}^x_{n_1+\frac{1}{2},n_2}}\right]$$

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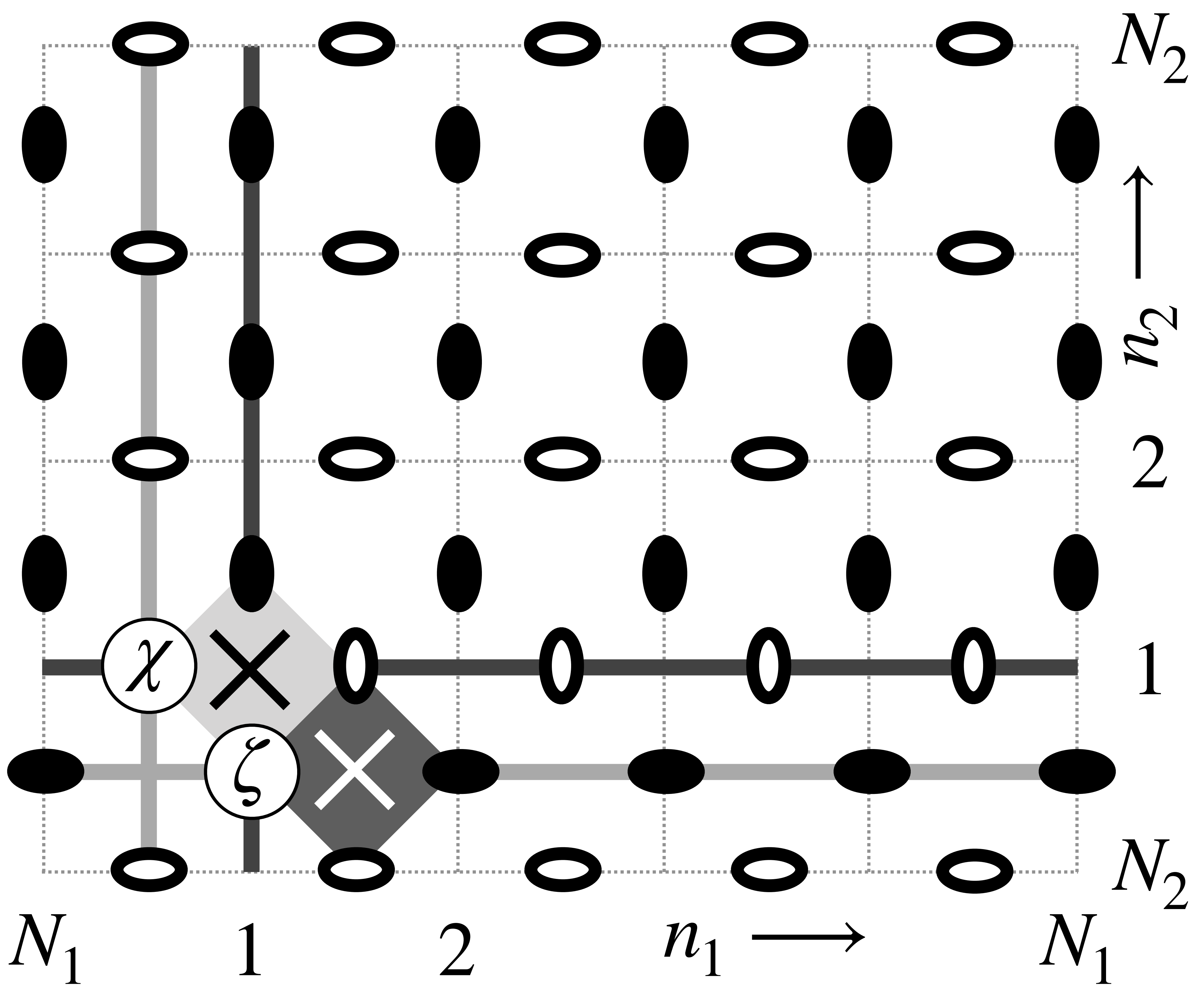

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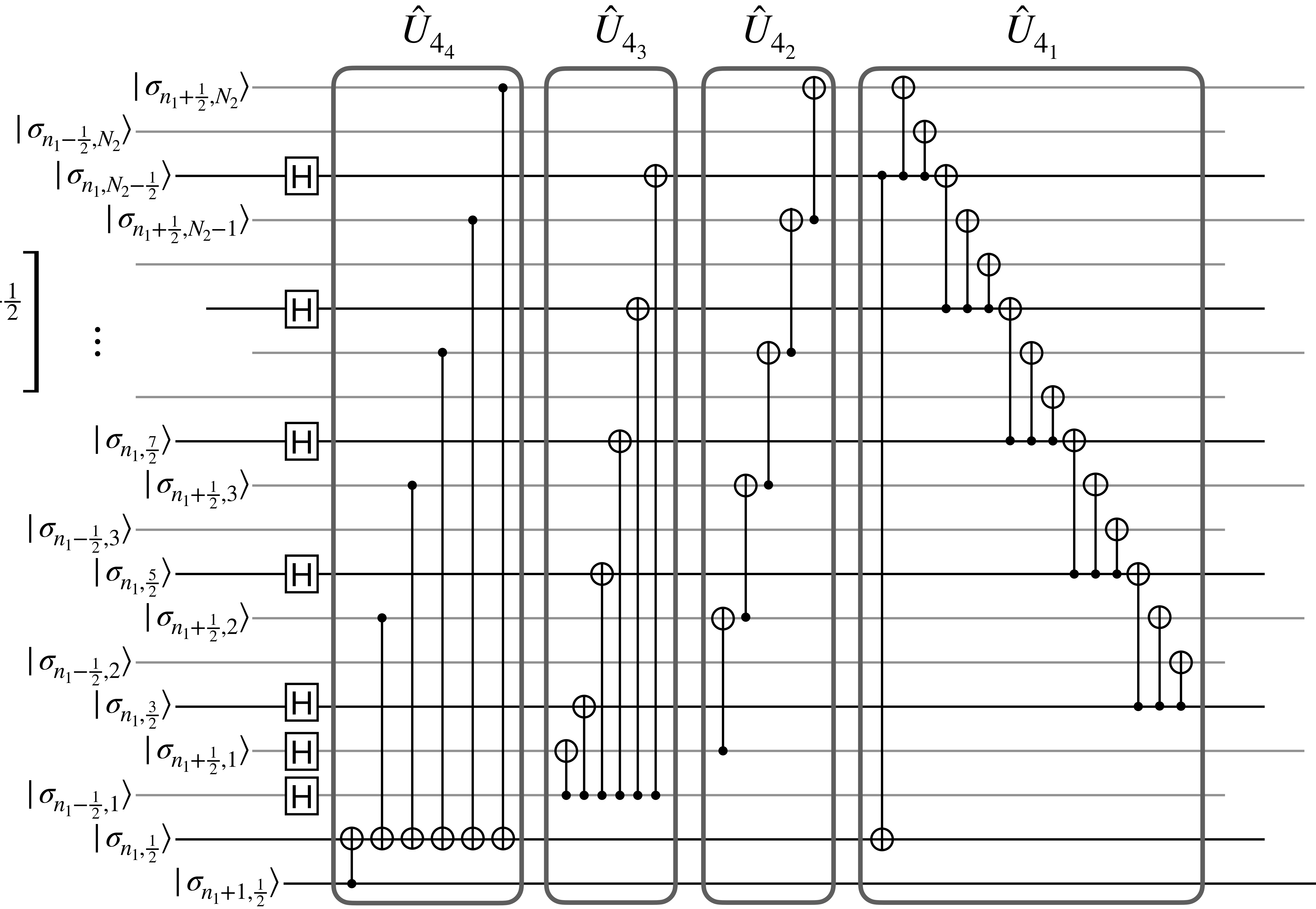


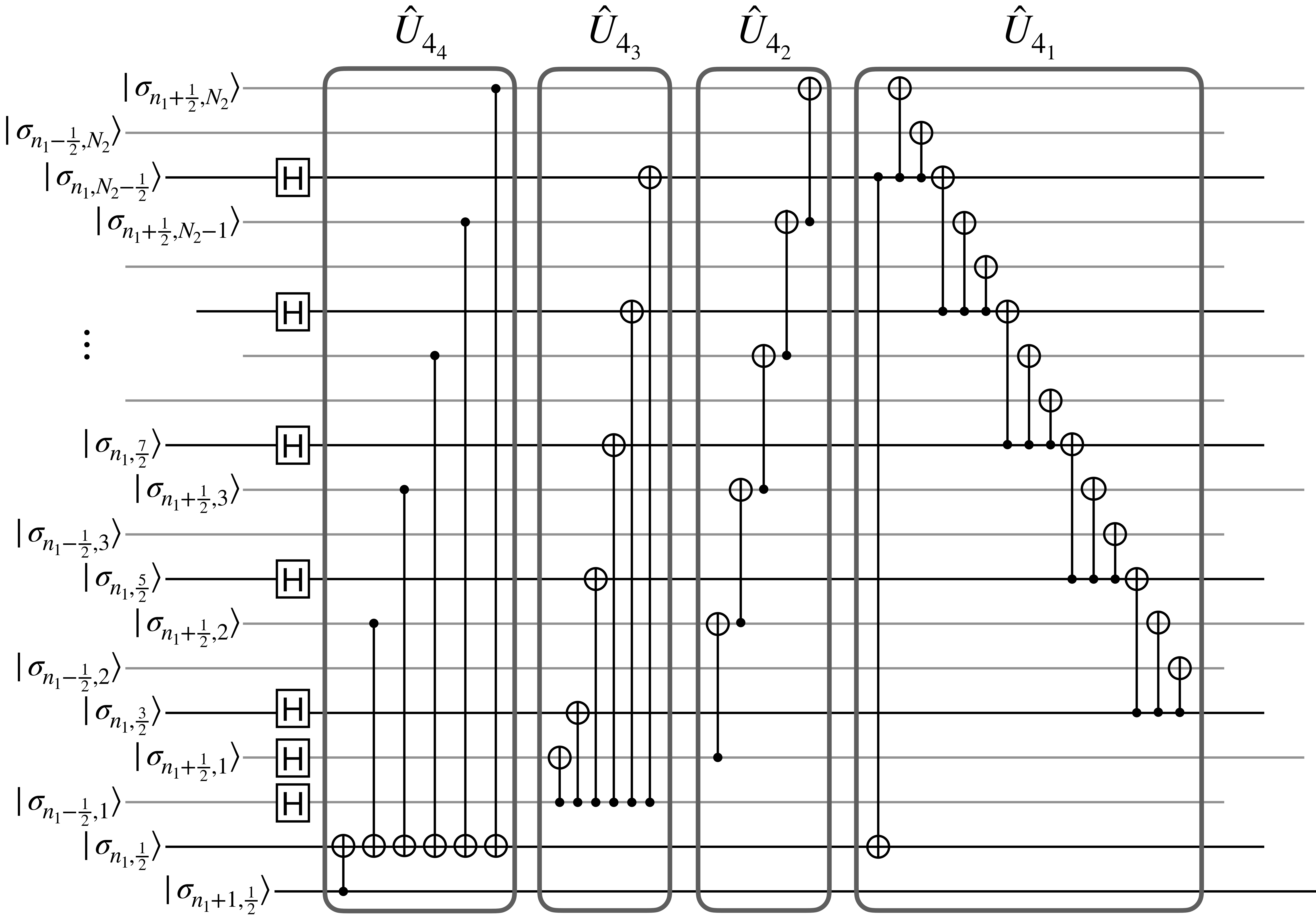

circuit on torus

# Exact Quantum Circuits

BK, JPSJ (2025)

unitary transformation for the toric code model

- CNOT gates required $\sim N_1 N_2$
- Circuit-depth $\sim N_2$ for the input state with all qubits in $|+\rangle$ state, or $\sim N_2 + N_1$ for input states with a few qubits in $|-\rangle$ state. In general, circuit-depth $\sim N_1 N_2$ on torus.
- Generates linear superposition in degenerate eigensubspace by applying rotation to the missing qubits




circuit on torus

# Summary

1. Toric code model rigorously transformed into independent qubits.

2. Complete set of eigenstates constructed exactly.

3. Exact quantum circuits devised for generating any toric code eigenstate.

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## Thank You