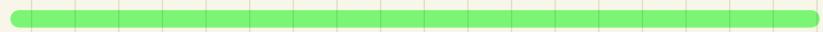


# Dissipative Euler flows satisfying the local energy inequality

$$\left\{ \begin{array}{l} \partial_t v_q + (v_q \cdot \nabla) v_q + \nabla p_q = \nu_q \Delta v_q \\ \operatorname{div} v_q = 0 \end{array} \right.$$

$$E(t) := \frac{1}{2} \int_{\mathbb{T}^3} |v_q|^2(x, t) dx$$

$$E'(t) = - \nu_q \int_{\mathbb{T}^3} |\nabla v_q|^2(x, t) dx$$



$$> 0$$

$$\nu_q \rightarrow 0$$

Theorem (Isett 2012)  $\forall \kappa < \frac{1}{3}$

If  $q = +\infty$ , i.e.  $v_q = 0$ .  $\exists$  weak solutions  $u$  s.t.

$$|u(x,t) - v(\varrho, t)| \leq C|x-\varrho|^\alpha$$

and  $E'(t) \neq 0$

Remarks  $u$  is achieved as  $\lim_{q \rightarrow +\infty} v_q$

$$\left\{ \begin{array}{l} \partial_t u_q + \operatorname{div} u (u_q \otimes u_q) + \nabla p_q = \operatorname{div} \overset{\circ}{R}_q \\ \operatorname{div} u_q = 0 \end{array} \right.$$

compare

$$\overset{\circ}{R}_q \text{ to } \nu_q D u_q$$

$$\text{tr } \hat{R}_q = 0$$

$\hat{R}_q$  is symmetric

$$\text{div} (\underline{D} u_q + \underline{D} u_q^T) = \Delta u_q$$

We should compare

$$\hat{R}_q$$

to

$$\underline{D} u_q + \underline{D} u_q^T$$

Now

$$u_q = \sum_{i=0}^q w_i$$

Fourier scale for  $w_i$ :  $a^{l-i}$   
 (actually it is  $a^{b-i}$  but 1)

$$u_{q+1}(x, t) = u_q(x, t) + \underbrace{W(u_q, \hat{R}_q, \lambda x, \lambda t)}_{\text{+ correction}}$$

$$W = \sum_j \alpha_j(u_q, \hat{R}_q) e^{i \lambda b_j x}$$

Not so important in  
 many instances

bew de  $\omega$  so that

$$\partial_t \omega_{q+1} + \operatorname{div} \omega (\omega_{q+1} \otimes \omega_{q+1}) + \nabla p_{q+1}$$

$= \operatorname{div} \overset{\circ}{R}_{q+1}$

$$\overset{\circ}{R}_{q+1} = \operatorname{div}^{-1} ( )$$

1)  $\omega$  solves a PDE in the fast

variables and kills  $\lambda_q(\cdot)$  terms in the green expression

2) slow derivatives are killed because

$$\operatorname{div}^{-1} \left( \sum_j b_j(t, x) e^{i \lambda_q k_j \cdot x} \right)$$

$$u \frac{1}{\lambda_q}$$

3) Except  $\operatorname{div}_s$  slow  $\sum_{j,k} \underbrace{a_{jk} w_j \otimes w_k + \overset{\circ}{R}_q}_{\text{Resonances "cancel" } \overset{\circ}{R}_q}$

A)  $\omega_{q+1}$  interacts "only" with  $\omega_q$   
and  $\omega_{q+2}$

B)  $L^\infty$  size of  $\omega_{q+1} \sim \lambda_{q+1}^{-1/3}$

$$\int |\omega_{q+1}|^2 \sim \lambda_{q+1}^{-2/3}$$

Energy spectrum  $\sim \lambda^{-5/3}$

c) Do  $\tilde{R}_q$  and  $V_q$   $D_{eq}$  coupled in H2e  
 $\|D_{eq}\|_\infty \sim \lambda_q^{2/3}$

$$\|\tilde{R}_q\|_\infty \sim \lambda_{q+1}^{-2/3}$$

$$V_q \sim \lambda_{q+1}^{-2/3} \lambda_q^{-2/3} \sim \lambda_q^{-4/3}$$

Nirenberg's scale !

— Energy identity

$$\partial_t \frac{|u_q|^2}{2} + \operatorname{div}(u_q \left( \frac{|u_q|^2}{2} + p_q \right)) = \left( \Delta \frac{|u_q|^2}{2} - |\nabla u_q|^2 \right) v_q$$

Suggests

Conjecture (Isent)  $\forall \alpha < \frac{1}{3} \exists v$

sol. of Euler s.t.

$$(E1) \quad \partial_t \frac{|v|^2}{2} + \operatorname{div}(v \left( \frac{|v|^2}{2} + p \right)) \leq 0$$

$$\text{and } v \in L^\infty_t(C_x^\alpha)$$

Theorem (Buckmaster, D., Steinelykke) (Vicol 2018)

$$\frac{d}{dt} \int \frac{|u|_x^2}{2} (x, t) dx < 0.$$

Theorem (D. Steinelykke 2002)  $\exists$  weak sol. est.

Theorem (Isella 2019) (EI) in  $L^\infty$

$\exists C$  solutions sol. (EI) if  $\alpha < \frac{1}{15}$

Theorem (D.-Ikwon 2020)  $\alpha < \frac{1}{7}$

$$\begin{aligned} - \left( \partial_t u_q + \operatorname{div}(u_q \otimes u_q) + \nabla p_q = \right. \\ \left. \operatorname{div} \tilde{R}_q \right) \\ \operatorname{div} u_q = 0 \\ \partial_t |u_q|^2 + \operatorname{div} \left( (|u_q|^2 + \tilde{R}_q) u_q \right) \\ \leq \operatorname{div} (\tilde{R}_q u_q) + \operatorname{div} \phi_q \\ \text{writing} \end{aligned}$$

Resources on the Celtic term

Wolfehill & Q