Unitary Symmetries of Open Quantum Systems Trajectories

Calum Brown

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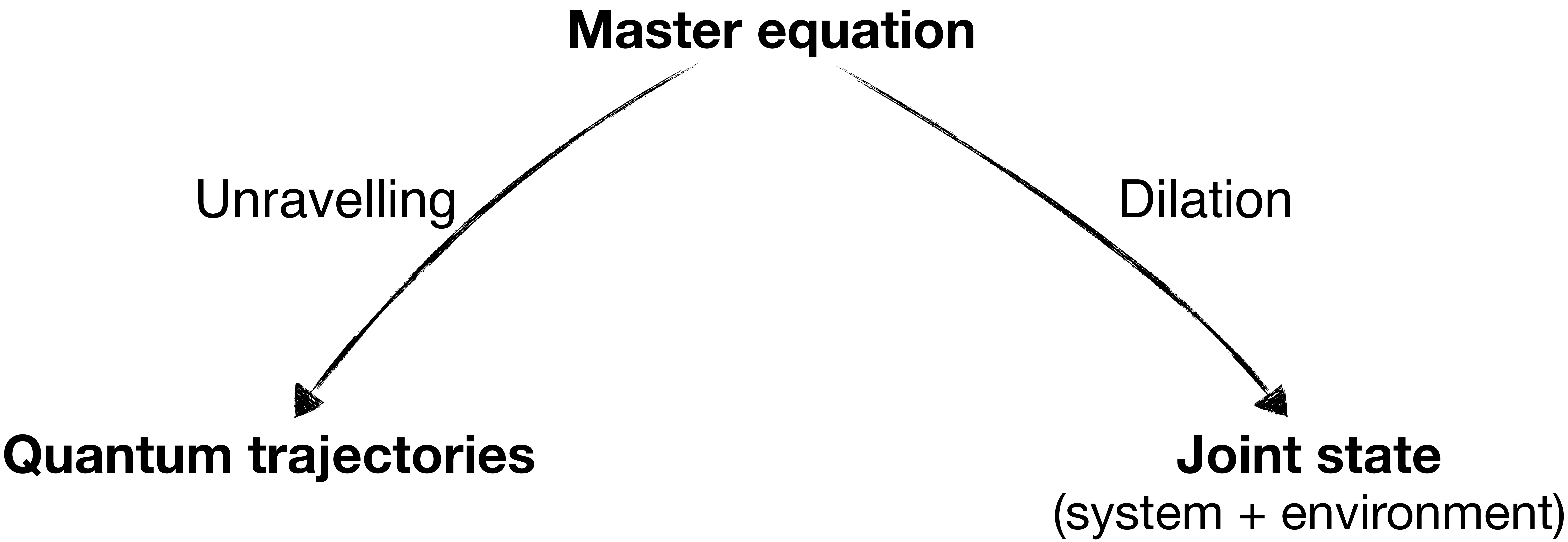
Outline


Master equation
Unravelling
Dilation
Quantum trajectories
Joint state
(system + environment)

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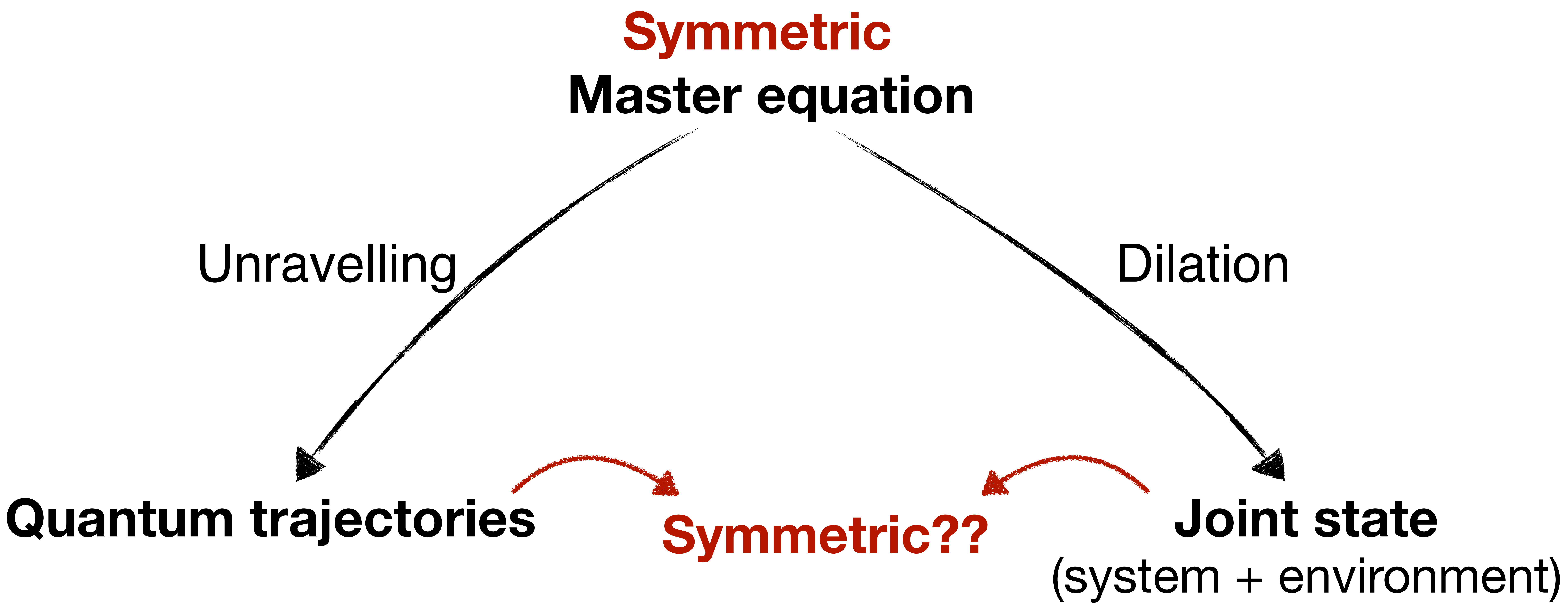
Symmetric
Master equation
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Quantum trajectories
Symmetric??
Joint state
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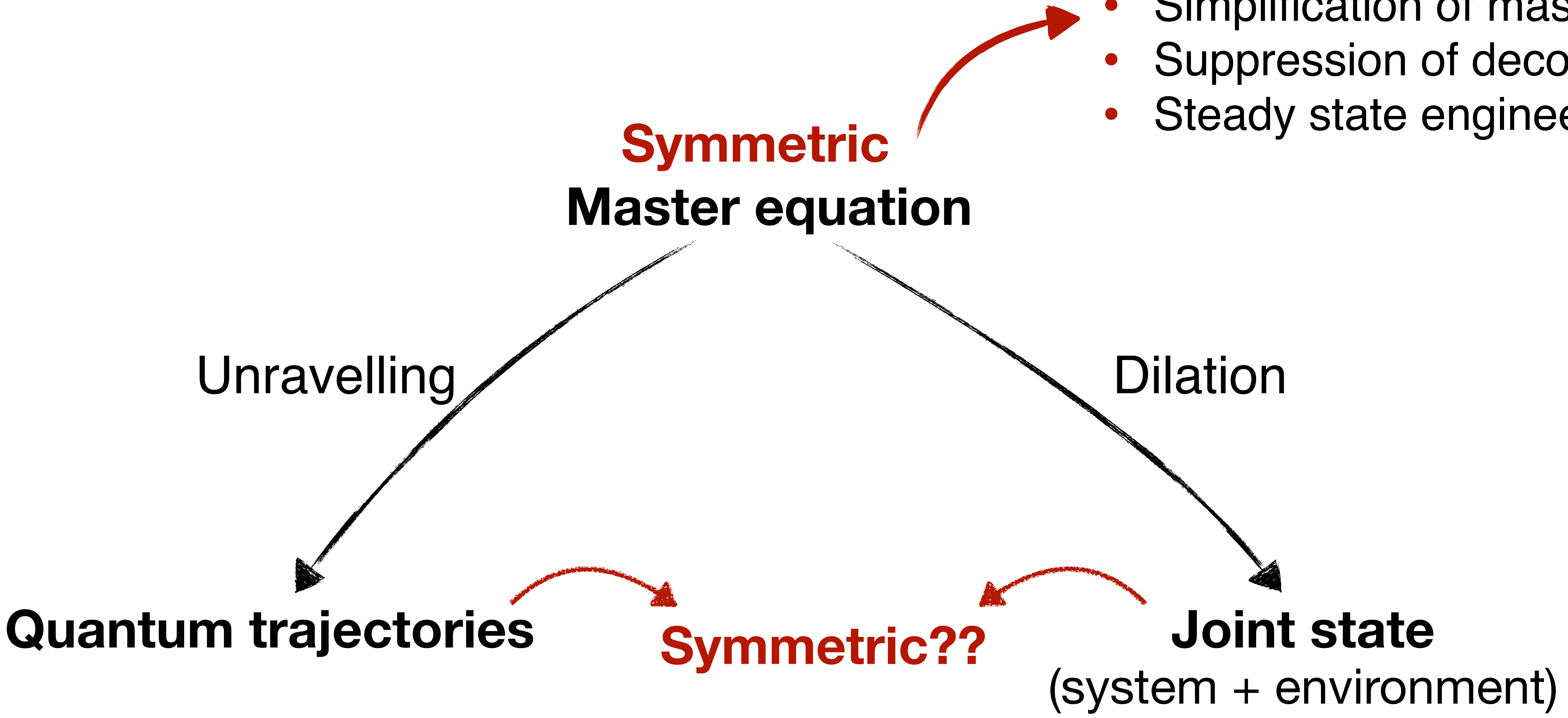
Outline

- Physical insight
- Reduce degrees of freedom
- Supply spectral information
- Simplification of master operators
- Suppression of decoherence
- Steady state engineering



Weak symmetry of master equation

- Quantum master equation (QME) $$\frac{d}{dt}\rho(t) = -i[H,\rho_t] + \sum_{k=1}^{d}\left(J_k\rho_t J_k^\dagger - \frac{1}{2}\{J_k^\dagger J_k,\rho_t\}\right) \equiv \mathcal{L}(\rho)$$

- Consider a unitary symmetry operator U, such that $U^N = \mathbb{1}$

- States transform under the symmetry as $\mathcal{U}(\rho) = U\rho U^\dagger$.

- The master operator, $\mathcal{L}$, is weakly symmetric with respect to unitary symmetry U, iff

$$\mathcal{U}\mathcal{L}\mathcal{U}^\dagger = \mathcal{L}$$

Symmetry condition I

[Buča, Prosen 2012], [Albert, Jiang 2014]

Quantum trajectories

- Conditional state evolves by stochastic Schrödinger equation:

$$d\psi_t = \mathcal{B}[\psi_t]dt + \sum_k \left(\frac{\mathcal{J}_k(\psi_t)}{\mathrm{Tr}[\mathcal{J}_k(\psi_t)]} - \psi_t \right) dq_{k,t}$$

[Dalibard, Castin, Mølmer 1992]
[Wiseman, Milburn 2010]

Quantum trajectories

$$\mathcal{B}[\psi] = -iH_{\text{eff}}\psi + i\psi H_{\text{eff}}^{\dagger} - \psi\,\mathrm{Tr}(-iH_{\text{eff}}\psi + i\psi H_{\text{eff}}^{\dagger}), \qquad H_{\text{eff}} = H - \frac{i}{2}\sum_k J_k^{\dagger}J_k$$

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$$\mathcal{J}_k(\psi) = J_k \psi J_k^{\dagger}, \qquad dq_{k,t} \in \{0,1\}, \qquad \mathbb{E}[dq_{k,t}] = \mathrm{Tr}[\mathcal{J}_k(\psi_t)]dt$$

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- Probability distribution for conditional state $P(\psi, t)$
- Evolves by the **unravelled quantum master equation (UQME)**:

$$\frac{\partial}{\partial t} P(\psi, t) = \mathcal{W}^\dagger P(\psi, t)$$

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'Unravelled generator'

[Dalibard, Castin, Mølmer 1992]
[Wiseman, Milburn 2010]

Symmetry of unravelled generator

(Symmetry of quantum trajectories)

- Consider unitary superoperator Υ, which acts as $\Upsilon P(\psi, t) = P(\mathcal{U}^\dagger \psi, t)$

- The unravelled generator, $\mathcal{W}^\dagger$, is symmetric with respect to unitary symmetry U, when

$$\Upsilon \mathcal{W}^\dagger \Upsilon^\dagger = \mathcal{W}^\dagger$$

Conditions on $H, J_1 \ldots, J_d$?

Recall symmetry of master operator:

$\mathcal{U}\mathcal{L}\mathcal{U}^\dagger = \mathcal{L}$ **Symmetry condition I**

Symmetry of unravelled generator

(Symmetry of quantum trajectories)

- Group jump operators with the same destinations (for all initial states) into sets such that

$$i, j \in S_\alpha \;\Leftrightarrow\; J_i|\psi\rangle \propto J_j|\psi\rangle \;\; \forall|\psi\rangle$$

- Collective action of each set is $\mathcal{A}_\alpha(\psi) = \sum_{j \in S_\alpha} \mathcal{J}_j(\psi)$

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Types of jump sets

Reset jumps: $J_k = \sqrt{\gamma_k}|\chi_\alpha\rangle\langle\xi_k| \;\text{ for } k \in S_\alpha$

Non-reset jumps: $J_k = \lambda_k J^{(\alpha)} \text{ for } k \in S_\alpha$

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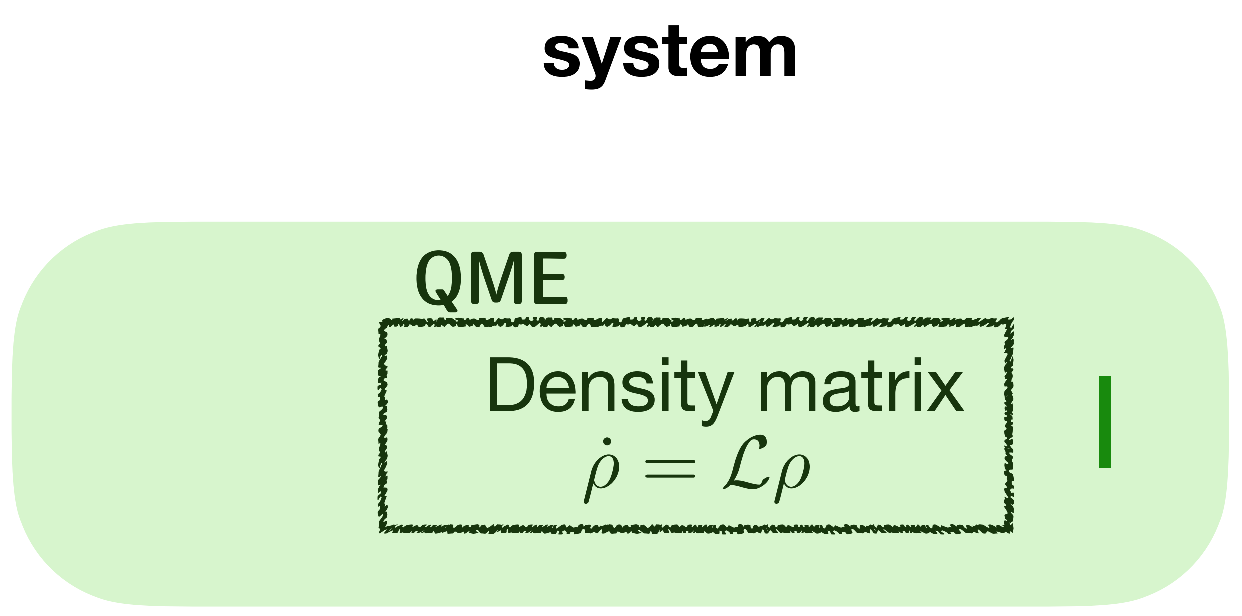
Symmetry condition II

Then $\Upsilon \mathcal{W}^\dagger \Upsilon^\dagger = \mathcal{W}^\dagger$, iff

$$\mathcal{U}(H) = H, \quad \mathcal{U}\mathcal{A}_\alpha\mathcal{U}^\dagger = \mathcal{A}_{\pi(\alpha)} \;\forall\alpha$$

(Current) Summary



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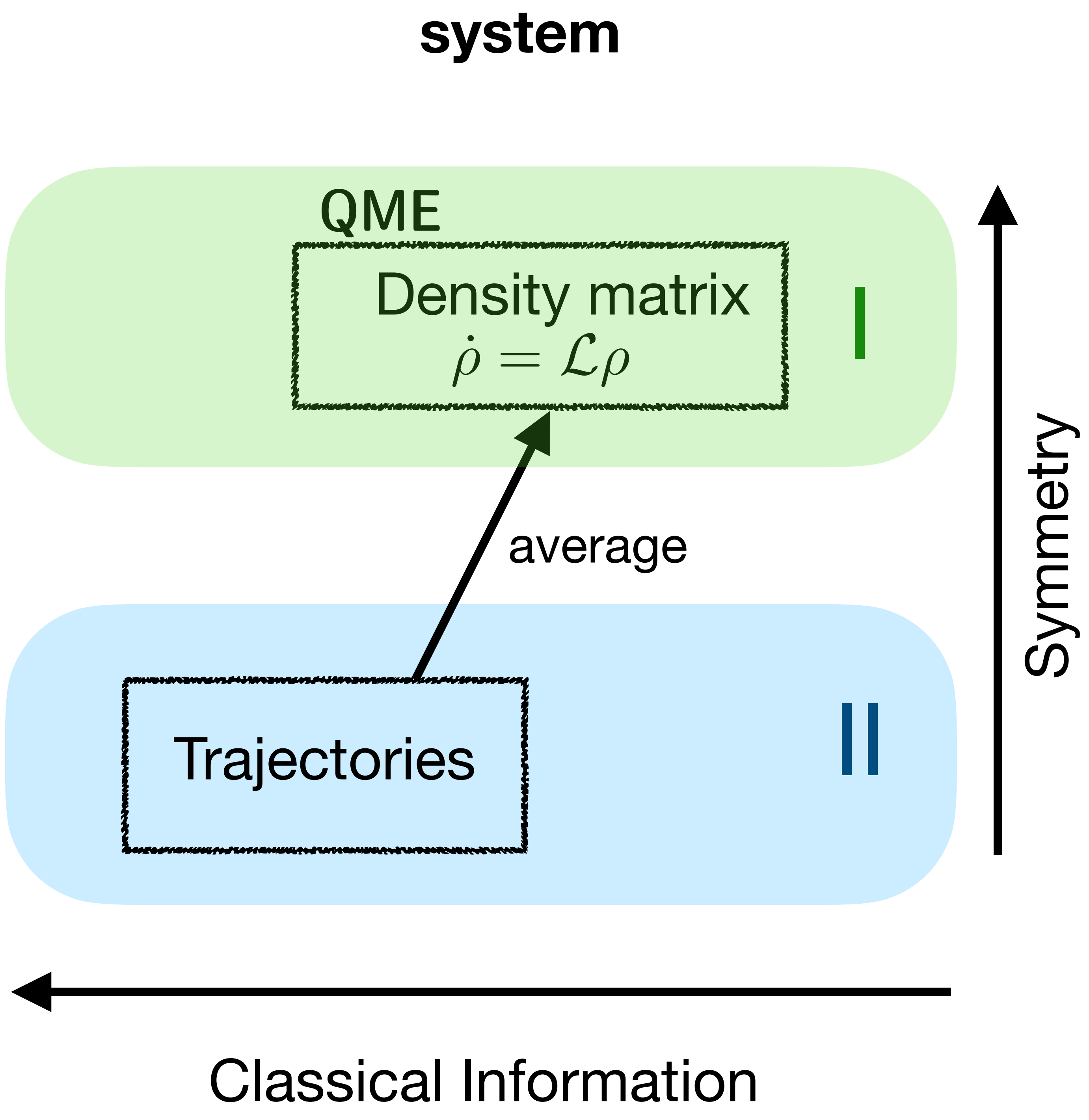

$$\text{I} \Longleftarrow \text{II}$$

Implications of symmetry on dynamics

Symmetric Master equation

$$\mathcal{U}\mathcal{L}\mathcal{U}^\dagger = \mathcal{L}$$

Symmetric unravelled generator

$$\Upsilon \mathcal{W}^\dagger \Upsilon^\dagger = \mathcal{W}^\dagger$$

Implications of symmetry on dynamics

Symmetric Master equation

$\mathcal{U}\mathcal{L}\mathcal{U}^\dagger = \mathcal{L}$

- For initial state ρ_0, solution of QME is given by path $\rho_{[0,\tau)}$
- For the symmetry transformed initial state $\mathcal{U}(\rho_0)$, the solution of QME is the path $\mathcal{U}(\rho_{[0,\tau)})$
- For symmetric initial state

$$\mathcal{U}(\rho_0) = \rho_0 \implies \mathcal{U}(\rho_t) = \rho_t$$

Symmetric unravelled generator

$\Upsilon \mathcal{W}^\dagger \Upsilon^\dagger = \mathcal{W}^\dagger$

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Symmetric unravelled generator

$\Upsilon \mathcal{W}^\dagger \Upsilon^\dagger = \mathcal{W}^\dagger$

- For initial distribution P_0, the solution of UQME is given by path $P_{[0,\tau)}$
- For the symmetry transformed initial distribution $\Upsilon(P_0)$, the solution of UQME is given by path $\Upsilon(P_{[0,\tau)})$
- For symmetric initial distribution

$$\Upsilon(P_0) = P_0 \implies \Upsilon(P_t) = P_t$$

- Symmetry of stochastic trajectories

$$p(\psi_{[0,t)}|\psi_0) = p\big(\mathcal{U}(\psi_{[0,t)})|\mathcal{U}(\psi_0)\big)$$

Trajectories with measurement records

- Now consider trajectories of $(\psi_t, \mathbf{q}_t)$

$$d\psi_t = \mathcal{B}[\psi_t]dt + \sum_k \left(\frac{\mathcal{J}_k(\psi_t)}{\mathrm{Tr}[\mathcal{J}_k(\psi_t)]} - \psi_t \right) dq_{k,t}$$

(SSE)

- These 'labelled trajectories' evolve with generator $\mathcal{W}_F^\dagger$,

$$\frac{\partial}{\partial t} P_t(\psi, \mathbf{q}) = \mathcal{W}_F^\dagger P_t(\psi, \mathbf{q})$$

- Introduce unitary symmetry operator Υ_F , which acts as $\Upsilon_F P_t(\psi, \mathbf{q}) = P_t(\mathcal{U}^\dagger \psi, \pi^{-1}(\mathbf{q}))$

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The generator is symmetric, $\Upsilon_F \mathcal{W}_F^\dagger \Upsilon_F^\dagger = \mathcal{W}_F^\dagger$, iff

$\mathcal{U}(H) = H, \qquad \mathcal{U}(J_k) = J_{\pi(k)} e^{i\phi_k} \quad \forall k$ **Symmetry condition III**

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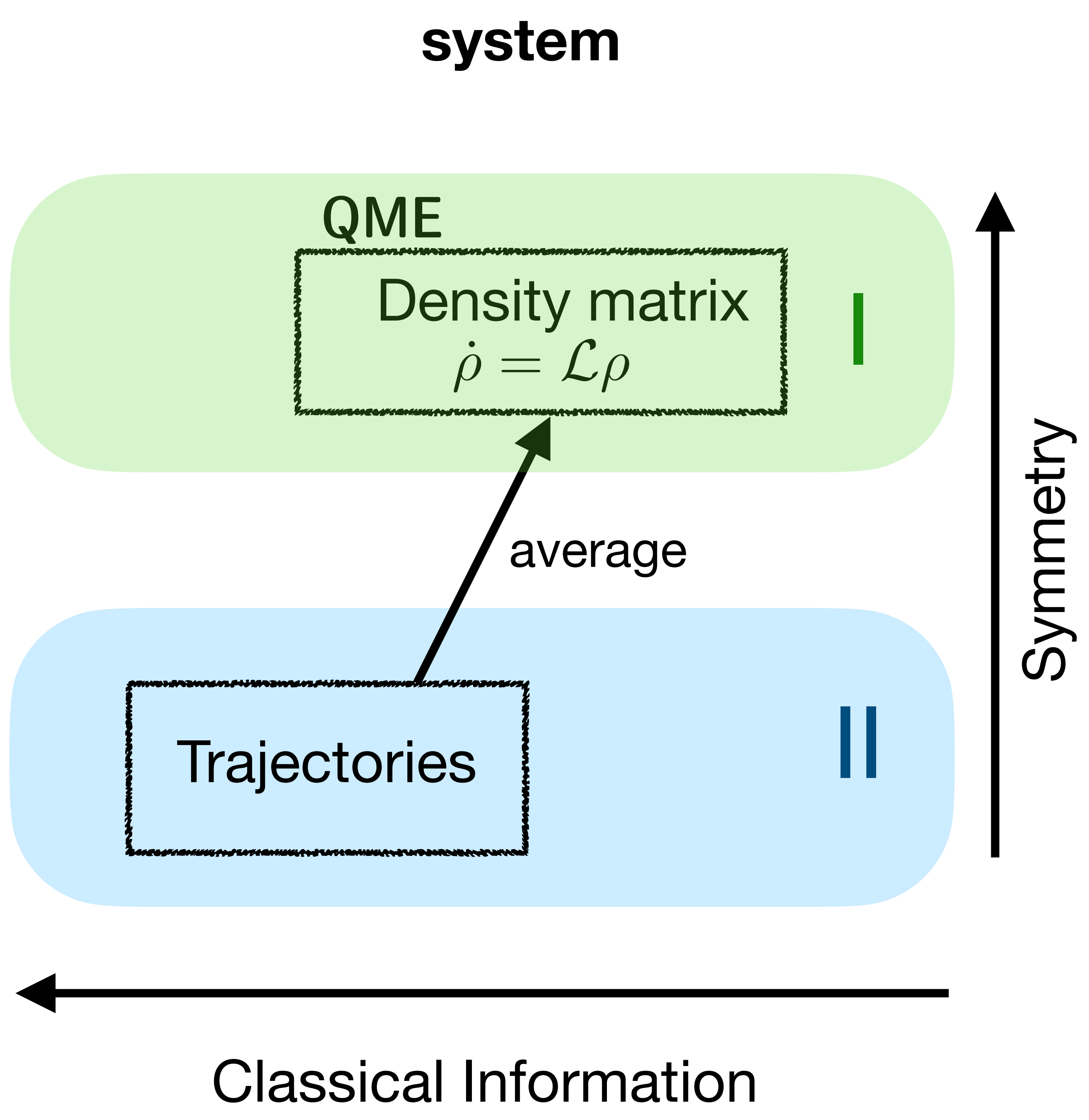

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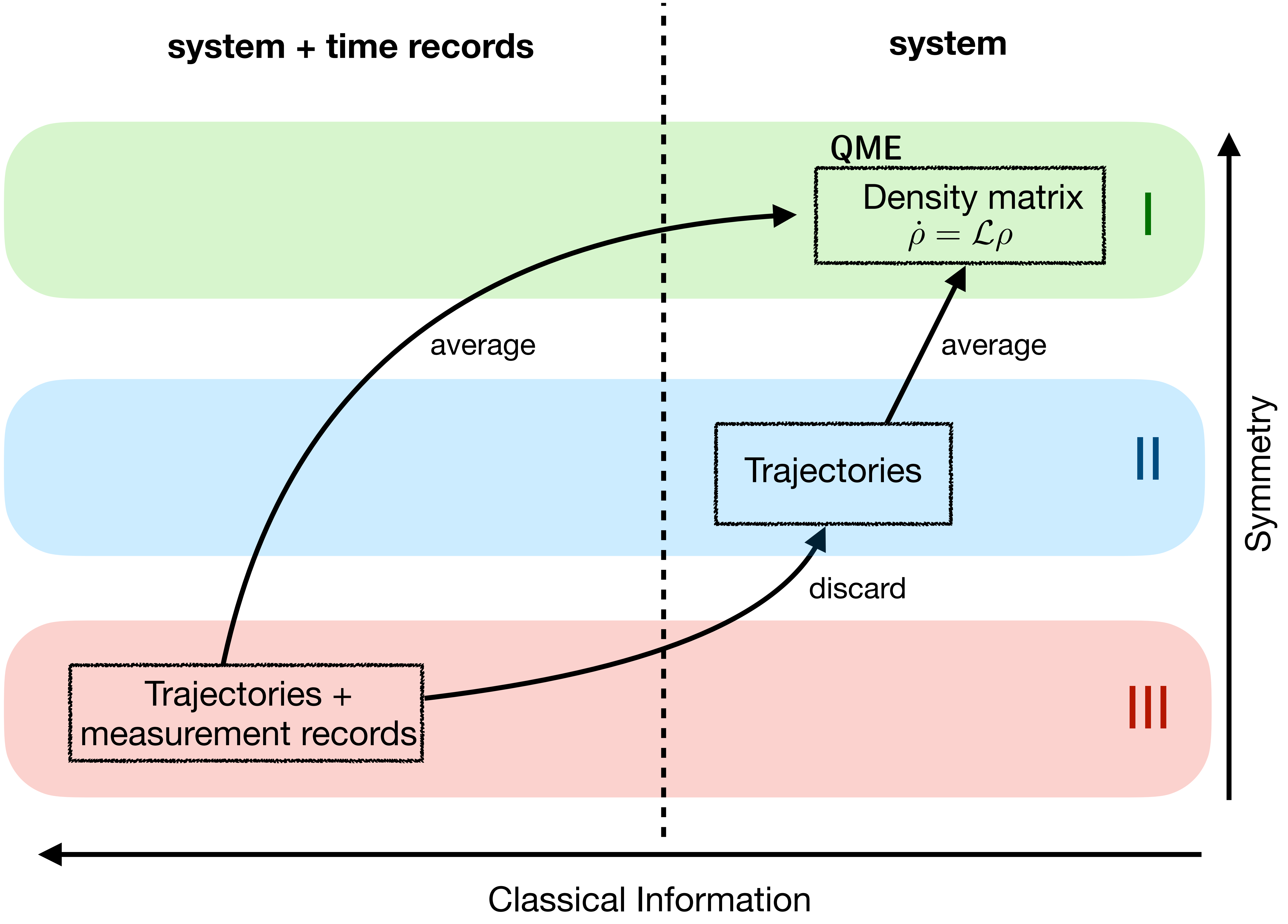

$$\text{I} \Longleftarrow \text{II} \Longleftarrow \text{III}$$

Joint state

- Dilation of the QME leads to unitary evolution of the system + environment joint state, given by the **continuous matrix product state (cMPS)**

$$|\Psi_t\rangle = \int_{\mathbf{m}_t} |\varphi_t(\mathbf{m}_t)\rangle \otimes |d\mathbf{m}_t\rangle$$

$|\varphi_t(\mathbf{m_t})\rangle = e^{-iH_{\text{eff}}(t-t_n)} J_{j_n} \dots e^{-iH_{\text{eff}}(t_2-t_1)} J_1 e^{-iH_{\text{eff}} t_1} |\psi_0\rangle$, $\quad |d\mathbf{m}_t\rangle = dB^\dagger_{j_n,t_n} \dots dB^\dagger_{j_1,t_1} |\text{vac}\rangle$, $\quad dB_{j,t} dB^\dagger_{k,t} = \delta_{jk} dt$

- Evolves as $d|\Psi_t\rangle = -i dH_t |\Psi_t\rangle$

with Hamiltonian $dH_t \equiv H \otimes \mathbb{1}_E \, dt + i \sum_{j=1}^{d} \left(J_j \otimes dB^\dagger_{j,t} - J_j^\dagger \otimes dB_{j,t} \right)$

[Hudson, Parthasarathy 1984]
[Verstraete, Cirac 2010]

Symmetry of cMPS

- Define unitary operation on environment $\mathcal{U}_E(\cdot) = U_E(\cdot)U_E^\dagger$
- cMPS is symmetric with respect to symmetry $\mathcal{U} \otimes \mathcal{U}_E$, that is $\mathcal{U} \otimes \mathcal{U}_E(dH'_t) = dH'_t$ iff

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$\mathcal{U}\mathcal{L}\mathcal{U}^\dagger = \mathcal{L}$ and $\mathcal{U}_E$ is taken as $\mathcal{U}_E(dB_{j,t}) = \sum_k \mathbf{U}_{jk} dB_{k,t}$ **Symmetry condition I**

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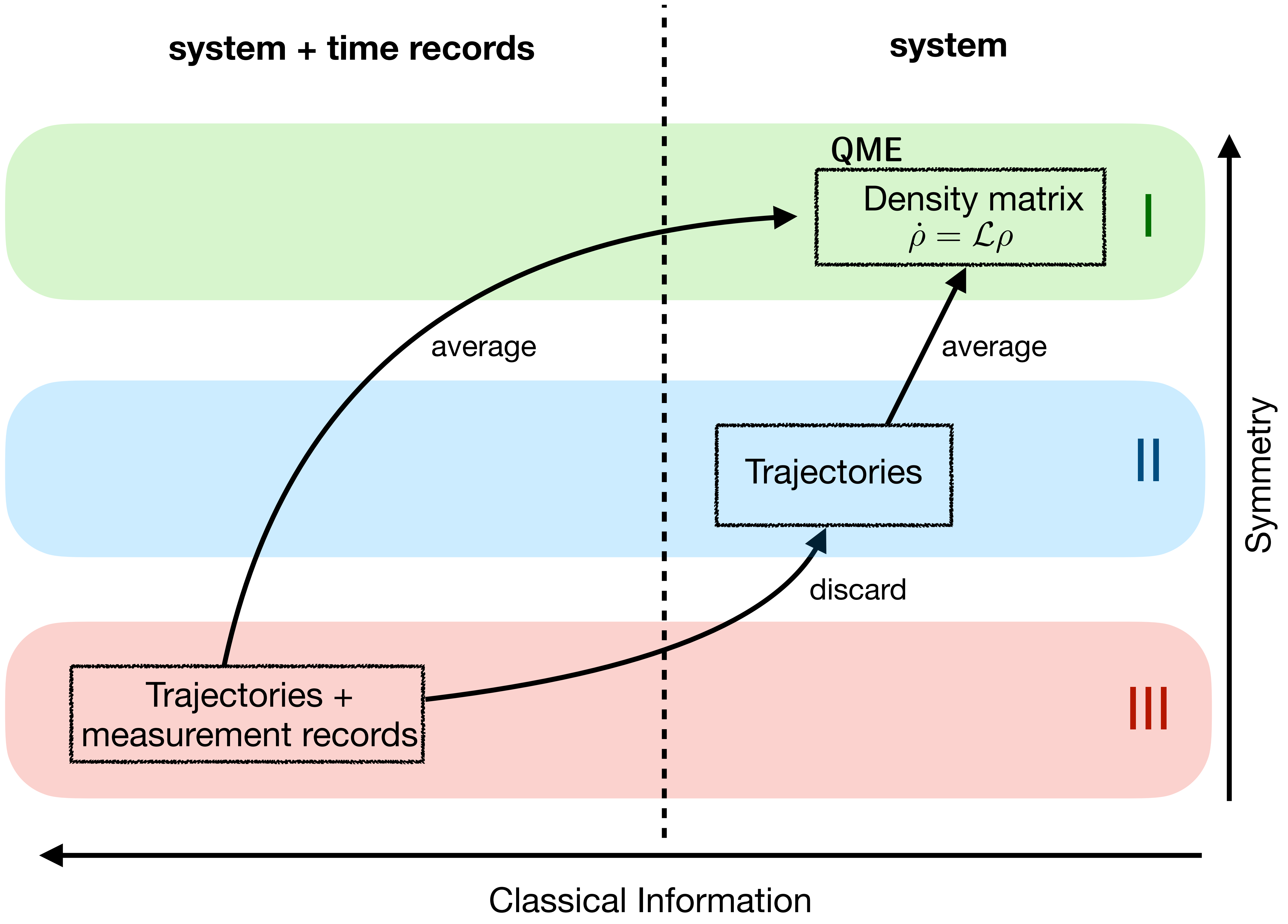
$\Updownarrow$

$$J_j' = J_j - \frac{\mathbb{1}}{d_s}\mathrm{Tr}(J_j), \quad \mathcal{U}(J_j') = \sum_k \mathbf{U}_{jk} J_k'$$

Symmetry condition I

(Current) Summary



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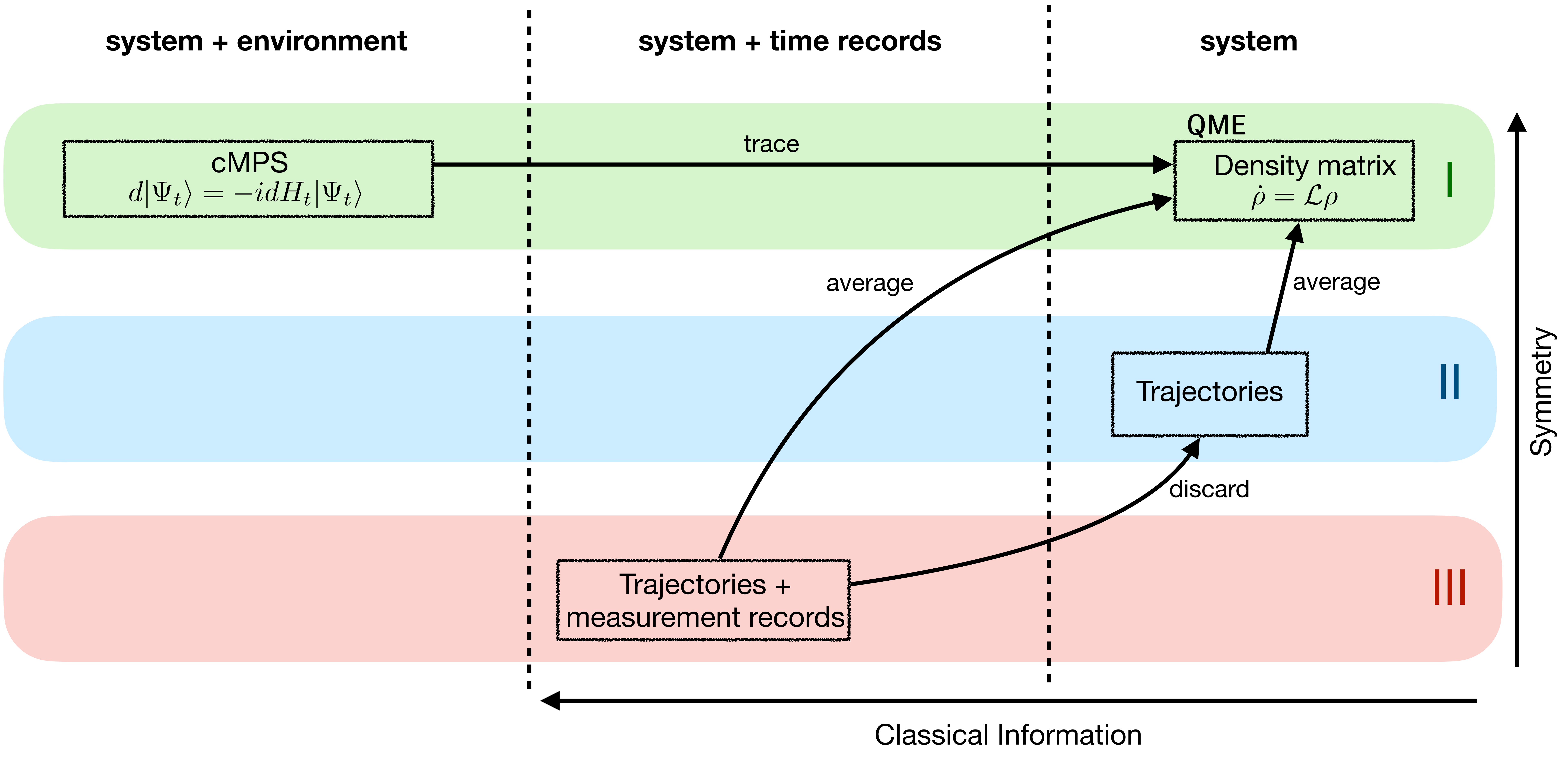

$I \Longleftarrow II \Longleftarrow III$

Objects in each layer *either* all symmetric *or* all non-symmetric

Dephasing of cMPS (measured CMPO)

- Quanta detected via projective measurement in the environment basis formed by dB's.
- Continuous matrix product operator (cMPO) for dephased joint state

$$R_t$$

$$R_t = \int p(\mathbf{m}_t|\psi_0)\,\psi_t(\mathbf{m}_t) \otimes |d\mathbf{m}_t\rangle\langle d\mathbf{m}_t|$$

- Evolution $dR_t = d\mathbb{L}_t(R_t)$

$$d\mathbb{L}_t(R_t) = -i\left[(H_{\text{eff}} \otimes \mathbb{1}_E)R_t - R_t(H_{\text{eff}}^\dagger \otimes \mathbb{1}_E)\right]dt$$

$$+\sum_{k=1}^{d}(J_k \otimes dB_{k,t}^\dagger)R_t(J_k^\dagger \otimes dB_{k,t})$$

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Symmetry

- Symmetric measured cMPO has $(\mathcal{U} \otimes \mathcal{U}_E) d\mathbb{L}_t (\mathcal{U} \otimes \mathcal{U}_E)^\dagger = d\mathbb{L}_t$
- This occurs iff $\mathcal{U}(H) = H, \qquad \mathcal{U}(J_k) = J_{\pi(k)} e^{i\phi_k} \quad \forall k$ **Symmetry condition III**

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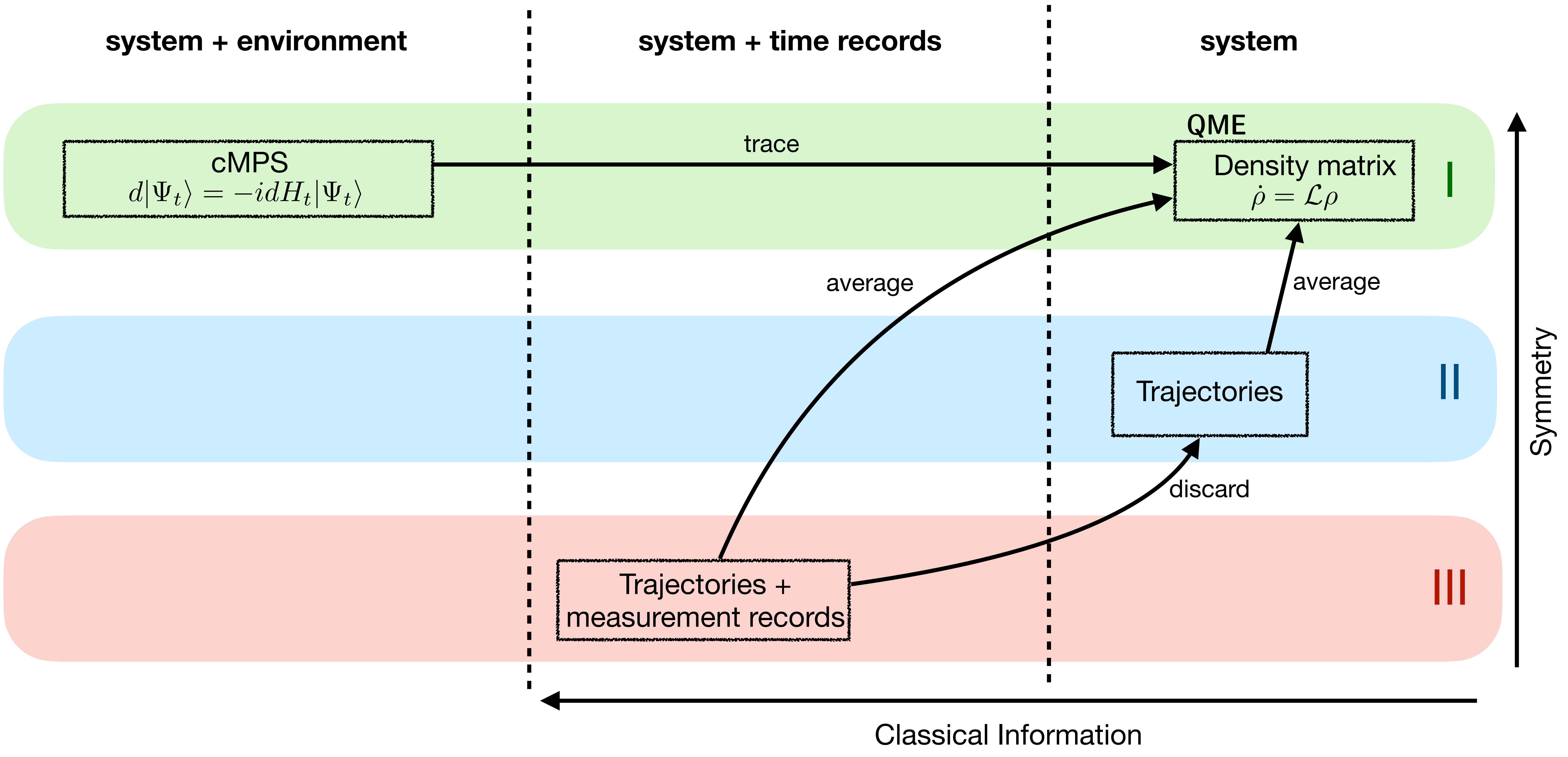

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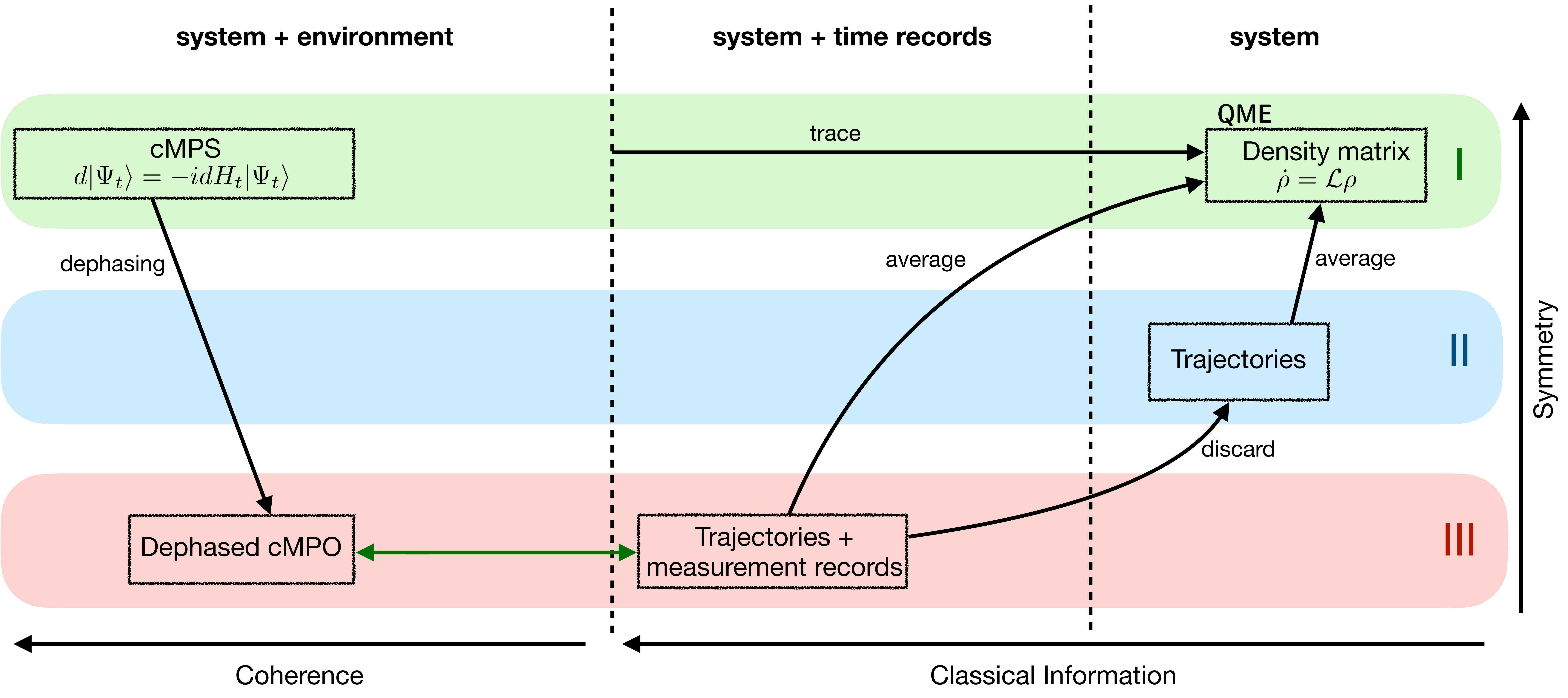

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Summary

- Characterised the conditions for weak unitary symmetry to be present in quantum trajectories and their measurement records. [By considering gauge freedoms of unravelled generator]
- Showed that there always exists a dilation of a symmetric QME such that the corresponding cMPS has a separable symmetry (and vice versa).
- Considered dephasing of the cMPS, which corresponds to quantum trajectories and their measurement records, in which these objects share symmetry conditions.
- Applications: eigenfunctions of generator, numerical simplification, support of operators, physical insight
- Future outlook: non-unitary symmetries, approximate symmetry, resource theory of asymmetry

Appearing on arXiv soon:

[Brown, Jack, Macieszczak]
[Brown, Macieszczak, Jack]

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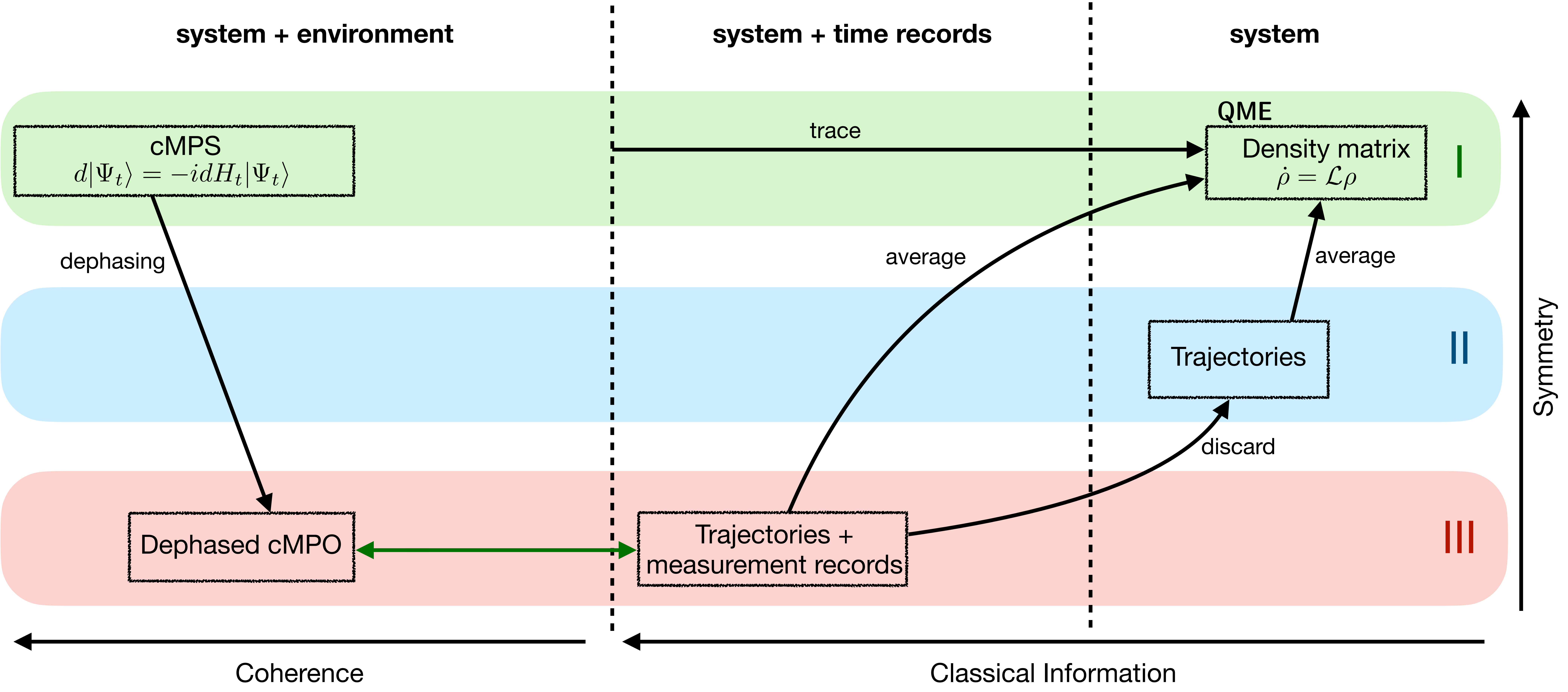

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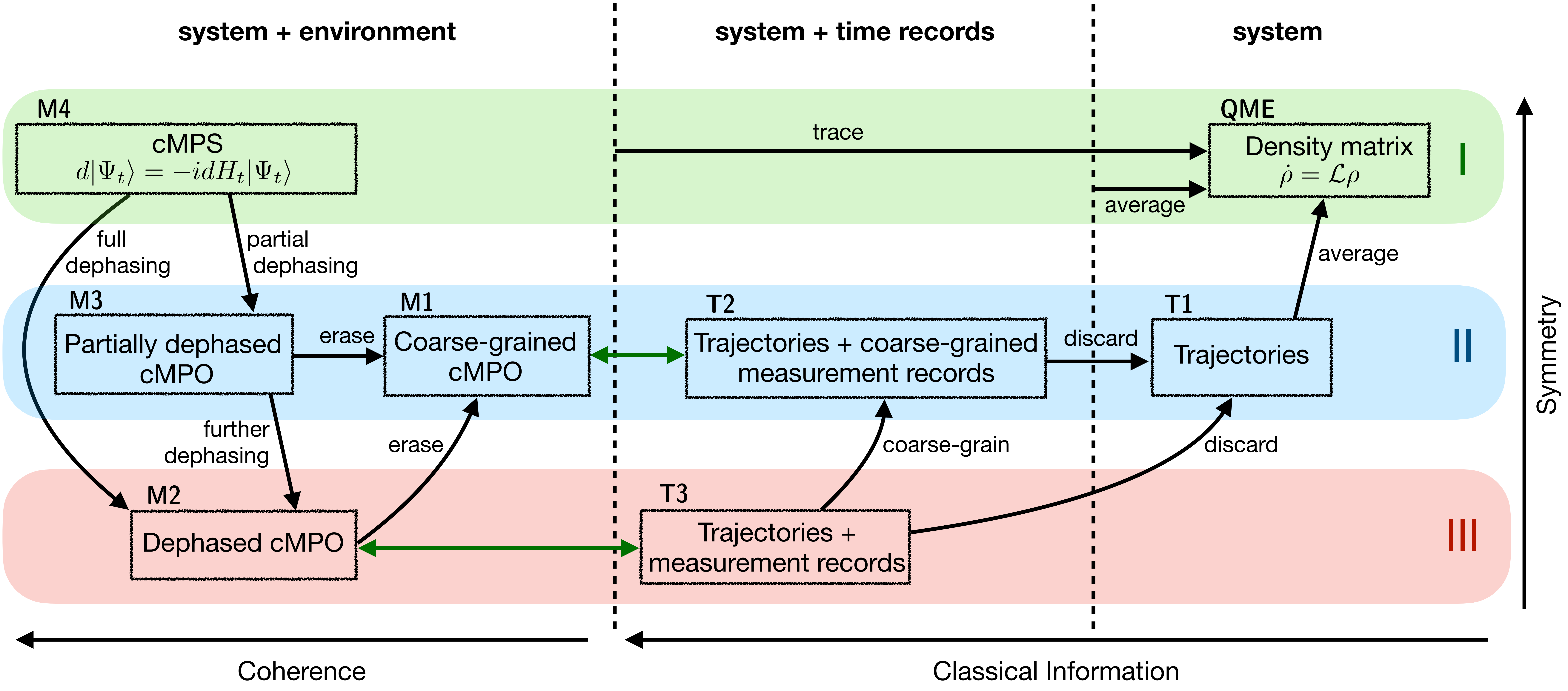

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Unravelled generator

$$\mathcal{W}^\dagger[P_t(\psi)] = -\nabla \cdot [P_t(\psi)\mathcal{B}(\psi)] + \int d\psi' \, [P_t(\psi')W(\psi', \psi) - P_t(\psi)W(\psi, \psi')]$$

$$\mathcal{B}(\psi) = -iH_{\text{eff}}\psi + i\psi H_{\text{eff}}^\dagger - \psi \text{Tr}(-iH_{\text{eff}}\psi + i\psi H_{\text{eff}}^\dagger)$$

$$W(\psi, \psi') = \sum_j \delta\left(\psi' - \frac{\mathcal{J}_j(\psi)}{\text{Tr}[\mathcal{J}_j(\psi)]}\right) \text{Tr}[\mathcal{J}_j(\psi)]$$

Gauge transformations of unravelled generator

When do different continuous measurements yield identical quantum trajectories?

Gauge transformations of unravelled generator

When do different continuous measurements yield identical quantum trajectories?

Symmetry condition II:

- We have that $\Upsilon \mathcal{W}^\dagger \Upsilon^\dagger = \mathcal{W}$, iff

$$\mathcal{U}(H) = H, \quad \mathcal{U}\mathcal{A}_\alpha \mathcal{U}^\dagger = \mathcal{A}_{\pi(\alpha)} \ \forall \alpha$$

- Two representations of $\mathcal{L}$, given by $\tilde{H}, \tilde{J}_1, \ldots, \tilde{J}_{\tilde{d}}$ and $H, J_1, \ldots, J_d$ have equal trajectory generators, $\tilde{\mathcal{W}} = \mathcal{W}$, iff

$$\tilde{H} = H + r\mathbb{1}, \quad \tilde{\mathcal{A}}_\alpha = \mathcal{A}_{\pi(\alpha)} \ \forall \alpha$$

Recall: $i, j \in S_\alpha \ \Leftrightarrow \ J_i|\psi\rangle \propto J_j|\psi\rangle \ \forall |\psi\rangle$

$$\mathcal{A}_\alpha(\psi) = \sum_{j \in S_\alpha} \mathcal{J}_j(\psi)$$

Gauge transformations of unravelled generator

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$$\tilde{H} = \mathcal{U}(H), \;\; \tilde{\mathcal{J}}_j \equiv \mathcal{U}\mathcal{J}_k\mathcal{U}^\dagger$$

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$$\tilde{H} = H + r\mathbb{1} \quad \text{and} \quad \tilde{J}_j = \sum_{k=1}^{d} \mathbf{V}_{jk} J'_k$$

where $$\mathbf{V} = \sum_\alpha \mathbf{V}^{(\alpha)}$$

$$\mathbf{V}^{(\alpha)}_{jk} = 0 \quad \text{unless} \quad j \in \tilde{S}_\alpha,\, k \in S_{\pi(\alpha)}$$

$$\sum_{j \in \tilde{S}_\alpha} (\mathbf{V}^{(\alpha)}_{jk})^* \mathbf{V}^{(\alpha)}_{jk'} = \delta_{kk'} \quad \text{for} \quad k, k' \in S_{\pi(\alpha)}$$