Currents

$$j^a_\mu = i\,(\phi^\dagger T^a \partial_\mu \phi - \partial_\mu \phi^\dagger T^a \phi)$$

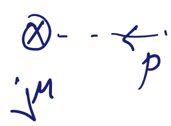
O(N) currents.

$$= i(v+\rho)^2 \left[n_0^T \xi^T T^a \partial_\mu \xi n_0 - n_0^T \partial \xi^T T^a \xi n_0 \right]$$

$$\sim i v^2 \left[n_0^T T^a \partial_\mu \pi\, n_0 + n_0^T \partial_\mu \pi\, T^a n_0 \right]$$

$$\sim -2v\, \partial_\mu \pi^a \quad \text{for broken generators}$$

$$\langle 0 | j^\mu | \pi \rangle = 2 i v p_\mu$$



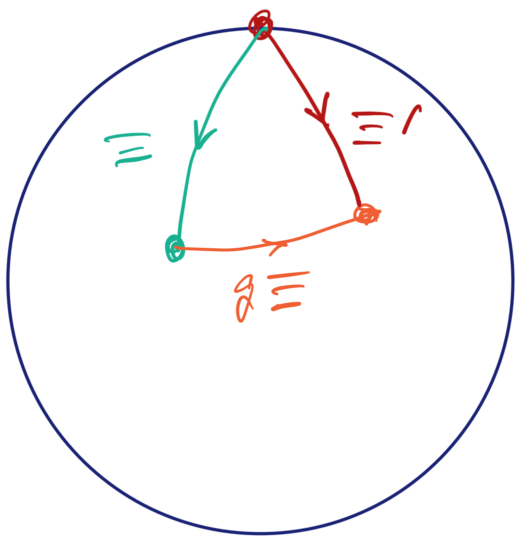
broken currents create GB from vacuum.

CCWZ

Generalize the construction to G/H breaking in an arbitrary theory:

$$\Xi(x) = e^{i X \cdot \pi} \qquad (\text{CCWZ uses } \xi)$$

parameterizes the local vacuum. (near N pole. global structure not studied here)



under global G transformation

$$g \Xi = \Xi' h$$

i.e. $g \Xi(x) = \Xi'(x)\, h(x)$

h <u>depends on x</u> because it depends on g and Ξ $h(g, \Xi)$.

i.e. $\Xi \to g \Xi h^{-1}$

and we want a Lagrangian invariant under this symmetry. The symmetry is realized non-linearly through exponentials of generators.

For the $O(N)$ model $\quad h n_0 = n_0$

so $\quad \Xi n_0 \to g \Xi n_0$

<u>Note</u> : g is global (does not depend on x)

h is local (depends on x)

QCD

$$\mathcal{L} = \sum_r \left(\bar{q}_r \, i \not{D} \, q_r - m_r \bar{q}_r q_r \right) - \frac{1}{4} G^A_{\mu\nu} G^{\mu\nu A} + \frac{\theta g^2}{32\pi^2} G^A_{\mu\nu} \tilde{G}^A_{\mu\nu}$$

For light quarks (u, d, s) we can treat m_r as a perturbation and expand in m

$$\bar{q} \, i \not{D} \, q = \bar{q}_L \, i \not{D} \, q_L + \bar{q}_R \, i \not{D} \, q_R$$

$$\bar{q} q = \bar{q}_L q_R + \bar{q}_R q_L$$

$$q = \begin{pmatrix} u \\ d \\ s \end{pmatrix}$$

$$\mathcal{L} = \bar{q} \, i \not{D} \, q = \bar{q}_L \, i \not{D} \, q_L + \bar{q}_R \, i \not{D} \, q_R$$

Theory has an $SU(3)_L \otimes SU(3)_R$ global chiral symmetry

$$q_L(x) \to L \, q_L(x) \qquad q_R(x) \to R \, q_R(x)$$

There are also two $U(1)$ symmetries

$U(1)_V$: $q(x) \to e^{i\alpha} q(x)$: Baryon number

$U(1)_A$: $q_L(x) \to e^{i\alpha} q_L(x)$, $q_R(x) \to e^{-i\alpha} q_R(x)$: broken due to the anomaly

$$\partial_\mu \left(\bar{q} \gamma^\mu \gamma_5 q \right) = N_f \frac{g^2}{32\pi^2} G^A_{\mu\nu} \tilde{G}^{\mu\nu A} \qquad \epsilon_{0123} = +1$$

For $U(1)$ we have $U(1)_L$ and $U(1)_R$ or $U(1)_V$ and $U(1)_A$. But <u>not</u> for the non-abelian part $SU(N)_A$ is <u>not</u> a group. The commutator of two axial generators is a vector generator

The other quarks c, b, t are heavy quarks, and treated using HQET as an expansion in $1/m$.

QCD has a $SU(3)_L \times SU(3)_R = G$ symmetry. This is spontaneously broken to $H = SU(3)_V$ <u>non-perturbatively</u> by a condensate $\langle \bar{q}_r q_s \rangle = C\,\delta_{rs}$ so we have the symmetry breaking $SU(3)_L \times SU(3)_R \to SU(3)_V$ and 8 goldstone bosons (π^+, π^-, π^0) (K^+, K^0) $(\bar{K}^0, K^-)$ and η. There is <u>no</u> 9^{th} GB for the $U(1)_A$ $\longrightarrow$ (was the η' puzzle)

(Can use chiral rotations to move ALPS into M and compute masses etc using χPT)

$$\mathcal{L} = \bar{q}_L i \not{D} q_L + \bar{q}_R i \not{D} q_R - \bar{q}_L M q_R - \bar{q}_R M^\dagger q_L$$

treat $M = \begin{pmatrix} m_u & & \\ & m_d & \\ & & m_s \end{pmatrix}$ as a perturbation

Can include general background sources : gauge fields, masses

Gasser, Leutwyler : Ann. Phys 158 (1984) 142

notation : Bijnens, Colangelo, Ecker JHEP 02 (1999) 020

come back to this.

$$G/H = \frac{SU(3)_L \times SU(3)_R}{SU(3)_V} \sim SU(3)$$ is a group (this is a special case)

$$\frac{SU(N_f) \times SU(N_f)}{SU(N_f)}$$

CCWZ formalism: $g = (L, R) = SU(3)_L \times SU(3)_R$ transformation

$h = (h, h) = SU(3)_V$ transformation

G generators $\quad T^a_L = T^a \otimes 1 \qquad T^a_R = 1 \otimes T^a$

H generators $\quad T^a = T^a_L + T^a_R$

Broken generators $\quad X^a = T^a_L - T^a_R$

no unique definition. can always shift by an unbroken generator.

unbroken generators form a vector space: $T^a |\Omega\rangle = 0$

Normalise so that $\quad \mathrm{tr}\, T^a T^b = \frac{1}{2} \delta^{ab}$

$$\Xi = e^{i X^a \pi^a / f} = (\xi, \xi^\dagger) \qquad \xi = e^{i T^a \pi^a / f}$$

$$\pi^a = \frac{1}{\sqrt{2}} \begin{bmatrix} \frac{\pi^0}{\sqrt{2}} + \frac{\eta}{\sqrt{6}} & \pi^+ & K^+ \\ \pi^- & -\frac{1}{\sqrt{2}}\pi^0 + \frac{\eta}{\sqrt{6}} & K^0 \\ K^- & \bar{K}^0 & -\frac{2\eta}{\sqrt{6}} \end{bmatrix}$$

[isospin multiplets are $(\pi^-, \pi^0, -\pi^+)$

(K^0, K^+)

$(K^-, -\bar{K}^0)$

η^0]

$$\Xi' = g\,\Xi\,h^\dagger \qquad (\xi', \xi'^\dagger) = (L, R)\cdot(\xi, \xi^\dagger)\cdot(h^\dagger, h^\dagger)$$

$$\xi' = L\,\xi\,h^\dagger$$
$$\xi'^\dagger = R\,\xi^\dagger h^\dagger \quad \Rightarrow \quad \boxed{\begin{aligned}\xi' &= L\,\xi\,h^\dagger \\ &= h\,\xi\,R^\dagger\end{aligned}}$$

$$\Sigma = \xi^2 = e^{2iT^a\pi^a/f}$$

$$\boxed{\Sigma(x) \to L\,\Sigma(x)\,R^\dagger}$$

Bijnens et al: $\quad u = e^{i\pi/f} \qquad U = u^2 = e^{2i\pi/f}$

and $\quad U \to R\,U\,L^\dagger \qquad u' = R\,u\,h^\dagger = h\,u\,L^\dagger$

so $\quad \xi = u^\dagger \qquad \Sigma = u^\dagger \quad$ and $\quad \pi \leftrightarrow -\pi$

---

parity $\quad L \leftrightarrow R \qquad \pi \leftrightarrow -\pi \qquad U \leftrightarrow U^\dagger \qquad u \leftrightarrow u^\dagger$

① Write down most general Lagrangian with this transformation law.

② Make field redefs and use IBP to simplify Lagrangian

③ Cayley-Hamilton identities (see below)

④ Need a power counting: powers of p.

$$u \sim 1 \qquad U \sim 1$$

$$\partial \sim p$$

$$M \sim p^2 \qquad (m_\pi^2 \propto m_q \propto p^2)$$

Exact chiral symmetry with no sources

First start with no external sources. Then L, R are global and do not depend on x.

N_f = number of flavors

$N_f = 2$ $\quad SU(2)$ $\quad \chi PT$ $\quad \pi$

$N_f = 3$ $\quad SU(3)$ $\quad \chi PT$ $\quad \pi, K, \eta$

N_f arbitrary

zero derivatives: $\langle \ \rangle \equiv$ Trace $\qquad \langle \Sigma \Sigma^\dagger \rangle = \langle \mathbb{1} \rangle$ = constant, etc

these are all constant $\Rightarrow$ all interactions involve derivatives.

Easier to work with Σ since h depends on x.

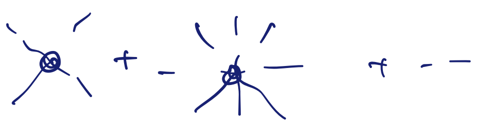
$\mathcal{L} = \frac{f^2}{4} \langle \partial_\mu \Sigma^\dagger \partial^\mu \Sigma \rangle$ is only term at order p^2

$$\Sigma = e^X = 1 + X - \cdots \qquad \partial_\mu \Sigma = \partial_\mu X + \cdots \qquad X = \frac{2i\pi}{f}$$

$$L = \frac{f^2}{4} \operatorname{tr}\left(-2i \frac{\partial\pi}{f}\right)\left(2i\frac{\partial\pi}{f}\right)$$

$$= \operatorname{tr} \partial\pi\, \partial\pi = \frac{1}{2}(\partial_\mu \pi)^a (\partial_\mu \pi)^a + \cdots$$

$$L = \operatorname{Tr} \partial_\mu \pi\, \partial^\mu \pi + \frac{1}{3f^2} \operatorname{Tr}[\pi, \partial_\mu \pi]^2 + \cdots$$



$$\left(-\frac{1}{6f^2}\right) f^{abc} f^{agh} \pi^b \partial_\mu \pi^c \pi^g \partial_\mu \pi^h$$

for $SU(2)$ $\qquad + \frac{1}{6f^2}\left[(\pi \cdot \partial\pi)^2 - (\pi\cdot\pi)(\partial\pi\cdot\partial\pi)\right]$

4π, 6π, ... all interactions determined in terms of f.

non-linear realization of symmetry relates processes with <u>different</u> numbers of pions.

$$\mathcal{L} = \frac{1}{2} \underset{\smile}{g_{ab}(\pi)}\, \partial_\mu \pi^a \partial_\mu \pi^b$$

non-trivial metric

$$g_{ab} = \delta_{ab} - \frac{1}{3f^2} f^{gac} f^{gbd} \pi^c \pi^d + \dots$$

At p^4 for $SU(N)$ χPT

$$\mathcal{L} = \hat{L}_1 \langle \partial_\mu \Sigma^\dagger \partial^\mu \Sigma \rangle^2 + \hat{L}_2 \langle \partial_\nu \Sigma^\dagger \partial_\nu \Sigma \rangle \langle \partial_\mu \Sigma^\dagger \partial_\nu \Sigma \rangle$$
$$+ \hat{L}_3 \langle \partial_\mu \Sigma^\dagger \partial^\mu \Sigma \partial_\nu \Sigma^\dagger \partial_\nu \Sigma \rangle + \hat{L}_0 \langle \partial_\mu \Sigma^\dagger \partial_\nu \Sigma \partial_\mu \Sigma^\dagger \partial_\nu \Sigma \rangle$$

① can integrate by parts.

② Cayley - Hamilton theorem

For 2×2: $\frac{1}{6} \langle A \rangle^3 - \frac{1}{2} \langle A \rangle \langle A^2 \rangle + \frac{1}{3} \langle A^3 \rangle = 0$

$$\frac{\delta}{\delta A} \quad \Rightarrow \quad A^2 - A \langle A \rangle - \frac{1}{2} \langle A^2 \rangle \mathbb{1} + \frac{1}{2} \langle A \rangle^2 \mathbb{1} = 0$$

$$AB + BA - A\langle B \rangle - B \langle A \rangle - \langle AB \rangle \mathbb{1} + \langle A \rangle \langle B \rangle \mathbb{1} = 0$$

using $A \longrightarrow A + B$

For 3×3:

$$\langle A^4\rangle - \frac{4}{3}\langle A^3\rangle\langle A\rangle - \frac{1}{2}\langle A^2\rangle^2 + \langle A^2\rangle\langle A\rangle^2 - \frac{1}{6}\langle A\rangle^4 = 0$$

and similar identities. BCE Eq (3.1)

For $SU(3)$:

$$L_1 = \frac{\hat{L}_0}{2} + \hat{L}_1$$

$$L_2 = \hat{L}_0 + \hat{L}_2$$

$$L_3 = -2\hat{L}_0 + \hat{L}_3$$

For $SU(2)$

$$l_1 = -2\hat{L}_0 + 4\hat{L}_1 + 2\hat{L}_3$$

$$l_2 = 4(\hat{L}_0 + \hat{L}_2)$$

$$\mathcal{L} = \frac{l_1}{4}\langle \partial_\mu\Sigma\, \partial^\mu\Sigma^\dagger\rangle^2 + \frac{l_2}{4}\langle \partial_\mu\Sigma^\dagger \partial_\nu\Sigma\rangle\langle \partial_\mu\Sigma^\dagger \partial_\nu\Sigma\rangle$$

Currents :

$$q_L \to e^{i\epsilon_L^a T^a} q_L$$

$$\delta \mathcal{L} = -\partial_\mu \epsilon_L^a \bar{q}_L \gamma^\mu T^a q_L = -\partial_\mu \epsilon_L^a j_L^{\mu a}$$

$$j_L^{\mu a} = \bar{q}_L \gamma^\mu T^a q_L \qquad j_R^{\mu a} = \bar{q}_L \gamma^\mu T^a q_L$$

$$\delta \mathcal{L}_{EFT} = \delta \frac{f^2}{4} \langle \partial_\mu u^\dagger \partial^\mu u \rangle$$

j_R : $\quad u \to u e^{-i\epsilon_R^a T^a} \qquad u^\dagger \to e^{i\epsilon_R^a T^a} u^\dagger$

$$\delta \mathcal{L} = \frac{f^2}{4} i \partial_\mu \epsilon_R^a \langle T^a u^\dagger \partial^\mu u - \partial^\mu u^\dagger u T^a \rangle$$

$$j_R^{\mu a} = \frac{i f^2}{4} \langle -T^a u^\dagger \partial^\mu u + T^a \partial_\mu u^\dagger u \rangle$$

$$= -\frac{i f^2}{2} \langle T^a u^\dagger \partial_\mu u \rangle$$

$$j_L^{\mu a} = -\frac{i f^2}{2} \langle T^a u \partial_\mu u^\dagger \rangle = \frac{i f^2}{2} \langle T^a \partial_\mu u \, u^\dagger \rangle$$

$$j_R^\mu = f \langle T^a \partial_\mu \pi \rangle - i \langle T^a [\pi, \partial\pi] \rangle - \frac{2}{3f} \langle T^a [\pi, [\pi, \partial\pi]] \rangle + \cdots$$

$$= \frac{f}{2} \partial_\mu \pi^a + \frac{1}{2} f^{abc} \pi^b \partial\pi^c + \frac{1}{3f} f^{gab} f^{gcd} \pi^b \pi^c \partial\pi^d$$

For $SU(2)$: $j_R^\mu = \frac{f}{2} \partial\pi^a + \frac{1}{2} \epsilon^{abc} \pi^b \partial\pi^c + \frac{1}{3f} \left(\pi^a (\pi \cdot \partial\pi) - (\pi \cdot \pi) \partial\pi^a \right)$

$+ \cdots$

j_L^μ : $\quad \pi \to -\pi$

$$j^{\mu a}_V = f^{abc}\, \pi^b\, \partial \pi^c + \cdots \qquad \text{even in } \pi$$

$$= \epsilon^{abc}\, \pi^b\, \partial \pi^c + \cdots \qquad \text{for } SU(2)$$

$$j^{\mu a}_A = f\, \partial_\mu \pi^a + \frac{2}{3f} f^{gab} f^{gcd}\, \pi^b \pi^c \partial \pi^d + \cdots \qquad \text{odd in } \pi$$

$$= f\, \partial_\mu \pi^a + \frac{2}{3f}\left[\pi^a (\pi \cdot \partial \pi) - \partial \pi^a (\pi \cdot \pi) \right] + \cdots$$

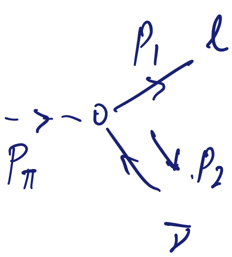
$$\langle 0 |\, j^{\mu a}_A\, | \pi^b(p) \rangle = -i f p^\mu \delta^{ab}$$

$f \simeq 93$ MeV.
creates π from vacuum

$\underline{\pi \to \ell \nu}$

$$\mathcal{L} = -\frac{4G_F}{\sqrt{2}} V_{ud}\, (\bar{u} \gamma^\mu P_L d)(\bar{l} \gamma^\mu P_L \nu_l) \qquad l = e, \mu$$

$$A = -i \frac{4G_F}{\sqrt{2}} V_{ud}\, \langle 0 |\, \bar{u} \gamma^\mu P_L d\, | \pi^- \rangle\, \bar{u}(p_1) \gamma_\mu P_L v(p_2)$$



$$T^1 + i T^2 = \frac{1}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + \frac{i}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

$$j^\mu_L = j^{\mu 1}_L + i\, j^{\mu 2}_L \qquad \pi^- = \frac{\pi^1 - i \pi^2}{\sqrt{2}}$$

$$\langle 0 |\, \bar{u} \gamma^\mu P_L d\, | \pi^- \rangle = -i \frac{f}{2} p_\pi^\mu \cdot \sqrt{2} \qquad \otimes \cdots\cdots$$

$$A = \frac{4G_F}{\sqrt{2}} V_{ud}\, f \left(-\frac{1}{\sqrt{2}}\right) \underbrace{p_\pi^\mu}_{(p_1+p_2)^\mu} \bar{u}(p_1) \gamma_\mu P_L v(p_2)$$

$$= -2 G_F f V_{ud}\, \underbrace{\bar{u}(p_1)(\not{p}_1 + \not{p}_2) P_L v(p_2)}_{m_l\, \bar{u}(p_1) P_L v(p_2)}$$

$$\sum_{\text{spin}} |A|^2 = 4 G_F^2 f^2 |V_{ud}|^2 m_\ell^2 \, \mathrm{Tr}\,(\not{p}_1 + m_\ell) P_L \not{p}_2 P_R$$

$$2 p_1 \cdot p_2 = (p_1 + p_2)^2 - p_1^2 - p_2^2 = m_\pi^2 - m_\ell^2$$

$$= 4 G_F^2 f^2 |V_{ud}|^2 m_\pi^2 m_\ell^2 \left(1 - \frac{m_\ell^2}{m_\pi^2}\right)$$

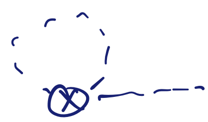
$$\Gamma = \frac{G_F^2}{4\pi} f^2 |V_{ud}|^2 m_\pi m_\ell^2 \left(1 - \frac{m_\ell^2}{m_\pi^2}\right)^2$$

$$\frac{\Gamma(\pi \to \mu\nu)}{\Gamma(\pi \to e\nu)} = \left(\frac{m_\mu}{m_e}\right)^2 \frac{\left(1 - \frac{m_\mu^2}{m_\pi^2}\right)^2}{\left(1 - \frac{m_e^2}{m_\pi^2}\right)^2} \gg 1.$$

$$\bar{\nu} \longleftarrow \pi \longrightarrow \ell$$ need a helicity flip

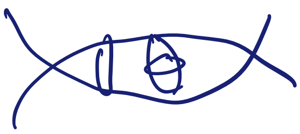
right handed left-handed

gives $f \sim 93$ MeV

<u>corrections</u>:  etc. to chiral matrix element

Weinberg Power Counting

$\mathcal{L} = \sum_{k \geq 2} \mathcal{L}_k$ $\qquad$ $\mathcal{L}_k$ terms of order p^k



generic loop graph

$$A \sim \int \left[\frac{d^4 p}{(2\pi)^4} \right]^L \frac{1}{(p^2)^I} \cdot \prod_k (p^k)^{V_k}$$

V_k vertices of order p^k. $\quad$ $p \sim$ internal or external momentum

In dim reg, μ only enters as $\log \mu$ from expanding μ^ϵ. Therefore $A \sim p^D$ (p some external momentum)

$$D = 4L - 2I + \sum k V_k$$

Now $\quad V - I + L = 1$ for any connected graph

$$D = 2L + 2 - 2V + \sum k V_k$$

$$\boxed{D - 2 = 2L + \sum (k-2) V_k}$$

since $k \geq 2$ all terms on r.h.s. ≥ 0.

$\Rightarrow D \geq 2$

$D = 2 \Rightarrow$ tree graphs with insertions of V_2

$D = 4 \Rightarrow$ 1-loop graphs with V_2

or tree grap with one V_4 and V_2

$D=6 \Rightarrow$ 2 loop graphs with V_2

1 loop graphs with one V_4

tree graphs with 2 V_2 or one V_6.

The loop expansion is combined with the p expansion.

one loop from $O(p^2)$ Lagrangian is running of $O(p^4)$

$$\mu \frac{d\hat{L}_i}{d\mu} = -\frac{\hat{\Gamma}_i}{16\pi^2} \qquad \hat{\Gamma}_0 = \frac{n_f}{48} \quad \hat{\Gamma}_1 = \frac{1}{16} \quad \hat{\Gamma}_2 = \frac{1}{8} \quad \hat{\Gamma}_3 = \frac{n_f}{24}$$

$$\mu \frac{dL_i}{d\mu} = -\frac{\Gamma_i}{16\pi^2} \qquad \Gamma_1 = \frac{3}{32} \quad \Gamma_2 = \frac{3}{16} \quad \Gamma_3 = 0$$

$$\mu \frac{dl_i}{d\mu} = -\frac{\gamma_i}{16\pi^2} \qquad \gamma_1 = \frac{1}{3} \quad \gamma_2 = \frac{2}{3}$$

anomalous dim are pure numbers.

Large N_c $\quad f \propto \sqrt{N_c} \qquad \mathcal{L} = \frac{N_c}{\hbar} \langle \partial_\mu \Sigma \partial^\mu \Sigma^\dagger \rangle$

The loop expansion $= \frac{1}{N_c}$ expansion

see Les Houches lectures hep-ph/9802419