Ph. D. Synopsis

Aspects of open quantum field theories

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1 Introduction

Effective field theory (EFT) has been a fundamental tool in our understanding of various physical systems - starting from standard model of elementary particles, study of critical and near critical phenomena, fluid dynamics, to our current understanding of cosmology etc. But, there are other areas of physics involving dissipation and decoherence which we do not have good understanding of from an EFT point of view.

The study of systems as above assumes the fundamental processes to be closed (unitary). But a real system is not closed - interact with its surrounding at each instant of time. Real systems that appear in cold atom physics, fluid dynamics, plasma physics, heavy ion collision etc. do not possess an equilibrated and unitary description. Quantum information in these systems is exchanged with the environment and we see decoherence and dissipation while course graining. It may seem that the openness, if small, can be neglected, but physicists often are interested in long time behaviour of a system. Thus the effect of the environment should be taken into account to understand certain aspects of physical systems.

Open systems, due to their non-unitary nature, are described by mixed states which are characterised by the density matrix. A quantum mechanical treatment of time evolution for a density matrix was started by Schwinger [1] and formalised by Keldysh [2], commonly called Schwinger-Keldysh (SK) prescription or closed-time-path or in-in prescription. Feynman and Vernon in [3] carried out a systematic study of an open system, interacting with a harmonic bath via bilinear coupling. They constructed an open effective action which they call influence phase of the system, obtained by integrating out the bath. The next major advancement in the direction is the seminal papers by Gorini-Kossakowski-Sudarshan [4] and Lindblad [5]. They came up with a master equation to describe Markovian (memoryless) open systems. The master equation can be promoted to path-integral to get the Feynman-Vernon influence phase of an open system.

The broad motivation for studying open field theory is to understand dissipation from an EFT point of view. Is there a Wilsonian renormalisation way of understanding dissipation while course-graining? We ask rather a simpler question: can we find a UV-complete open field theory by computing the flow equation of the coupling constants? There are some work in this direction in the non-relativistic context [6]. But, a relativistic treatment is absent in the literature, which will be a check of the non-relativistic models and vice versa. A consistent model for relativistic open field theory can be implemented in the study of the visible universe as an open system [7–11]. A gravitational SK construction, which we will come to in §5, might help us understand better the physics of disordered non-Fermi liquids [12, 13] (and references therein), non equilibrium steady state processes [14–18] etc.

The central theme of the synopsis is to study interacting open field theory in a systematic way. We use two different tools for that: Feynman diagrammatic way and holography. In the first approach, we start with a local, Lorentz invariant EFT of open systems and study its one loop renormalisability. We find that one loop correction to the parameters in such theories are non-local divergent. We do not have a resolution to these divergences. Rest of this note deals with derivation of an open EFT for a probe field by integrating out a thermal CFT bath. We implement holographic tools to do that. The EFT has various notable features. An exciting feature is that it points to the existence of a generalisation of fluctuation-dissipation relation at the non-linear level.

The organisation of the synopsis is the following. We start by a brief discussion on SK formalism followed by an application to open QFT in §2. In §3 we explore open ϕ^4 theory, an EFT model of open scalar field. We compute one loop correction to various coupling constants. §4 deals with the new type of divergence (non-local) at generic one scalar loop diagrams. Then we use a holographic prescription to study open ϕ^4 theory in

§5. We derive various fluctuation-dissipation relations at the non-linear level obeyed by a probe scalar field. We end with few concluding remarks and open problems in §6.

2 Schwinger-Keldysh formalism and introduction to open QFT

A quantum system is described by its state and observables (e.g. correlation functions) are computed w.r.t. that state. People are often interested in real-time correlation functions which can be computed by using the SK prescription. SK prescription is a path integral prescription for the time evolution of density matrix. In this chapter we introduce SK prescription for closed systems and then generalise it to open systems. A detailed discussion of SK path integral and its application to non-equilibrium systems can be found in [6, 19–21].

2.1 A brief discussion on Schwinger-Keldysh formalism

The density matrix consists of both bra and ket degrees of freedom (d.o.f.). The SK prescription is a path-integral prescription of doubled d.o.f. corresponding to bra and ket. The SK generating functional of a quantum particle w.r.t. an initial state ρ_i is given by

$$\mathcal{Z}_{SK}[J_R, J_L] = \int_{\rho_i}^{x_R = x_L|_{t=\infty}} \mathcal{D}x_R\, \mathcal{D}x_L\, e^{iS_{SK}[x_R,x_L] + i\int dt (J_R x_R - J_L x_L)}\,, \tag{2.1}$$

where x_R and x_L are SK d.o.f. (eg. position) of the particle, evolving along the forward and backward time respectively. ρ_i is the initial density matrix and J_R, J_L are the sources for x_R, x_L respectively. S_{SK} is the SK action which in unitary system is given by

$$S_{SK}[x_R, x_L] = S[x_R] - S[x_L]\,, \tag{2.2}$$

where $S[x_{R,L}]$ is the action for a system. A diagrammatic interpretation of the SK path-integral is shown in figure 1. At the future infinity, the R and L d.o.f. are identified, which is known as the SK boundary condition.

Now, we describe the procedure to relate the SK correlators to real-time correlators. The SK correlators are obtained by inserting operators on the SK contour in figure 1. An n-point SK correlation function of operator $\mathcal{O}$ (R and L type) can be written as

$$\langle \mathcal{O}_L(t_n) ... \mathcal{O}_L(t_{k+1} \mathcal{O}_R(t_k) ... \mathcal{O}_R(t_1) \rangle = \text{tr}\left[U_c(-\infty, -\infty) \mathcal{O}(t_n) ... \mathcal{O}(t_1) \rho_i(-\infty)\right]\,. \tag{2.3}$$

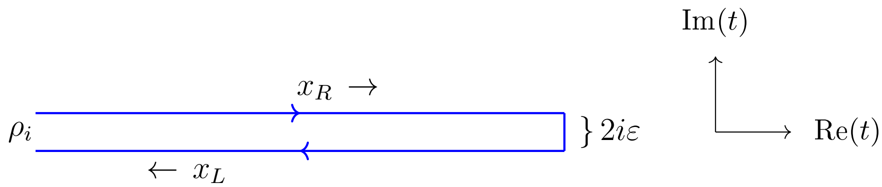


Figure 1: The Schwinger-Keldysh contour - R, L d.o.f. evolve along the forward and backward time respectively. The initial state is described by the density matrix ρ_i.

Here ρ_i is the initial density matrix defined at past infinity and U_c is the contour ordered time evolution operator. Since the operators are contour ordered, operators on R and L branch should be time ordered and anti-time ordered respectively. Thus 2.3 can be written as

$$\begin{aligned}&\langle \mathcal{O}_L(t_n)...\mathcal{O}_L(t_{k+1}\mathcal{O}_R(t_k)...\mathcal{O}_R(t_1)\rangle \\ &= \text{tr}\left[\bar{\mathcal{T}}\Big(U(-\infty,\infty)\mathcal{O}(t_n)...\mathcal{O}(t_{k+1})\Big)\,\mathcal{T}\Big(U(\infty,-\infty)\mathcal{O}(t_k)...\mathcal{O}(t_1)\Big)\rho_i(-\infty)\right] .\end{aligned} \tag{2.4}$$

A recent review on various aspects of SK correlators can be found in [22].

We end this section with a short discussion of SK two point functions. Since the d.o.f.s are doubled, we expect to get at most four SK two-point correlators:

$$\begin{aligned}\langle \mathcal{O}_R(t_2)\mathcal{O}_R(t_1)\rangle &= \langle \mathcal{T}\mathcal{O}(t_2)\mathcal{O}(t_1)\rangle\,, \qquad \langle \mathcal{O}_L(t_2)\mathcal{O}_R(t_1)\rangle = \langle \mathcal{O}(t_2)\mathcal{O}(t_1)\rangle\,, \\ \langle \mathcal{O}_L(t_2)\mathcal{O}_L(t_1)\rangle &= \langle \bar{\mathcal{T}}\mathcal{O}(t_2)\mathcal{O}(t_1)\rangle\,, \qquad \langle \mathcal{O}_R(t_2)\mathcal{O}_L(t_1)\rangle = \langle \mathcal{O}(t_1)\mathcal{O}(t_2)\rangle\,.\end{aligned} \tag{2.5}$$

The RR and LL ones are the time-ordered and the anti-time-ordered two point correlators. The rest of the correlators do not have any time-ordering, thus are the Wightman correlators.

It is often convenient to adopt a basis called the *average-difference* (a-d)/Keldysh basis, defined as

$$\mathcal{O}_a = \frac{1}{2}(\mathcal{O}_R + \mathcal{O}_L) \qquad \mathcal{O}_d = \mathcal{O}_R - \mathcal{O}_L\,. \tag{2.6}$$

$\mathcal{O}_a$ can be interpreted as classical d.o.f. and $\mathcal{O}_d$ as fluctuation d.o.f. of the system. One advantage of using this basis is that the number of correlators is less than that of R-L

basis. The two-point correlators in this basis are given as

$$\begin{aligned} \langle \mathcal{O}_a(t_2)\mathcal{O}_d(t_1)\rangle &= \Theta(t_2-t_1)\,\langle[\mathcal{O}(t_2),\mathcal{O}(t_1)]\rangle\ , \qquad \langle \mathcal{O}_a(t_2)\mathcal{O}_a(t_1)\rangle = \langle\{\mathcal{O}(t_2),\mathcal{O}(t_1)\}\rangle\ , \\ \langle \mathcal{O}_d(t_2)\mathcal{O}_a(t_1)\rangle &= -\Theta(t_2-t_1)\,\langle[\mathcal{O}(t_2),\mathcal{O}(t_1)]\rangle\ , \qquad \langle \mathcal{O}_d(t_2)\mathcal{O}_d(t_1)\rangle = 0\,. \end{aligned} \tag{2.7}$$

We see that the dd correlator is identically zero[1]. Later we will argue that correlators with only d-type operator are zero (A proof can be found in [22]). Another advantage of a-d basis is the following. The SK path-integral in this basis has a dual classical stochastic description[24].

2.2 Introduction to open quantum field theories

Open systems, as described earlier, are obtained by integrating out the bath. A local description for the system requires the bath to be forgetful, hence strongly coupled [25]. Integrating a strongly coupled bath is a difficult task in practice. We will rather follow two alternate procedures: 1. take an ansatz for open EFT action and study its dynamics, 2. integrate out the strongly coupled bath holographically (§5). To write an ansatz, we consider first the SK action (2.2) for a closed system. R and L d.o.f. do not mix in an unitary QFT, for example in (2.2). The introduction of R-L mixing terms in the above SK action naively breaks unitarity and leads to openness. This is indeed true. However, the R-L mixing terms can not be arbitrary; a specific form is allowed. We call that to be the *Lindblad form*[26, 27].

The *Lindblad form* of the SK action can be derived from Gorini-Kossakowski-Sudarshan-Lindblad (GKSL) equation[4, 5]. GKSL equation is a differential equation for Markovian non-unitary time evolution of the density matrix, given by

$$\frac{d\rho}{dt} = -i[H,\rho] + \sum_k \gamma_k \left[L_k \rho L_k^\dagger - \frac{1}{2}\left\{ L_k^\dagger L_k\,,\,\rho\right\}\right]\,. \tag{2.8}$$

The commutator on the RHS evolves the system unitarily, whereas the terms within the summation are responsible for dissipation. The operators L_k are positive semi-definite

[1] The d-d correlator being zero is a manifestation of the *largest-time equation*, which is used to prove unitarity in QFTs. A discussion on this can be found in [23]

operators, called Lindblad operators and γ is a positive constant. A textbook treatment of the Lindblad equation can be found in [28, 29].

Before we write the path-integral form of the GKSL equation, let us interpret the Lindblad operators in terms of SK d.o.f.. Since the density matrix has a ket-bra form, the operators on the left side of the density matrix in (2.8) act on ket (forward time evolving). We thus expect those operators to be R type SK fields. Similarly the operators on the right side of the density matrix are of L type.

Following the above interpretation one can show that the path-integral for a local open QFT, corresponding to (2.8), is given by

$$\int \mathcal{D}\phi_R\,\mathcal{D}\phi_L\phi_R\,e^{iS_{SK}}\,, \tag{2.9}$$

where $\phi_R(\phi_L)$ are $R(L)$-type quantum field(s) and the action in d space-time dimensions,

$$\begin{aligned} S_{SK} = \int d^dx\Big\{&\mathcal{L}(\phi_R) - \mathcal{L}(\phi_L) \\ &+ i\sum_k \gamma_k\Big[L_k(\phi_R)\,L_k^\dagger(\phi_L) - \frac{1}{2}(L_k^\dagger L_k)(\phi_R) - \frac{1}{2}(L_k^\dagger L_k)(\phi_L)\Big]\Big\}\,. \end{aligned} \tag{2.10}$$

$\mathcal{L}$ in the above action is the Lagrangian for a closed system and $\mathcal{L}(\phi_R) - \mathcal{L}(\phi_L)$ is the corresponding SK Lagrangian. The rest of the terms are responsible for openness. Notice that these terms also contain interaction terms with only ϕ_R and ϕ_L but with imaginary couplings.

3 Renormalisation of open ϕ^4 theory

From our understanding of QFTs, physical parameters at different energy scale are related by the renormalisation group flow equation. We also know from open system literature that an open system interacts with its bath at the microscopic level (high energy) and it dissipates energy at a coarse-grained scale (low energy). We move one step further and ask: is it possible to understand decoherence at different energy scales as a renormalisation flow equation of correlation functions?

We study renormalisation of open ϕ^4 theory. We shall follow the textbook procedure of renormalisation and compute the beta functions of the couplings. An ansatz for Lorentz invariant action for open ϕ^4 theory is given by[27]

$$
\begin{aligned}
S_{SK} = &- \int d^4x \left[\frac{z}{2}(\partial\phi_R)^2 + \frac{m^2}{2}\phi_R^2 + \frac{\lambda}{4!}\phi_R^4 + \frac{\sigma}{3!}\phi_R^3\phi_L \right] \\
&+ \int d^4x \left[\frac{z^*}{2}(\partial\phi_L)^2 + \frac{m^{*2}}{2}\phi_L^2 + \frac{\lambda^*}{4!}\phi_L^4 + \frac{\sigma^*}{3!}\phi_L^3\phi_R \right] \\
&+ i \int d^4x \left[z_\Delta \partial\phi_R \partial\phi_L + m_\Delta^2 \phi_R\phi_L + \frac{\lambda_\Delta}{2!2!}\phi_R^2\phi_L^2 \right] .
\end{aligned} \tag{3.1}
$$

The coupling constants z, m^2, λ, σ are complex and $z_\Delta, m_\Delta^2, \lambda_\Delta$ are real.

We can write the above open action in the Lindblad form (2.10), and get a set of relations among the coupling constants. Those relations can be termed as Lindblad conditions and are given by[27]

$$
\mathrm{Im}[z] - z_\Delta = 0\,, \tag{3.2a}
$$

$$
\mathrm{Im}[m^2] - m_\Delta^2 = 0\,, \tag{3.2b}
$$

$$
\mathrm{Im}[\lambda + 4\sigma] - 3\lambda_\Delta = 0\,. \tag{3.2c}
$$

The combinations of the above couplings on the LHS can be called as Lindblad violating couplings. One would expect that the above conditions should also be satisfied under loop correction to preserve the Lindblad structure. So, we compute loop correction to the coupling constants and check whether these constraints are preserved along with the study of renormalisation.

To justify perturbation theory, we assume all quartic coupling constants to be small ($<< 1$) and $\mathrm{Im}[m^2] << \mathrm{Re}[m^2]$ & $\mathrm{Im}[z] << \mathrm{Re}[z]$. With these assumptions, we get the free closed SK action as our unperturbed SK action. Using the free closed SK action we get the SK propagators[27]. The propagators in momentum space along with their diagrammatic representation are the following.

$$
\begin{aligned}
\langle \phi_R(-k)\phi_L(k) \rangle &= \frac{-i}{k^2 + \mathrm{Re}[m^2] - i\varepsilon} : \quad \text{(solid line, } k \to) \quad : R \\
\langle \phi_L(-k)\phi_L(k) \rangle &= \frac{i}{k^2 + \mathrm{Re}[m^2] + i\varepsilon} : \quad \text{(dashed line, } k \to) \quad : L \\
\langle \phi_L(-k)\phi_R(k) \rangle &= 2\pi\Theta(k^0)\,\delta(k^2 + \mathrm{Re}[m^2]) : \quad \text{(solid–dashed line, } k \to) \quad : P \\
\langle \phi_R(-k)\phi_L(k) \rangle &= 2\pi\Theta(-k^0)\,\delta(k^2 + \mathrm{Re}[m^2]) : \quad \text{(dashed–solid line, } k \to) \quad : M
\end{aligned} \tag{3.3}
$$

For short hand we have used R, L, P, M for the SK propagators.

Action in the *average-difference* basis

We had emphasised in §2.1 that physical properties are more apparent in the a-d basis, defined in (2.6). We write the action in this basis as

$$
\begin{aligned}
S_{SK} = - \int d^4x \Bigg\{ & \frac{i}{2\times 2!} \left(\mathrm{Im}[z] + z_\Delta\right) (\partial\phi_d)^2 + \mathrm{Re}[z]\, \partial\phi_a \partial\phi_d + i \left(\mathrm{Im}[z] - z_\Delta\right) (\partial\phi_a)^2 \\
& + \frac{i}{2\times 2!} \left(\mathrm{Im}[m^2] + m_\Delta^2\right) \phi_d^2 + \mathrm{Re}[m^2] \phi_a \phi_d + i \left(\mathrm{Im}[m^2] - m_\Delta^2\right) \phi_a^2 \\
& + \mathrm{Re}[\lambda + 2\sigma] \frac{\phi_a^3 \phi_d}{3!} + \frac{i}{2} \left(\mathrm{Im}[\lambda] + \lambda_\Delta\right) \frac{\phi_a^2 \phi_d^2}{2!2!} + \frac{1}{4} \mathrm{Re}[\lambda - 2\sigma] \frac{\phi_a \phi_d^3}{3!} \\
& + \frac{i}{8} \left(\mathrm{Im}[\lambda - 4\sigma] - 3\lambda_\Delta\right) \frac{\phi_d^4}{4!} + 2i \left(\mathrm{Im}[\lambda_4 + 4\sigma_4] - 3\lambda_\Delta\right) \frac{\phi_a^4}{4!} \Bigg\} .
\end{aligned}
\tag{3.4}
$$

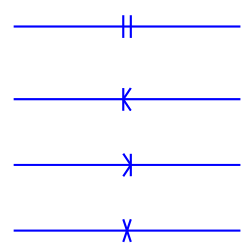
The ubiquitous feature of this basis is that the Lindblad violating couplings (given in eqn. (3.2)) appear as coefficients of pure *average* terms in the above action. Thus calculating the loop correction of pure *average* terms would suffice to check the Lindblad conditions at loop levels.

3.1 One loop correction to the parameters

We resume our discussion on loop correction in the a-d basis. We first compute one loop corrections to the Lindblad violating couplings (3.2) and show that the β-function for those couplings are indeed zero. Then we state the β-function for individual couplings ($m^2, m_\Delta^2, \lambda, \sigma$ and λ_Δ).

To compute loop correction, we need propagators in the a-d basis and determine the Feynman rules for the interaction vertices. The propagators in the a-d basis along with their diagrammatic representation are given below.

$$
\begin{aligned}
\langle \phi_a(-k) \phi_a(k) \rangle &= iG_K = \frac{1}{2}(R + L) \; : \\
\langle \phi_d(-k) \phi_a(k) \rangle &= iG_{adv} = R - P \; : \\
\langle \phi_a(-k) \phi_d(k) \rangle &= iG_{ret} = R - M \; : \\
\langle \phi_d(-k) \phi_d(k) \rangle &= 0 \; :
\end{aligned}
\tag{3.5}
$$

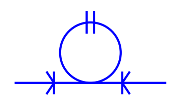


R, L, P, M are shorthand notation for SK propagators in the R-L basis. The Feynman rules for the vertices can be inferred from the action (3.4).

From our knowledge of closed ϕ^4 theory, there is no field renormalisation at one loop level. This also holds true in open ϕ^4 theory, i.e., there is no one loop correction to z and z_Δ[27]. Thus the first Lindblad condition in (3.2) is trivially preserved at one loop. We will now compute one loop correction to rest of the Lindblad violating terms.

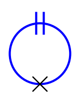
Loop correction to Lindblad violating couplings:

The total contribution for the correction to $\text{Im}[m^2] - m_\Delta^2$ is given by

$2(\text{Im}[m^2] - m_\Delta^2) \times$  $+2(\text{Im}[\lambda + 4\sigma] - 3\lambda_\Delta)\text{Re}[m^2] \times$  (3.6)

The rest two of the tadpole loops do not contribute because those consist of advanced or retarded propagator, which has poles in one side of the complex loop energy plane. Closing the contour on the other side of the complex plane and assuming no contribution from infinity (which is true in open ϕ^4 theory), we get zero for those loops.

The tadpole diagram in the above equation is divergent and in dimensional regularisation it is given by

 $= \frac{\text{Re}[m^2]}{(4\pi)^2}\left[\frac{2}{d-4} + \ln\frac{\text{Re}[m^2]}{4\pi\mu^2 e^{-\gamma_E}} - 1\right] .$ (3.7)

This divergence can be cancelled by introducing a counter ϕ_a^2 term in the action. The counter term depends on the energy scale μ due to loop correction. The corresponding β-function for $\text{Im}[m^2] - m_\Delta^2$ is given by

$$\beta_{\text{Im}[m^2]-m_\Delta^2} \equiv \frac{d}{d\ln\mu}(\text{Im}[m^2] - m_\Delta^2) = \frac{\text{Re}[m^2]}{(4\pi)^2}\left(\text{Im}[\lambda + 4\sigma] - 3\lambda_\Delta\right) . \tag{3.8}$$

We find that the beta function is proportional to a Lindblad violating term $(\text{Im}[\lambda + 4\sigma] - 3\lambda_\Delta)$. Setting this Lindblad violating term to zero at tree level prevents $(\text{Im}[m^2] - m_\Delta^2)$ to flow and gain a non-zero value at different energy scales.

A similar analysis can be carried out for $(\text{Im}[\lambda+4\sigma]-3\lambda_\Delta)$ vertex. The corresponding β-function is given by

$$\beta_{\text{Im}[\lambda+4\sigma]-3\lambda_\Delta} \equiv \frac{d}{d\ln\mu}(\text{Im}[\lambda + 4\sigma] - 3\lambda_\Delta) = \frac{6}{(4\pi)^2}\text{Re}[\lambda + 2\sigma]\ \left(\text{Im}[\lambda + 4\sigma] - 3\lambda_\Delta\right) . \tag{3.9}$$

We again find that the β-functions of the quartic Lindblad violating term is proportional to itself. It does not flow at different scales if set to zero at tree level. Hence we conclude that the Lindblad violating couplings are protected under one loop renormalisation.

The beta functions for all of the couplings can be found in [27] and are given by

$$\begin{aligned} \frac{dm^2}{d\ln\mu} &= \frac{m^2}{(4\pi)^2}\Big[\lambda + 2\sigma - i\lambda_\Delta\ \Big]\,, \\ \frac{dm_\Delta^2}{d\ln\mu} &= \frac{2}{(4\pi)^2}\mathrm{Re}\Big[m^2(\lambda_\Delta + i\sigma)\Big] \end{aligned} \tag{3.10}$$

for the mass terms and

$$\begin{aligned} \frac{d\lambda}{d\ln\mu} &= \frac{3}{(4\pi)^2}(\lambda + 2\ \sigma - i\lambda_\Delta)(\lambda + i\lambda_\Delta)\,, \\ \frac{d\sigma}{d\ln\mu} &= \frac{3}{(4\pi)^2}(\lambda + \sigma + \sigma^\star + i\lambda_\Delta)(\sigma - i\lambda_\Delta)\,, \\ \frac{d\lambda_\Delta}{d\ln\mu} &= \frac{1}{(4\pi)^2 i}\Big[(\lambda + 2\sigma^\star)(\sigma^\star + i\lambda_\Delta) + 3i\sigma\lambda_\Delta - c.c.\Big] \end{aligned} \tag{3.11}$$

for the quartic couplings.

One can study the running of the coupling constants by solving the above differential equations numerically. A thorough numerical analysis for various domain of couplings can be found in [27]. One can also find the fixed point(s) of the flow equations. Around the fixed point(s), flow equations linearise. Thus the near fixed point behaviour of the couplings can be studied analytically by solving the linearised equations.

We move on to other aspects of open QFTs. We find new type of divergences at loop level which are non-local. Thus those can not be cancelled by local counter-terms. We call those to be non-local divergences.

4 Non-local divergences

In this section, we compute more general SK one loop integrals and we find that those have non-local divergences. These integrals are convenient to compute in R-L basis. So we adopt R-L basis in this section.

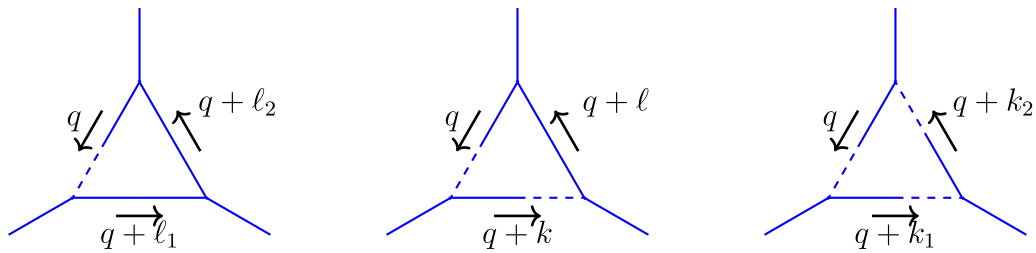


Figure 2: Triangle diagrams representing the loop integrals C_{PRR}, C_{PPR} and C_{PPP} respectively.

4.1 Non-local divergences in open scalar field theory

In the previous section, we have computed one loop correction to the coupling constants and computed the corresponding β-functions. But computing one loop β-functions are not enough to show one loop renormalisability. One should also ensure that higher order diagrams (such as triangle, box etc.) are convergent. We find that the triangle diagrams (that appear in one loop correction to cubic vertex in ϕ^3 theory) have unusual non-local divergences.

One can show that all triangle loop integrals (having non-local divergences) in R-L basis can be classified into three triangle loop integrals: C_{PRR}, C_{PPR} and C_{PPP}, shown in figure 2. Here we name triangle loop integrals as C-type integrals following the Passarino-Veltman convention[30]. The subscript bears the nature of SK loop propagators in a counter-clock-wise fashion to avoid confusion.

The above triangle loop integrals, shown in figure 2, can be computed using hard cut-off regularisation. The integral expression for C_{PRR} and its leading order divergence are the following.

$$\begin{aligned} C_{PRR} =& (-i)^2 \int \frac{d^4q}{(2\pi)^4} \frac{2\pi\Theta(q^0)\delta(q^2+\text{Re}[m^2])}{[(q+k_1)^2+\text{Re}[m^2]-i\varepsilon]\,[(q+k_2)^2+\text{Re}[m^2]-i\varepsilon]} \\ =& \frac{1}{(4\pi)^2}\frac{\tanh^{-1}\left(\frac{\sqrt{-\Sigma^2}}{k_1\cdot k_2}\right)}{\sqrt{-\Sigma^2}}\log\Lambda+\dots, \end{aligned} \tag{4.1}$$

where $\Sigma \equiv \sqrt{k_1^2k_2^2-(k_1\cdot k_2)^2}$, is the area of the parallelogram formed by k_1^μ and k_2^μ. The integral expression for C_{PPR} is given by

$$C_{PPR} = -i\int \frac{d^4q}{(2\pi)^4}\frac{2\pi\Theta(q^0)\,\delta(q^2+\text{Re}[m^2])\times 2\pi\Theta(q^0+k_1^0)\,\delta((q+k_2)^2+\text{Re}[m^2])}{(q+k_2)^2+\text{Re}[m^2]-i\epsilon}\,, \tag{4.2}$$

which evaluates to

$$\mathrm{div}\,(C_{PPR}) = \begin{cases} \frac{i\pi}{(4\pi)^2}\frac{1}{\sqrt{-\Sigma^2}}\,\log\Lambda\,, & \text{if } k_1 \text{ is spacelike} \\ 0 & \text{if } k_1 \text{ is timelike.} \end{cases} \tag{4.3}$$

Finally the integral expression for C_{PPP} is given by

$$\begin{aligned} C_{PPP} = \int \frac{d^4q}{(2\pi)^4}\, 2\pi\Theta(q^0)\,\delta(q^2+\mathrm{Re}[m^2]) \times 2\pi\Theta(q^0+k_1^0)\,\delta((q+k_1)^2+\mathrm{Re}[m^2]) \\ \times 2\pi\Theta(q^0+k_2^0)\,\delta((q+k_2)^2+\mathrm{Re}[m^2])\,. \end{aligned} \tag{4.4}$$

The leading divergence of this integral is following.

$$\mathrm{div}\,(C_{PPP}) = \begin{cases} \frac{2\pi}{(4\pi)^2}\,\frac{1}{\Sigma_c}\,\log\Lambda\,, & \text{if } k_1 \text{ and } k_2 \text{ span a space-like hypersurface} \\ 0 & \text{otherwise}\,. \end{cases} \tag{4.5}$$

We see that in all of the C-type integrals, the leading divergence contains function of momenta (not polynomial) and therefore those are non-local. One might tempt to think that these divergences might cancel among different diagrams when one computes one loop correction to a correlation function. This is not true. We show this by computing a vertex correction in open ϕ^3 theory.

Non-local divergence in open ϕ^3 theory

The SK action for open ϕ^3 theory,

$$\begin{aligned} S_{SK}^{\phi^3} = &-\int d^4x\left[\frac{z}{2}(\partial\phi_R)^2 + \frac{m^2}{2}\phi_R^2 + \frac{\lambda_3}{3!}\phi_R^3 + \frac{\sigma_3}{2!}\phi_R^2\phi_L\right] \\ &+\int d^4x\left[\frac{z^*}{2}(\partial\phi_L)^2 + \frac{m^{*2}}{2}\phi_L^2 + \frac{\lambda_3^*}{3!}\phi_L^3 + \frac{\sigma_3^*}{3!}\phi_L^2\phi_R\right] \\ &+ i\int d^4x\left[z_\Delta\partial\phi_R\partial\phi_L + m_\Delta^2\phi_R\phi_L\right]\,. \end{aligned} \tag{4.6}$$

We compute one loop correction to σ_3 vertex to show the presence of non-local divergences. (One could have chosen the λ_3 vertex as well.) Only triangle diagrams contribute to one loop correction of σ_3. By keeping only non-local divergent integrals, the correction to σ_3

is given by[31]

$$
\begin{aligned}
&2i\Big[\sigma_3^3-\lambda_3\sigma_3^2-\lambda_3^*\sigma_3^2+\lambda_3^*\sigma_3\sigma_3^*+\sigma_3^2\sigma_3^*-\lambda_3\sigma_3^{*2}\Big]C_{RPR}(p_1,p_2,p_3)\\
&+i\Big[2\sigma_3(\lambda_3\sigma_3-\lambda_3\sigma_3^*+\sigma_3^{*2})-\sigma_3^{*3}+\sigma_3\sigma_3^*\lambda_3^*+\sigma_3^2\sigma_3^*+\lambda_3\sigma_3\lambda_3^*\Big]\\
&\qquad\times\left[C_{RRP}(p_1,p_2,p_3)+C_{PRR}(p_1,p_2,p_3)\right].
\end{aligned}
\tag{4.7}
$$

The loop integrals C_{RPR} and C_{RRP} in above equation are obtained by permuting the momenta p_1, p_2 and p_3 appropriately. We find the above result to be non-zero by substituting the divergence of C_{PRR} from (4.1). Thus can conclude that open ϕ^3 theory contains non-local divergence in one loop correction. We also suspect this to be generally true in all scalar field theories.

4.2 Non-local divergences in open Yukawa theory

Non-local divergences are also present in other field theories - the divergences shows up even in bubble diagrams (unlike bubble diagrams in scalar field theory). Let us elaborate this by considering open Yukawa theory[32]. The action is given by

$$
S_{\mathrm{Y}}=\int d^4x\left[\mathcal{L}_\phi+\mathcal{L}_\psi+\mathcal{L}_{\mathrm{Y}}\right]. \tag{4.8}
$$

$\mathcal{L}_\phi$, $\mathcal{L}_\psi$ and $\mathcal{L}_{\mathrm{Y}}$ are Lagrangians for scalar, spinor and Yukawa interaction respectively and are given by

$$
\begin{aligned}
\mathcal{L}_\phi=&-\left[\frac{1}{2}z_\phi\,(\partial\phi_R)^2+\frac{1}{2}m_\phi^2\phi_R^2+\frac{\lambda}{4!}\phi_R^4+\frac{\sigma}{3!}\phi_R^3\phi_L\right]\\
&+\left[\frac{1}{2}z_\phi^*(\partial\phi_L)^2+\frac{1}{2}m_\phi^{2*}\phi_L^2+\frac{\lambda^*}{4!}\phi_L^4+\frac{\sigma^*}{3!}\phi_L^3\phi_R\right]\\
&+i\Big[z_\Delta\,(\partial\phi_R)\cdot(\partial\phi_L)+m_{\phi\Delta}^2\phi_R\phi_L+\frac{\lambda_\Delta}{2!2!}\phi_R^2\phi_L^2\Big],
\end{aligned}
\tag{4.9}
$$

$$
\begin{aligned}
\mathcal{L}_\psi=&-\Big[z_\psi\bar{\psi}_R(-i\partial\!\!\!/)\psi_R+m_\psi\bar{\psi}_R\psi_R\Big]+\Big[z_\psi^*\bar{\psi}_L(-i\partial\!\!\!/)\psi_L+m_\psi^*\bar{\psi}_L\psi_L\Big]\\
&+i\Big[z_{\psi\Delta}\;\bar{\psi}_R(-i\partial\!\!\!/)\psi_L+m_{\psi\Delta}\bar{\psi}_R\psi_L\Big]+i\Big[\hat{z}_{\psi\Delta}\;\bar{\psi}_L(-i\partial\!\!\!/)\psi_R+\hat{m}_{\psi\Delta}\bar{\psi}_L\psi_R\Big]
\end{aligned}
\tag{4.10}
$$

and

$$
\begin{aligned}
\mathcal{L}_{\mathrm{Y}}=&-\Big[y\phi_R\bar{\psi}_R\psi_R+y_\sigma\phi_L\bar{\psi}_R\psi_R\Big]-\Big[y_\kappa\phi_R\bar{\psi}_R\psi_L+y_\rho\phi_R\bar{\psi}_L\psi_R\Big]\\
&+\Big[y^*\phi_L\bar{\psi}_L\psi_L+y_\sigma^*\phi_R\bar{\psi}_L\psi_L\Big]+\Big[y_\kappa^*\phi_L\bar{\psi}_R\psi_L+y_\rho^*\phi_L\bar{\psi}_L\psi_R\Big].
\end{aligned}
\tag{4.11}
$$

Here $y, y_\kappa, y_\sigma, y_\rho$ are all possible open Yukawa couplings.

To show the presence of non-local divergences, we compute one loop correction to the scalar and fermion masses. The one loop correction to the scalar field is not non-local divergent. The corresponding anomalous dimension of the field and beta function for m_ϕ^2 are the following.

$$\begin{aligned}\gamma_\phi \equiv \frac{1}{2}\frac{d\ln z_\phi}{d\ln\mu} &= -\frac{2}{(4\pi)^2}(y+y_\sigma^*)(y-y_\sigma^*+y_\kappa+y_\rho)\,,\\ \beta_{m_\phi^2} \equiv \frac{dm_\phi^2}{d\ln\mu} &= \frac{4m_\phi^2-24m_\psi^2}{(4\pi)^2}(y+y_\sigma^*)(y-y_\sigma^*+y_\kappa+y_\rho)+\frac{m_\phi^2}{(4\pi)^2}(\lambda-i\lambda_\Delta+2\sigma)\,.\end{aligned} \tag{4.12}$$

However, one loop correction to the fermion self-energy is non-local divergent. These divergences come from bubble diagrams consisting one fermion and one scalar propagator. Here we report the divergent piece of the fermion self-energy correction in the equal scalar and fermion mass limit ($m_\phi = m_\chi = m$):

$$\begin{aligned}&-(m-\not{k}/2)\Big[(y+y_\rho^*)(y+y_\sigma+2i\mathrm{Im}[y_\kappa])+(y+y_\kappa^*)(y+y_\sigma+2i\mathrm{Im}[y_\rho])\Big]\frac{1}{d-4}\frac{i}{(4\pi)^2}\\ &-\frac{\not{k}}{k^2}\Big[(y_\sigma-y_\rho)(y+y_\sigma+2i\mathrm{Im}[y_\kappa])+(y_\sigma-y_\kappa)(y+y_\sigma+2i\mathrm{Im}[y_\rho])\Big]\frac{1}{d-4}\frac{i(m^2)}{(4\pi)^2}+\dots\,.\end{aligned} \tag{4.13}$$

Notice that the second line of the above expression is non-local divergent.

4.3 Enhancement of divergence in generic scalar one loop

In this section, we comment on the divergence structure of generic scalar one loop integrals. We discuss two types of loop integral here: $(D-1)$-gon Wightman loop and Mixed R-P loop shown in figure 3. The rest of the non-local divergent integrals can be obtained in terms of those two integrals. We show that divergences appear only from certain domain of the external momenta. We point out that the divergences in these loops are enhanced than those of unitary loops. A detailed discussion on this can be found in [31].

$(D-1)$-gon Wightman loop:

We begin with the $(D-1)$-gon Wightman loop in D space-time dimension, which consists of only Wightman propagator P shown in figure 3. The integral expression for the Wightman

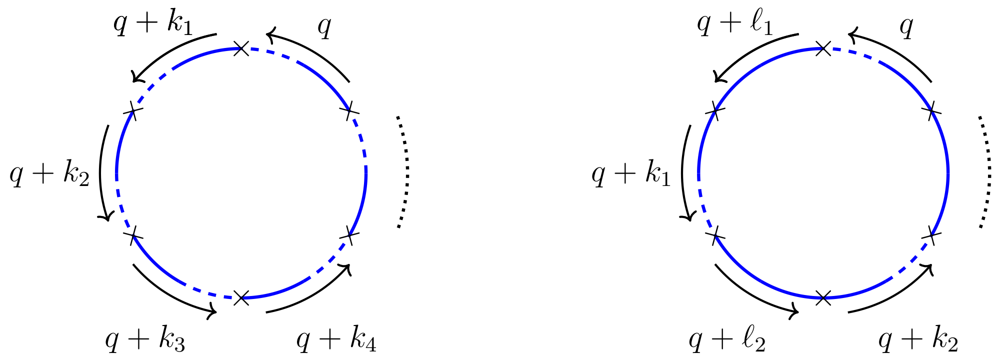


Figure 3: The left figure is a Wightman loop with only P propagators. The right one is a loop consisting R and P propagators.

loop is given by

$$\mathcal{I}[P^{D-1},\{k_i\}] = \int \frac{d^Dq}{(2\pi)^D}\, 2\pi\Theta(q^0)\delta[q^2+m^2]\prod_{j=1}^{D-2} 2\pi\Theta(q^0+k_j^0)\delta[(q+k_j)^2+m_j^2]\,. \quad (4.14)$$

The superficial degree of divergence of the above integral is $D-2(D-1) = 2-D$ by dimensional analysis. So, for $D>2$ this integral should converge. But the naive expectation is not correct. The correct degree of divergence is obtained after using the Dirac delta function which is zero, thus points to a logarithmic divergence. The leading divergence of this integral is given by

$$\mathcal{I}[P^{D-1},\{k_i\}] \approx \int \frac{d^Dq}{(2\pi)^D} 2\pi\Theta(q^0)\delta[q^2]\prod_{j=1}^{D-2} 2\pi\delta[2q\cdot k_j] \approx \frac{1}{2^{D-2}(2\pi)\Sigma_c}\log\Lambda\,, \quad (4.15)$$

where $\Sigma_c \sim \frac{1}{k^{D-2}}$, volume of a parallelotope formed by $\{k_i\}$. Thus the divergence of the above integral is non-local. However, the divergence appears only from certain domain of the external momenta. It comes from space-like k_is, transverse to the loop momenta integrated over the null cone as shown in figure 4.

Mixed loop

The mixed loop consists of only R and P propagators shown in figure 3. The integral expression for this loop is given by

$$\mathcal{I}[P^{N_c},R^{N-N_c},\{k_i,\ell_j\}] = \int \frac{d^Dq}{(2\pi)^D}\, 2\pi\delta_+[q^2+m^2]\frac{\prod_{j=1}^{N_c-1} 2\pi\delta_+[(q+k_j)^2+m^2]}{\prod_{j=1}^{N-N_c} i[(q+\ell_j)^2+m^2-i\varepsilon]}\,, \quad (4.16)$$

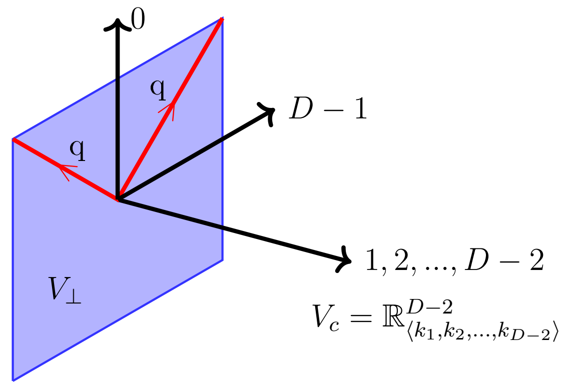


Figure 4: Geometry of P^{D-1} loop : The external momenta form a $\mathbb{R}^{D-2}$ hyperplane denoted here by V_c. The UV part of the loop integral comes from light-like excitations on the transverse Minkowski plane $V_\perp$.

where $\delta_+(q^2+m^2) \equiv \Theta(q^0)\delta(q^2+m^2)$. The correct degree of divergence is $D-N-1$ which is different from the naive counting of divergence. The leading divergence structure of this integral $\sim \frac{\Lambda^{D-N-1}}{k^{N_c-1}\ell^{N-N_c}}$. The divergence again comes from the loop momentum integrated over null cone, transverse to k_i^μ. Note that the presence of divergence does not restrict ℓ_j^μ which flow along the R propagators; it constrains momenta flowing only along P propagators.

4.4 Comment on non-local divergences

The appearance of non-local divergence in one loop in open QFTs is not yet well-understood. To decipher the physics of it, we should revisit the validity of the assumptions.

1. First of all, we assumed our theories to be Lorentz invariant. Thus any odd derivative term in the action is forbidden by Lorentz invariance. We justify this assumption by the following argument. We are interested in relativistic open QFTs and their UV completion. When the energy scale is high enough, the vacuum contribution dominate over the excited states in the absence of non-unitary couplings. The Lorentz invariance breaking terms at such high energy is sub-dominant. We choose this Lorentz invariant vacuum as our initial state. Then we turn on Lorentz invariant non-unitary coupling perturbatively.

The above assumption has computational advantage - we do not know in general how to compute Green's function in field theory w.r.t. a generic state. But, in a Lorentz invariant open QFT, we can make use of the vacuum propagators to do loop computation as done in §3.

2. The second assumption is locality of the effective action. There is no prior reason for us to believe that the action should be local. However, we shall show in the next chapter, that one can find local open QFTs (probe) in a bath of thermal CFT. If the bath temperature is high enough one obtain a local description of the probe QFT. But, if one pushes the temperature of the bath towards zero, locality breaks down.

If we stick to our assumptions, then the divergences should not be physical and there should be a way of absorbing these. For example, in a holographic construction one open QFTs (see §5), we find some unusual divergences which can be absorbed by a renormalisation procedure for open systems. A renormalisation of this sort might eliminate the non-local divergences.

One might also speculate that the idea of effective field theory breaks down, once a system is coupled to a bath. So, some modes of the bath correlators do not decay. If we could separate out those modes then a well behaved description of the system can be obtained.

5 Derivation of open effective action via holography

We now move on to a holographic study of open systems. We construct an open EFT for a probe scalar in a bath of thermal CFT which has a gravitational dual. We integrate out the bath by holographic means and find a local EFT for the probe in long wavelength and frequency limit. Let us begin with a discussion on the holographic SK prescription which is implemented to integrate out the bath.

5.1 Holographic SK prescription: gravitational SK saddle

The gravitational SK prescription is a geometric construction to compute real time correlators (SK correlators) using holography. The first proposal in this direction was given in [33]. They computed real-time boundary Green's function by imposing ingoing bound-

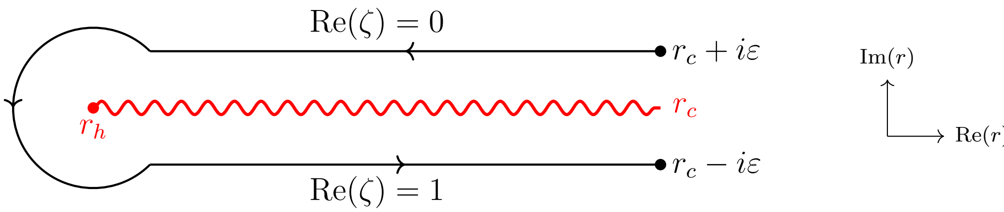


Figure 5: The complex r plane with the locations of the two boundaries and the horizon marked. The gravitational SK contour is a codimension-1 surface in this plane (drawn at fixed v). The direction of the contour is as indicated counter-clockwise encircling the branch point at the horizon.

ary condition at the future horizon. However they neglected the horizon contribution to Green's function, which was justified in [34] using the maximal Kruskal extension of the black hole geometry. One limitation of this approach is that it was well adopted to two point functions, but it left implicit how to obtain higher point functions.

A more detailed prescription for real-time correlators was addressed in [35, 36]. They argued that if given a boundary correlator one can find a SK contour.[2] The bulk construction of this contour is a set of piecewise smooth geometries: the real-time segment of the boundary contour gets filled in with Lorentzian geometries, and imaginary-time segments with Euclidean geometries. These geometries are glued together along codimension-1 space-like slices. However, if one asks questions about dynamically evolving geometries, one realises that piecewise smooth geometries pose a potential issue in the presence of horizons. While a complete answer to this question is still unclear, an interesting prescription was recently given by [37] to address this gap in the probe limit.

The prescription of [37] postulates that the gravitational dual of the asymptotic SK contour at the boundary is given by a complex two-sheeted space-time. This geometry is made of two copies of the black hole exterior smoothly glued together across the future horizon, with a particular monodromy condition. The validity of this prescription was tested by the authors themselves by constructing a quadratic action for a probe field at

[2]In general all real-time correlators can not be accommodated in SK contour. SK contour is a one fold closed time contour, which contains the information of certain real-time correlators. But for arbitrary time-ordered correlators, one would require OTO contour.

the boundary. The prescription has also passed a non-linear check in [38]. The authors in [38] studied non-linear Langevin dynamics of a particle using this holographic construction. A constant time slice of the gravitational SK (grSK) geometry is drawn in 5 and a brief discussion of the prescription is given below.

Let us consider the Schwarzschild-AdS$_{d+1}$ black hole in the ingoing Eddington-Finkelstein coordinates, given by

$$ds^2 = -r^2 f(r) dv^2 + 2dv\, dr + r^2 d\mathbf{x}^2\,, \qquad f(r) = 1 - \frac{r_h^d}{r^d}\,. \tag{5.1}$$

We define a new coordinate ζ, which we refer to as the *mock tortoise* coordinate:

$$\frac{dr}{d\zeta} = \frac{i\beta}{2} r^2\, f(r)\,, \tag{5.2}$$

where $\beta = \frac{4\pi}{d r_h}$ is the inverse temperature of the black hole. The coordinate ζ can be viewed as parameterising a two-sheeted surface, each of which can be thought of as the bulk extension of the SK contour. The real part of ζ differentiates the two sheets and is given by the monodromy around the horizon. By convention we choose one of the sheets to have vanishing real part. Our choice of normalisation fixes the real part to unity. We also cut-off the AdS geometry at a radial cut-off $r = r_c$ for computational purposes. With this choice,

$$\zeta(r_c + i\varepsilon) = 0\,, \qquad \zeta(r_c - i\epsilon) = 1\,. \tag{5.3}$$

The metric in the *mock tortoise* coordinate takes the form,

$$ds^2 = -r^2\, f(r)\, dv^2 + i\,\beta\, r^2\, f(r)\, dv\, d\zeta + r^2\, d\mathbf{x}^2\,. \tag{5.4}$$

Before proceeding further, it is useful to note a feature of the grSK geometries. These geometries are not time reversal invariant. The time reversal, as described in [39], is realised by the following transformation.

$$v \to i\,\beta\,\zeta - v\,, \qquad \omega \to -\omega\,. \tag{5.5}$$

This transformation is used to map the ingoing solution to the outgoing ones.

5.2 Open ϕ^4 theory in CFT bath

Consider a scalar probe $\Psi(x)$ coupled to a d-dimensional CFT consisting fields $\{X(x)\}$. We choose the CFT to be in thermal equilibrium and we let $\mathcal{O}[\{X\}]$ be a local gauge invariant operator in this theory. The action for the probe and the bath is given by

$$S = \int d^d x \Big(\mathcal{L}[\Psi] + \mathcal{L}[\{X\}] + \Psi(x)\, \mathcal{O}(x) \Big) \,. \tag{5.6}$$

We integrate out $\{X(x)\}$ and get an open EFT for Ψ. To do that we consider the CFT to have a gravitational dual, i.e., the CFT is a large N, strongly coupled gauge theory. In this regime the dual system is described by classical gravitational dynamics in AdS space. According to AdS/CFT dictionary, the operator $\mathcal{O}$ has a dual field Φ in AdS. Correspondingly, the conformal dimension Δ is mapped to the mass of the scalar field by $m^2 \ell^2_{AdS} = \Delta(\Delta - d)$. One can interpret the probe field as the source of $\mathcal{O}$ from (5.6). Then according to the dictionary, the boundary value of Φ is nothing but Ψ itself.

To get open effective action for Ψ, we integrate out $\mathcal{O}$ by computing its thermal correlators. The $\mathcal{O}$ correlators can be calculated by studying the dynamics of Φ. In this note, we model the bath such that Φ is minimally coupled, massless ($\mathcal{O}$ is marginal, i.e., $\Delta = d$) and self-interacting scalar.[3] The action for the scalar field in AdS,

$$S_\Phi = - \oint d\zeta \int d^d x \sqrt{-g} \left[\frac{1}{2} g^{AB} \partial_A \Phi \, \partial_B \Phi + \frac{\lambda}{4!} \Phi^4 \right] \,. \tag{5.7}$$

Here the integral over ζ is defined along the grSK contour. We compute the retarded and advanced Green's function and write the solution for Φ as a linear combination of retarded-advanced Green's functions. Upon substituting the solution back into the action (5.7) followed by integrating the bulk along the grSK contour, we get the EFT for the probe Ψ at the boundary.

5.2.1 Retarded and advanced Green's function

The e.o.m. for the field Φ is given by

$$\partial_A \left(\sqrt{-g} g^{AB} \partial_B \Phi \right) = 0 \,. \tag{5.8}$$

[3] A detailed discussion on bath relevant operator can be found in [39].

In Fourier domain, we define

$$\Phi = \int d^{d-1}k \, d\omega \, e^{-i\omega v + i\mathbf{k}\cdot x} \, \Phi_k \tag{5.9}$$

so that the e.o.m. can be written as

$$D_\zeta^+ \left(r^{d-1} D_\zeta^+ \Phi_k \right) + \frac{\beta^2}{4} r^{d-1} \left(f|\mathbf{k}|^2 - \omega^2 \right) \Phi_k = 0 \,, \tag{5.10}$$

where $D_\zeta^\pm$ is defined as the following.

$$D_\zeta^\pm \equiv \frac{\partial}{\partial \zeta} \pm \frac{\beta\omega}{2} \,. \tag{5.11}$$

The retarded (ingoing) Green's function is a solution to (5.10) obtained by imposing regularity at the horizon and normalised to unity at the cut-off boundary:

$$\frac{dG^+}{d\zeta}\Big|_{r_h} = 0 \,, \qquad G^+\big|_{r_c} = 1 \,. \tag{5.12}$$

By using (5.5), one can show that the advanced (outgoing) Green's function is given by

$$G_{out} = G^-(\omega, |\mathbf{k}|) \, e^{-\beta\omega\zeta} \,, \tag{5.13}$$

where $G^-(\omega, |\mathbf{k}|)$ is time-reversed solution to the e.o.m. and is defined as $G^-(\omega, |\mathbf{k}|) \equiv G^+(-\omega, |\mathbf{k}|)$. The full solution in the ingoing-outgoing basis is given by

$$\Phi(\zeta, \omega, \mathbf{k}) = C_+(\omega, \mathbf{k}) \, G^+(\zeta, \omega, \mathbf{k}) + C_-(\omega, \mathbf{k}) \, G^-(\zeta, \omega, \mathbf{k}) e^{-\beta\,\omega\,\zeta} \,. \tag{5.14}$$

We can now impose boundary conditions at the conformal boundary $r = r_c \pm i\,\varepsilon$ of the grSK geometry. We demand:

$$\Phi_k\big|_{\zeta=0} = \Psi_L(\omega, \ \mathbf{k}) \,, \qquad \Phi_k\big|_{\zeta=1} = \Psi_R(\omega, \ \mathbf{k}) \,. \tag{5.15}$$

Here the R, L are the boundary SK d.o.f. in R-L basis. Using the fact that G^+ and G^- are normalized to unity at these boundaries, we find

$$C_+(\omega, \mathbf{k}) + C_-(\omega, \ \mathbf{k}) = \Psi_L \,, \qquad C_+(\omega, \ \mathbf{k}) + C_-(\omega, \ \mathbf{k}) e^{-\beta\omega} = \Psi_R \,. \tag{5.16}$$

Substituting the solution for $C_+(\omega, \mathbf{k})$ and $C_-(\omega, \mathbf{k})$ in terms of Ψ_R and Ψ_L, the solution for (5.10) takes the form,

$$\Phi_k(\zeta, \omega, \ \mathbf{k}) = G^+(\zeta, \omega, \ \mathbf{k}) \left((1 + n_\omega) \, \Psi_R - n_\omega \, \Psi_L \right) - G^-(\omega, \ \mathbf{k}) \, e^{\beta\,\omega\,(1-\zeta)} \, n_\omega \left(\Psi_R - \Psi_L \right) , \tag{5.17}$$

where n_ω is the Bose-Einstein factor. The solution in the *average-difference* basis (defined in (2.6)) is given by

$$\Phi(\zeta,\omega,\mathbf{k}) = G^+\Psi_a + \frac{1}{2}\,G^H\,\Psi_d\,. \tag{5.18}$$

The average piece consists of only the retarded Green's function. Therefore it is analytic function of the bulk coordinate r. All non-analyticity is present only in G^H via ζ . This mode deserves to be called the *Hawking Green's function* and is given by

$$G^H \equiv \coth\left(\frac{\beta\omega}{2}\right)\,G^+ + e^{\frac{\beta\omega}{2}(1-2\zeta)}\,\operatorname{csch}\left(\frac{\beta\omega}{2}\right)\,G^-\,. \tag{5.19}$$

5.2.2 Influence phase in derivative expansion

The influence phase is defined as the bath contribution to the effective action of the probe field Ψ, which is obtained by substituting the solution (5.18) into the bulk action (5.7) and performing the ζ integral along the grSK contour. Now all we need is the explicit expressions for the retarded Green's function G^+. The retarded Green's function in general dimension can not be solved in a closed form. However, we can solve this in frequency and momentum expansion (derivative expansion in real space) such that $\beta\omega, \beta\mathbf{k} << 1$. This is also the Markovian limit which captures the low energy behaviour of the probe field. In the derivative expansion, we consider

$$G^+(\zeta,\omega,\mathbf{k}) = \sum_{n,m=0}^{\infty} G^+_{m,n}(\zeta)\left(\frac{\beta\omega}{2}\right)^m\left(\frac{\beta|\mathbf{k}|}{2}\right)^n\,, \tag{5.20}$$

and the boundary conditions, stated in (5.12), translate to

$$G^+_{0,0}\Big|_{r=r_c} = 1\,, \qquad G^+_{m,n}\Big|_{r=r_c} = \frac{dG^+_{m,n}}{d\zeta}\Big|_{r=r_h} = 0\,, \qquad \forall\; m,n\in\mathbb{Z}_+\,. \tag{5.21}$$

The explicit expressions for $G^{+}_{m,n}$s till second order are given by

$$
\begin{aligned}
G^{+}_{0,0} &= 1\,,\\
G^{+}_{1,0} &= -\int_0^{\zeta} d\zeta'\left(1-\left(\frac{r_h}{r'}\right)^{d-1}\right),\\
G^{+}_{0,1} &= 0\,,\\
G^{+}_{2,0} &= \int_0^{\zeta} d\zeta'\int_{\zeta_h}^{\zeta'} d\zeta''\left(1+\left(\frac{r''}{r'}\right)^{d-1}\right)\left(1-\left(\frac{r_h}{r''}\right)^{d-1}\right)\\
&\quad+\int_0^{\zeta_h} d\zeta'\left(1-\left(\frac{r_h}{r'}\right)^{d-1}\right)\int_0^{\zeta} d\zeta''\left(1-\left(\frac{r_h}{r''}\right)^{d-1}\right),\\
G^{+}_{0,2} &= -\int_0^{\zeta} d\zeta'\int_{\zeta_h}^{\zeta'} d\zeta''\left(\frac{r''}{r'}\right)^{d-1} f(r'')\,.
\end{aligned}
\tag{5.22}
$$

Quadratic influence phase

Substituting the solution for Φ_k into the action and subsequently integrating out the bulk, we get the influence phase. The quadratic influence phase in real space is given by

$$
S_{(2)} = \int d^d x\left[-\mathfrak{J}_{ad}\,\Psi_d\,\partial_t\Psi_a + i\,\mathfrak{J}_{dd}\,\frac{\Psi_d^2}{2!}\right], \tag{5.23}
$$

where

$$
\mathfrak{J}_{ad} = r_h^{d-1}\,,\qquad \mathfrak{J}_{dd} = \frac{2}{\beta}\,r_h^{d-1}\,,\qquad \frac{\mathfrak{J}_{dd}}{\mathfrak{J}_{ad}} = \frac{2}{\beta}\,. \tag{5.24}
$$

In the above expression, we have kept till first order derivative terms. Ψ_d^2 is a pure fluctuation term, thus can be thought of as the leading order contribution due to Hawking radiation. The derivative term is the damping term causing dissipation of energy from the probe. The ratio of $\mathfrak{J}_{dd}$ and $\mathfrak{J}_{ad}$, given in (5.24), is called the linear *fluctuation-dissipation* relation (FDR) produced in several papers [40–44].

Quartic influence phase

The quartic influence phase in not divergence free. We shall follow the usual procedure to get rid of the divergences: we write a bare quartic influence phase along with a counter-term influence phase. The bare influence phase takes the following form.[4]

$$S^{\rm bare}_{(4)} = -\lambda \int d^dx \sum_{k=1}^{4} \frac{1}{(4-k)!} \left(\Psi^{\rm b}_a + \frac{i}{8}\beta\partial_t\Psi^{\rm b}_d\right)^{4-k} \left[F^{\rm b}_k \frac{(\Psi^{\rm b}_d)^k}{k!} - F^{\rm b}_{k+1}\frac{(\Psi^{\rm b}_d)^{k-1}}{(k-1)!}\frac{i}{2}\beta\partial_t\Psi^{\rm b}_d\right] . \tag{5.25}$$

Here

$$F^{\rm b}_k \equiv \oint d\zeta\sqrt{-g}\, \left(\zeta - \frac{1}{2} + G^+_{1,0}\right)^k \tag{5.26}$$

are divergent functions. The functions $F^{\rm b}_{2k+1}$ are power-law divergent whereas $F^{\rm b}_{2k}$ diverges logarithmically as the following.

$$F^{\rm b}_{2k+1} = F^{\rm r}_{2k+1} + \frac{\Lambda_d}{4^k}\, , \qquad F^{\rm b}_{2k} = F^{\rm r}_{2k} + k\frac{\Lambda_l}{4^{k-1}}\, , \tag{5.27}$$

where

$$\Lambda_d \equiv \frac{r_c^d}{d}\, , \qquad \Lambda_l \equiv \frac{i}{\pi} r_h^d \log\frac{r_c}{r_h}\, . \tag{5.28}$$

Given the divergence structure above, we assert that the renormalised probe should be defined as the following.

$$\Psi^{\rm b}_a \equiv \Psi^{\rm r}_a - \frac{\lambda_l}{2\Lambda_d}\Psi^{\rm r}_d\, , \qquad \Psi^{\rm b}_d \equiv \Psi^{\rm r}_d + \frac{\lambda_l}{2\Lambda_d} i\,\beta\,\partial_t\Psi^{\rm r}_d\, . \tag{5.29}$$

Note that as $r_c \to \infty$ we have $\lim_{r_c\to\infty}\frac{\Lambda_l}{\Lambda_d} = 0$, i.e., the bare and the renormalised sources agree when the cut-off is removed. With respect to the renormalised source, the counter-term influence phase is the following:

$$S^{\rm c.t.} = \frac{\lambda}{4!}\int d^dx \left[\left(\Psi^{\rm r}_a + \frac{1}{2}\Psi^{\rm r}_d\right)^4 - \left(\Psi^{\rm r}_a - \frac{1}{2}\Psi^{\rm r}_d\right)^4\right] . \tag{5.30}$$

This is the usual counter-term in SK language that appear in closed system. The renormalised influence phase is given by

$$S_{(4)} = -\int d^dx \sum_{k=1}^{4} \frac{\lambda}{(4-k)!} \left(\Psi^{\rm r}_a + \frac{i}{8}\beta\partial_t\Psi^{\rm r}_d\right)^{4-k} \left[F^{\rm r}_k \frac{(\Psi^{\rm r}_d)^k}{k!} - F^{\rm r}_{k+1}\frac{(\Psi^{\rm r}_d)^{k-1}}{(k-1)!}\frac{i}{2}\beta\partial_t\Psi^{\rm r}_d\right] . \tag{5.31}$$

[4] We have kept higher order derivative terms in order to write the quartic influence phase in a compact form. One should keep in mind that this result is correct only till first order derivative terms.

Substituting the numerical values of F^{r}_k [39] and keeping till first order derivate terms, we get the following renormalised quartic influence phase:

$$S_{(4)} = -\sum_{k=1}^{4} \int d^d x \left[\frac{(i\Psi_d)^k}{k!} (\theta_k + \bar{\theta}_k \partial_t) \frac{\Psi_a^{n-k}}{(4-k)!} \right] . \tag{5.32}$$

where

$$\begin{aligned} \theta_1 &= \lambda \frac{r_h^d}{d} , \qquad \bar{\theta}_1 = \theta_2 = 0 , \qquad \bar{\theta}_2 = \lambda \frac{\beta r_h^d}{8d} , \\ \theta_3 &= -\lambda \frac{r_h^d}{2d} , \qquad \bar{\theta}_3 = \lambda \frac{\beta r_h^d}{2d} \frac{3\zeta(3)}{\pi^3} , \qquad \theta_4 = \lambda \frac{r_h^d}{d} \frac{3\zeta(3)}{\pi^3} . \end{aligned} \tag{5.33}$$

Notice that the parameters in the influence phase have certain relation among themselves given by

$$\theta_{k+1} + \frac{1}{4}\theta_{k-1} + \frac{2}{\beta}\bar{\theta}_k = 0 . \tag{5.34}$$

These relations are non-linear generalisation of the fluctuation-dissipation relation (FDR). The non-linear FDRs seem to be generic, since identical relations appear while integrating out marginal operator bath with n^{th} order interaction terms [39]. (In this note we have only considered $n = 4$ interaction term). These FDR also matches with the previously derived FDRs obeyed by a particle for $n = 4$ in [38, 45].

Note that the above influence phase (5.32) does not have $(\Psi^{\mathrm{r}}_a)^4$ term, which is a manifestation of the Lindblad condition stated in §3. Corresponding to this influence phase one can construct a stochastic dynamical equation[39] for the probe. In the stochastic description, $(\Psi^{\mathrm{r}}_d)^3\, \partial_t \Psi^{\mathrm{r}}_a$ term corresponds to the quartic correction to the damping term, and $(\Psi^{\mathrm{r}}_d)^4$ term brings in sub-leading contribution to the quadratic fluctuation.

6 Conclusion and open questions

We have studied open scalar field theory and its various aspects in SK formalism. We have discussed the systematic construction of open action in *R-L* and *average-difference* basis and computed one loop correction to the parameters in Lorentz invariant open scalar field theory. We have showed that at one loop order, there exists non-local divergences. We do not have a good understanding of these divergences. It is not clear to us yet whether these divergences can be removed by some renormalisation prescription.

However holography has given us a hope to construct action for open QFTs, although the original problem of renormalisation is not understood. In the holographic setup, we find the open effective action for a probe field in a thermal CFT bath. Following the AdS/CFT dictionary, we study the dynamics of a bulk scalar field dual to a CFT primary bath operator. We integrate out the bulk scalar field along the grSK contour. This allows us to compute influence phases for the probe field. The influence phases manifest the consistency conditions for open system and produce the linear and non-linear fluctuation-dissipation relations among certain parameters.

We conclude with some open problems.

1. The first problem is to understand the origin of non-local divergences and to regularise these. 2. In the hologrpahic setup, one can take this endeavour forward and find the EFTs for gauge fields & gravity and find a microscopic derivation for the theory of diffusion, Navier-Stokes equation, etc. Work on gauge theory has been attempted by [37], however various subtleties in gauge theory are not yet well-understood. 3. A prescription for the generalisation of the grSK contour to grOTO contour would allow us to study OTO correlations. 4. A limitation of the grSK construction is that it is ignorant of the initial state of the probe. The initial state is required along with the action to compute real-time correlators. Therefore a prescription with the incorporation of grSK saddle and the initial state would be a complete prescription to study many non-equilibrium processes.

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