Overview

- **Talk #1:**

Chaos in 1D (*is all you need!*) + Primer on Machine Learning

- **Talk #2 (Nov. 15th):**

Neurochaos Learning + Monsoon prediction

Chaos + Noise = Learning

#1 Chaos in 1D Maps *(is all you need!)*

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Complex Systems Programme

National Institute of Advanced Studies

Nov. 11, 2024

NIAS

Tempest in a Teapot

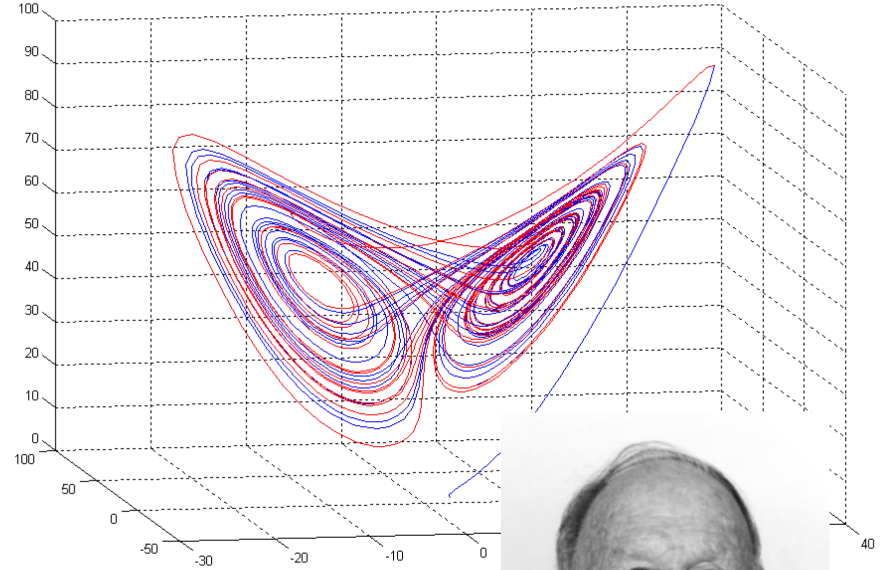
Can you trust your computer?

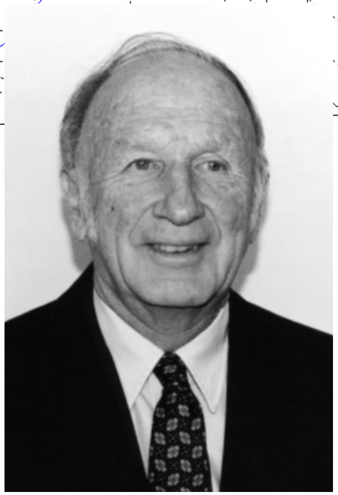
When Non-linearity meets Iteration

Determinism and Unpredictability co-exist!!

Even Newton's clock-work universe has unpredictability

"God does not need to play dice!!"





Edward Lorenz, 1963

BEWARE of ITERATIONS on your PC!!!

Preliminaries

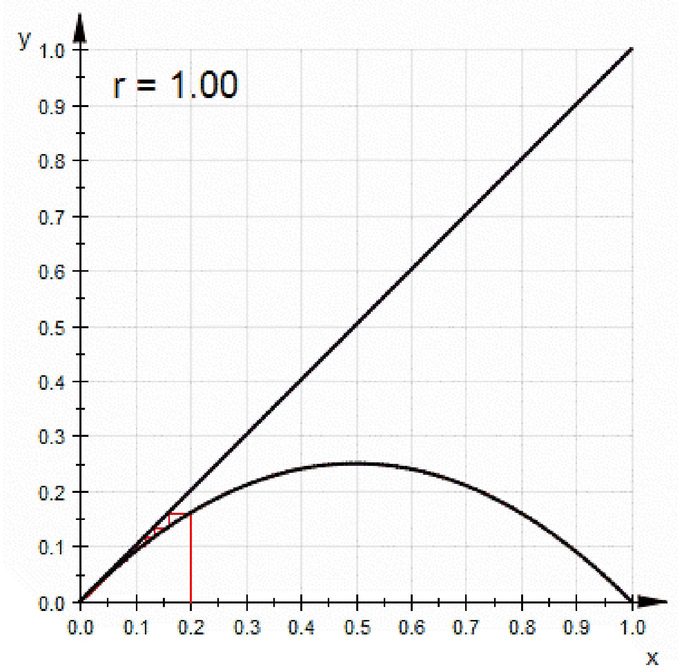
Notations and Definitions

Let us define…

- Chaos in 1D maps
- Symbolic Dynamics

Several candidates: Chaos in 1D maps

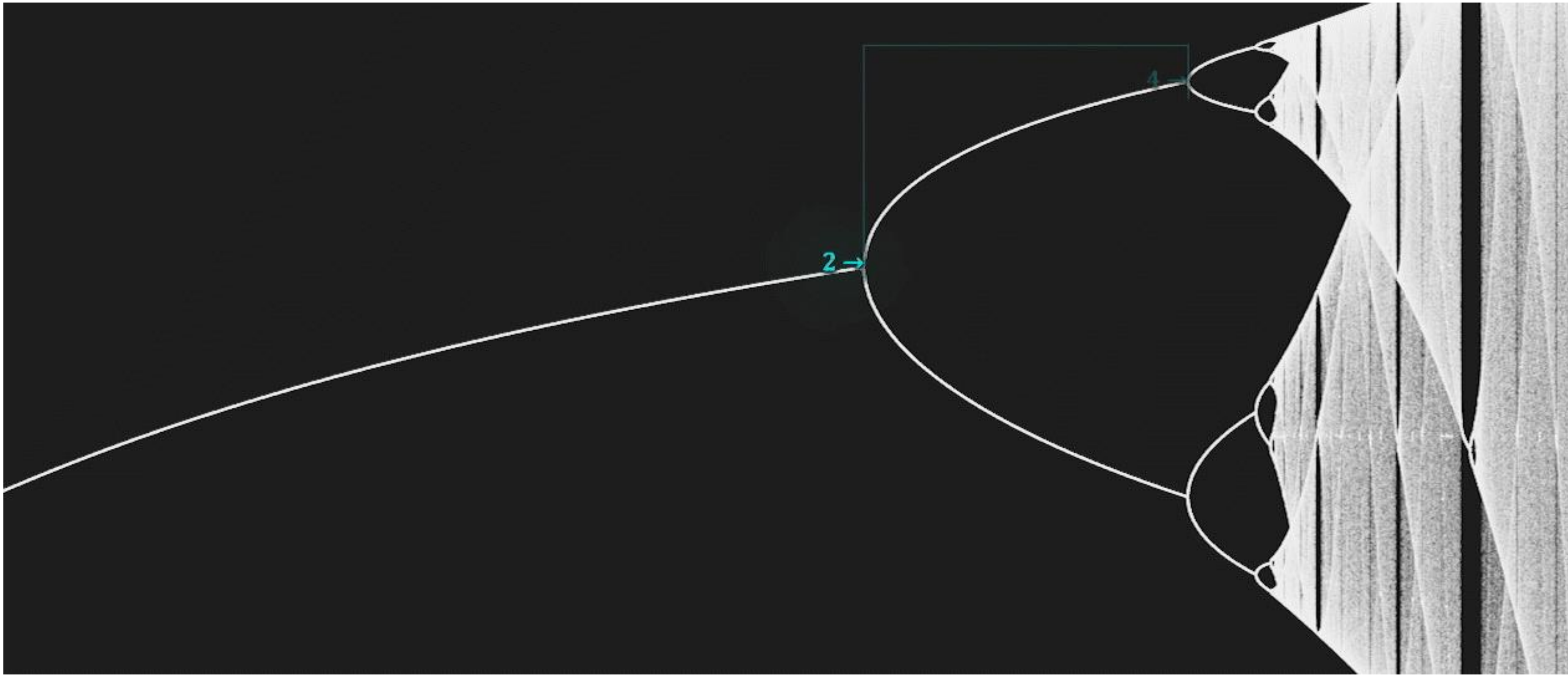
- Logistic map
- Tent map
- Standard map
- Circle map
- Ricker map
- Gauss map
- …



Source: Wiki

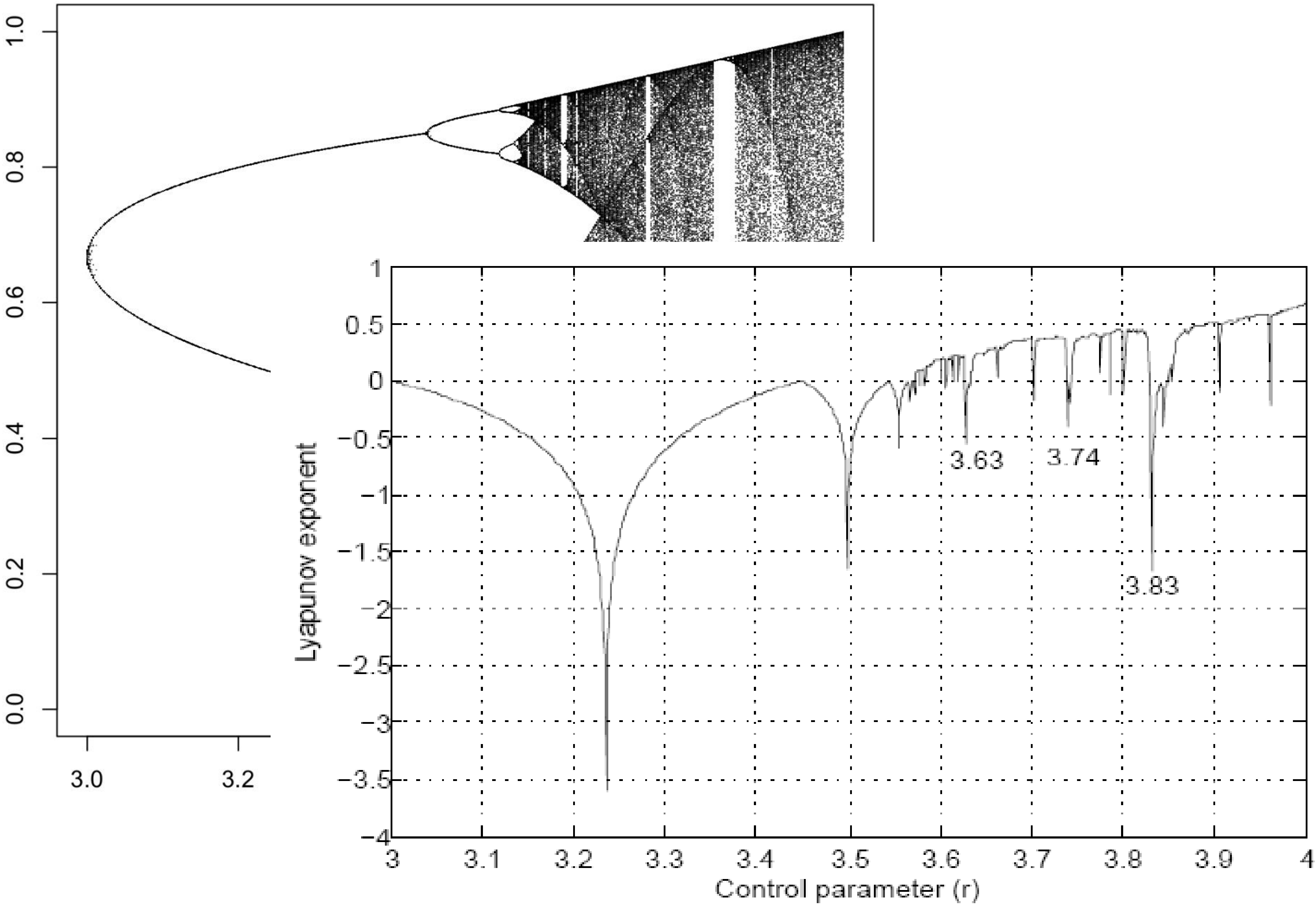
What is Chaos really?

For Artlovers…



Source: https://medium.com/codex/chaos-b1544ad03948
Author: Shameed Sait

For Physicists, Biologists, most Scientists…



For Mathematicians…

PERIOD THREE IMPLIES CHAOS

TIEN-YIEN LI AND JAMES A. YORKE

1. Introduction. The way phenomena or processes evolve or change in time is often described by differential equations or difference equations. One of the simplest mathematical situations occurs when the phenomenon can be described by a single number as, for example, when the number of children susceptible to some disease at the beginning of a school year can be estimated purely as a function of the number for the previous year. That is, when the number x_{n+1} at the beginning of the $n+1$st year (or time period) can be written

(1.1) $$x_{n+1} = F(x_n),$$

The American Mathematical Monthly, 1975

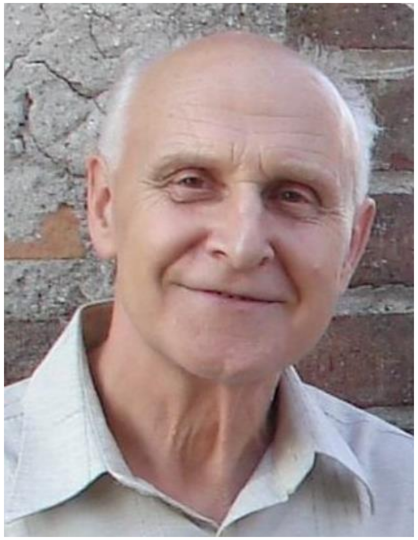
Sharkovsky’s Theorem (1964)

$$3 \triangleright 5 \triangleright 7 \triangleright 9 \triangleright 11 \triangleright \ldots$$

$$2 \cdot 3 \triangleright 2 \cdot 5 \triangleright 2 \cdot 7 \triangleright \ldots$$

$$2^2 \cdot 3 \triangleright 2^2 \cdot 5 \triangleright 2^2 \cdot 7 \triangleright \ldots$$

$$\ldots 2^4 \triangleright 2^3 \triangleright 2^2 \triangleright 2 \triangleright 1.$$



Oleksandr Sharkovsky

Sharkovskii, O. M. (1964). "**Co-existence of cycles of a continuous mapping of the line into itself**". *Ukrainian Math. J.* **16**: 61–71.

For the mathematically-challenged (like myself)…

From Intermediate Value Theorem To Chaos

XUN-CHENG HUANG
New Jersey Institute of Technology
Newark, NJ 07102

Xun-Cheng Huang, ***Mathematics Magazine***, Vol. 65, No. 2 (Apr., 1992), pp. 91-103

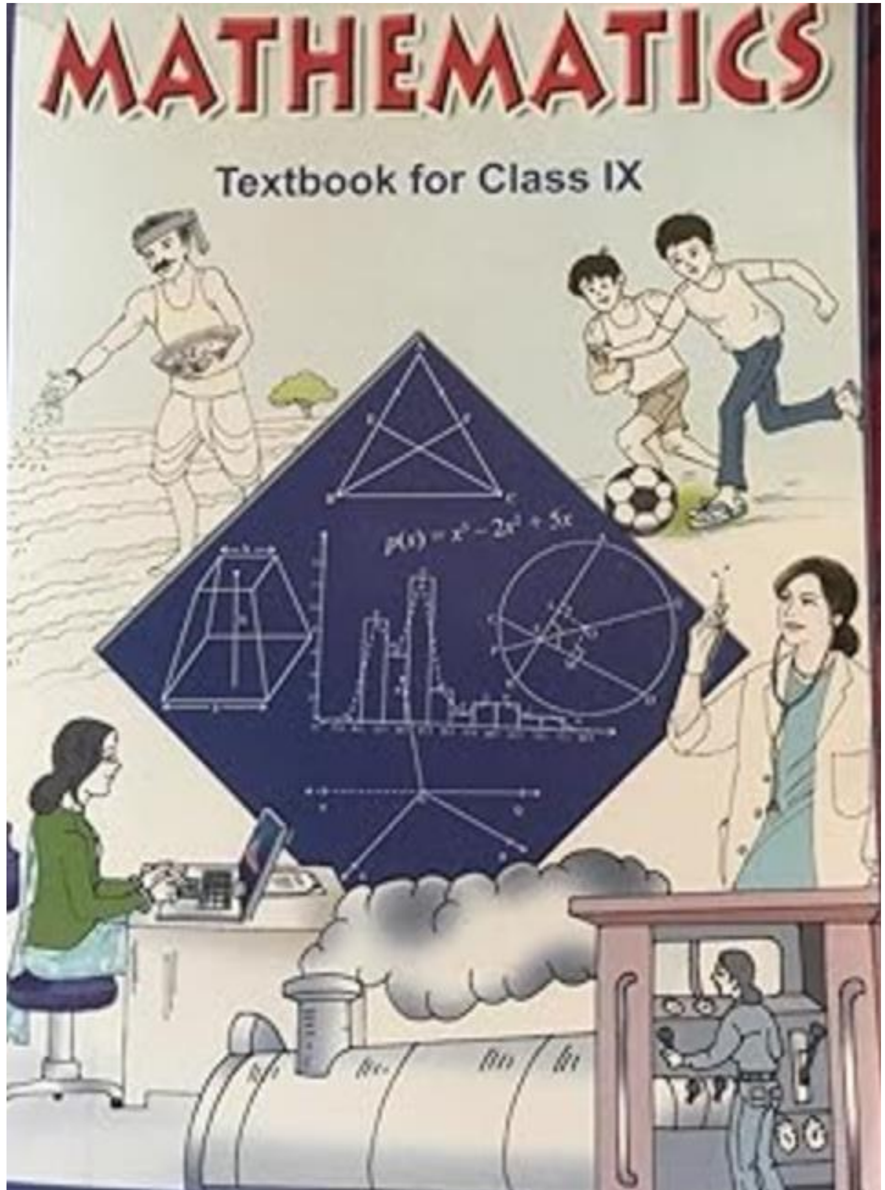
Is there an example of Chaos that is accessible to **ALL**?

Class IX Mathematics Text Book

विद्यया ऽमृतमश्नुते

एन सी ई आर टी

NCERT



Think of a real no. between 0 & 1…

1. x <- input
2. Output [10x]
3. x <- 10x-[10x]
4. Go to Step 2

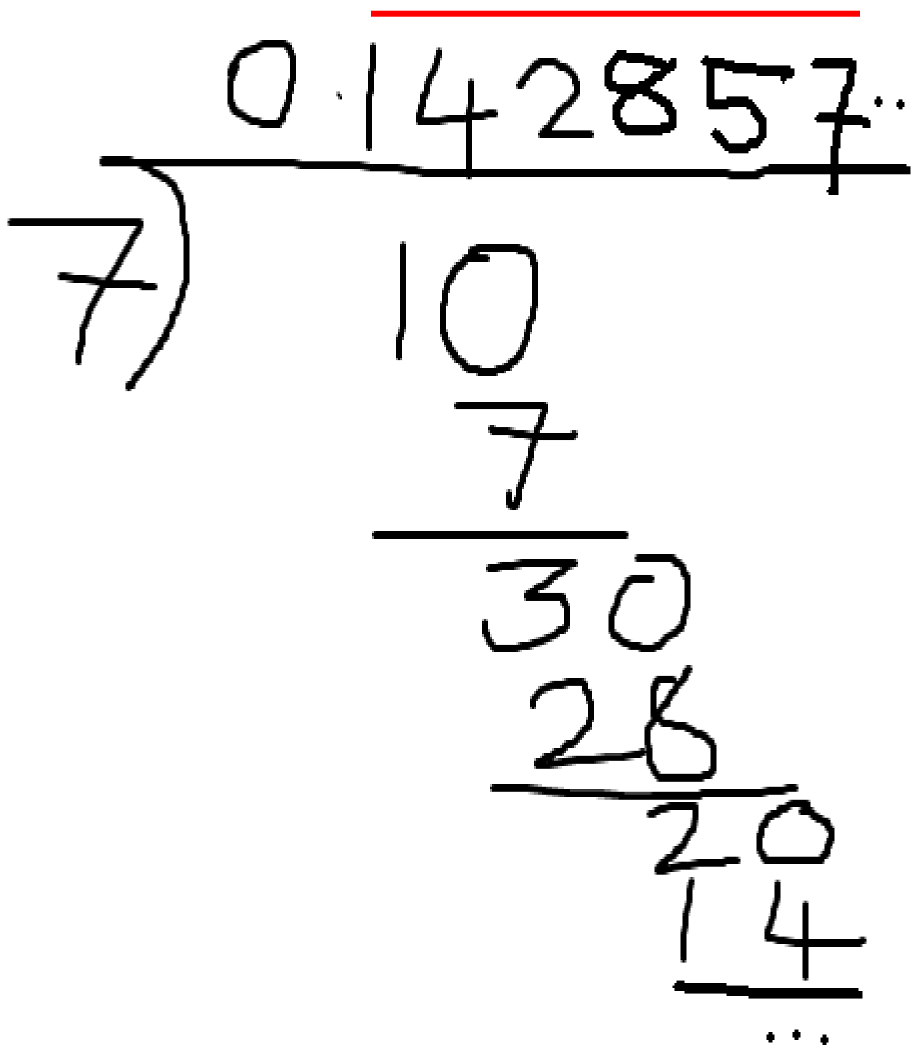
[.]: Integer part

Say, you take x = **1/7**

- Output **[10/7]** = **1**
- **x** $\longleftarrow$ **10/7 – [10/7] = 3/7**
- Output **[30/7]** = **4**
- **x** $\longleftarrow$ **30/7 – [30/7] = 2/7**
- Output **[20/7]** = **2**
- **x** $\longleftarrow$ **20/7 – [20/7] = 6/7**
- Output **[60/7]** = **8**
- **x** $\longleftarrow$ **60/7 – [60/7] = 4/7**
- **…**

$$1/7 \xrightarrow[1]{} 3/7 \xrightarrow[4]{} 2/7 \xrightarrow[2]{} 6/7 \xrightarrow[8]{} 4/7 \xrightarrow[5]{} 5/7 \xrightarrow[7]{} 1/7$$

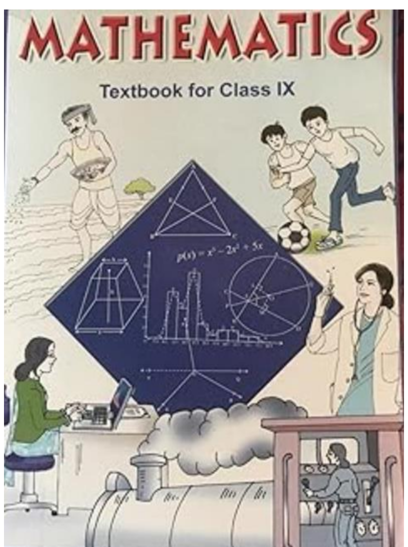
Long Division 1/7:



Example 5 : Find the decimal expansions of $\frac{10}{3}$, $\frac{7}{8}$ and $\frac{1}{7}$.

Solution :

$$\begin{array}{r|l}
 & 3.333... \\ \hline
3 & 10 \\
 & \;\,9 \\ \hline
 & \;\,10 \\
 & \;\;\,9 \\ \hline
 & \;\,10 \\
 & \;\;\;9 \\ \hline
 & \;\;\;10 \\
 & \;\;\;\;\,9 \\ \hline
 & \;\;\;\;\,1
\end{array}
\qquad
\begin{array}{r|l}
 & 0.875 \\ \hline
8 & 7.0 \\
 & 64 \\ \hline
 & \;\,60 \\
 & \;\,56 \\ \hline
 & \;\;\,40 \\
 & \;\;\,40 \\ \hline
 & \;\;\;\;0
\end{array}
\qquad
\begin{array}{r|l}
 & 0.142857... \\ \hline
7 & 1.0 \\
 & \;\,7 \\ \hline
 & \;\,30 \\
 & \;\,28 \\ \hline
 & \;\;\,20 \\
 & \;\;\,14 \\ \hline
 & \;\;\;60 \\
 & \;\;\;56 \\ \hline
 & \;\;\;\;40 \\
 & \;\;\;\;35 \\ \hline
 & \;\;\;\;\;50 \\
 & \;\;\;\;\;49 \\ \hline
 & \;\;\;\;\;\;1
\end{array}$$



Remainders : 1, 1, 1, 1, 1...
Divisor : 3

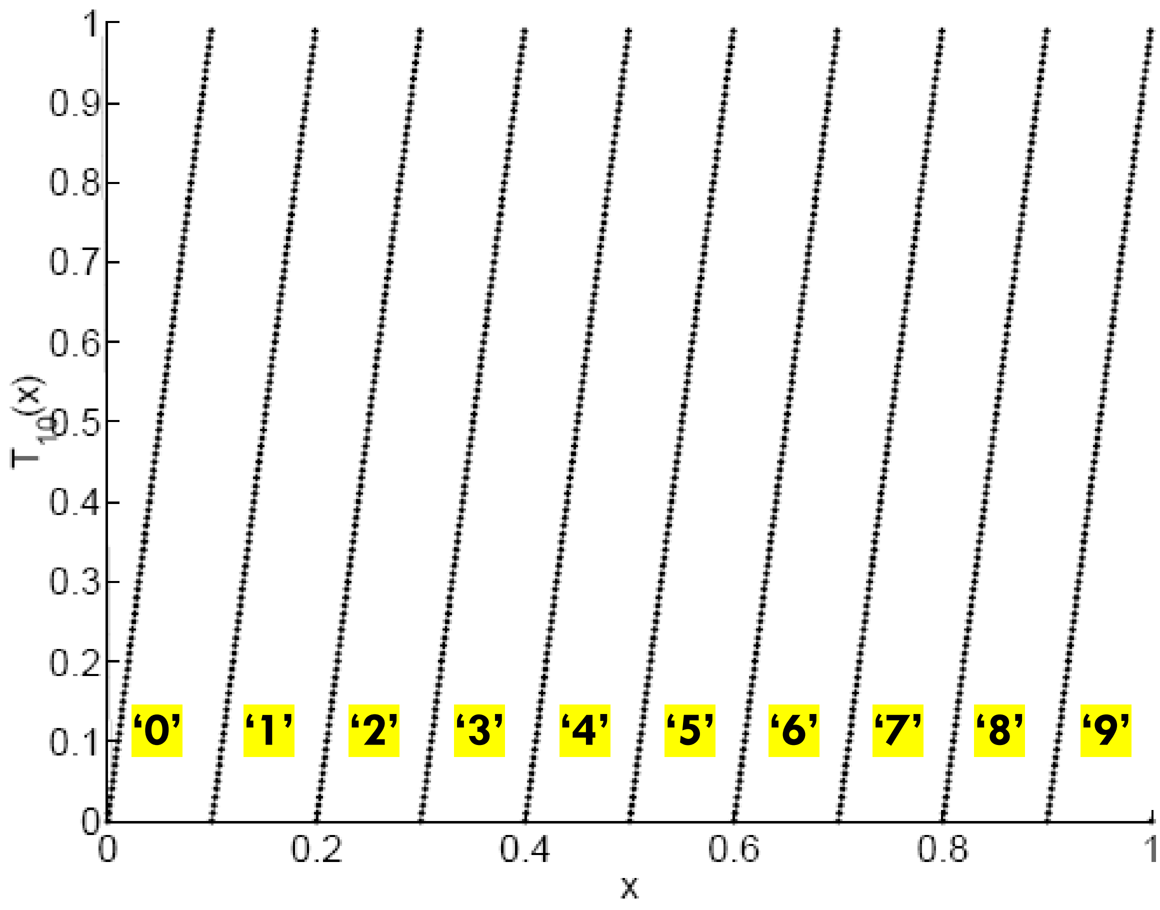
Remainders : 6, 4, 0
Divisor : 8

Remainders : 3, 2, 6, 4, 5, 1, 3, 2, 6, 4, 5, 1,...

Divisor : 7

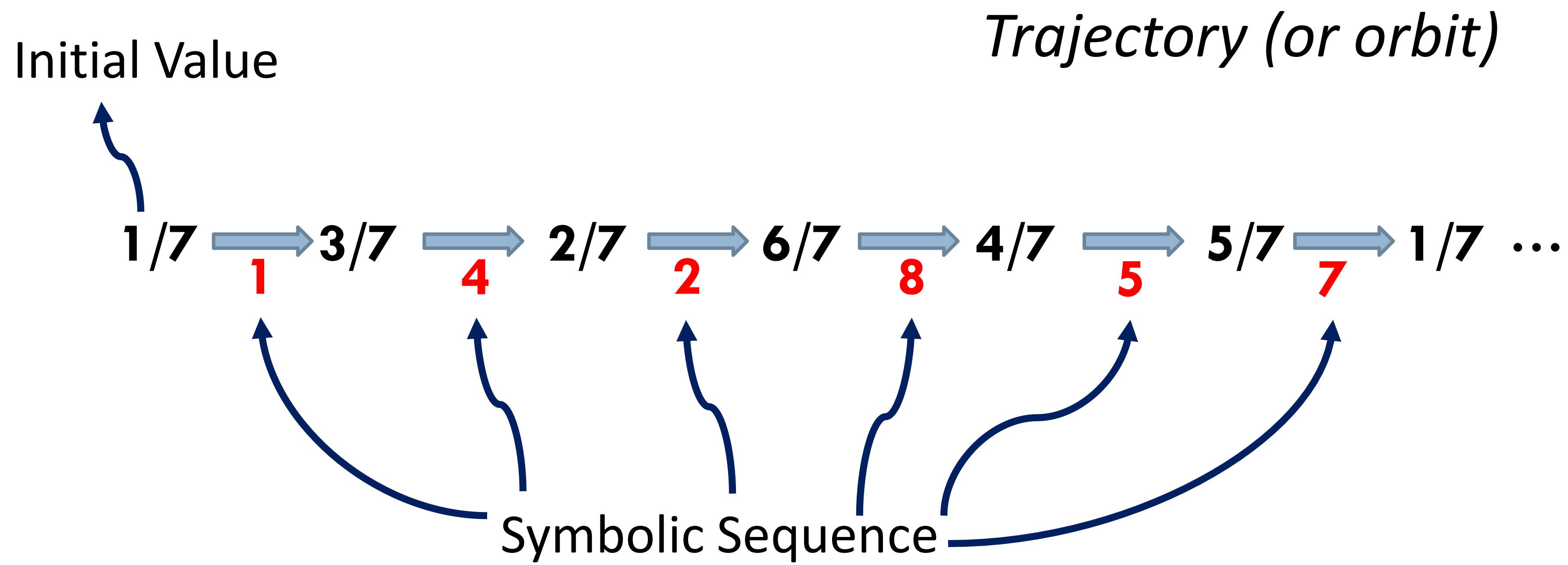
The Decimal Map

$$T_{10} : [0,1] \to [0,1] \qquad T_{10}(x) = 10x - [10x]$$



Ten symbols associated with the 10 intervals

Notations



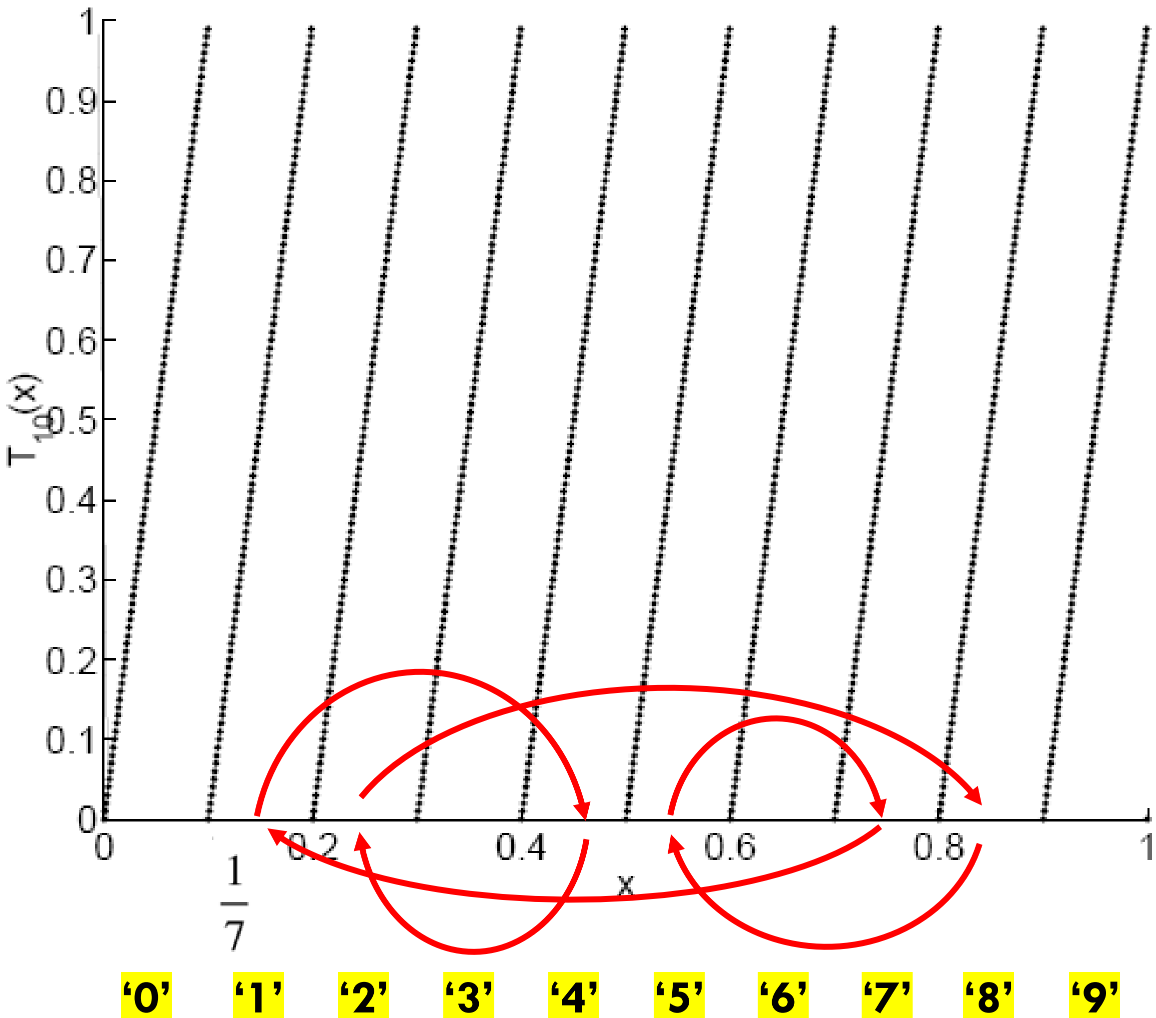
$0.\overline{142857}$, $0.\overline{428571}$, $0.\overline{285714}$, $0.\overline{857142}$, $0.\overline{571428}$, $0.\overline{714285}$, $0.\overline{142857}$, …

Notice the ‘shift’ at every iteration

Symbolic Dynamics

$$T_{10} : [0,1] \to [0,1]$$

$$T_{10}(x) = 10x - [10x]$$



Long Division = CHAOS

- 1/7 yields a purely periodic orbit (**period = 6**)

- 1/3 also gives a purely periodic orbit but with **period = 1**

- **1/125 = 0.008** -> 0.08 -> 0.8 -> 0 -> 0 -> 0 ->… **(eventually periodic with period = 1)**

- $\mathbf{0.12\overline{34}}$ -> $0.2\overline{34}$ -> $0.\overline{34}$ -> $0.\overline{43}$ -> $0.\overline{34}$ -> … **(eventually periodic with period = 2)**

Decimal expansions = CHAOS

- Rational numbers as initial conditions yield purely periodic (repeating) or eventually ***periodic orbits***.
- Irrational numbers yield non-repeating decimal expansion (***non-periodic orbits***). Example $1/\sqrt{2}$, $\pi-3$ etc.
- This map is also a type of ***Shift Map***. It is a CHAOTIC map (also ergodic).

So, every number with a non-terminating recurring decimal expansion can be expressed in the form $\frac{p}{q}$ $(q \neq 0)$, where p and q are integers. Let us summarise our results in the following form :

The decimal expansion of a rational number is either terminating or non-terminating recurring. Moreover, a number whose decimal expansion is terminating or non-terminating recurring is rational.

So, now we know what the decimal expansion of a rational number can be. What about the decimal expansion of irrational numbers? Because of the property above, we can conclude that their decimal expansions are *non-terminating non-recurring*.

So, the property for irrational numbers, similar to the property stated above for rational numbers, is

The decimal expansion of an irrational number is non-terminating non-recurring. Moreover, a number whose decimal expansion is non-terminating non-recurring is irrational.

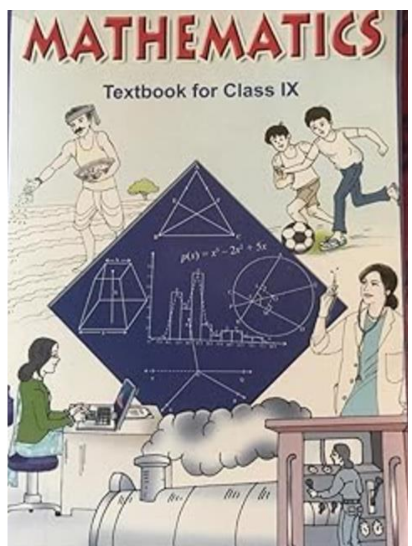
Recall s = 0.10110111011110... from the previous section. Notice that it is non-terminating and non-recurring. Therefore, from the property above, it is irrational. Moreover, notice that you can generate infinitely many irrationals similar to s.

What about the famous irrationals $\sqrt{2}$ and π? Here are their decimal expansions up to a certain stage.

$$\sqrt{2} = 1.4142135623730950488016887242096...$$

$$\pi = 3.14159265358979323846264338327950...$$

(Note that, we often take $\frac{22}{7}$ as an approximate value for π, but $\pi \neq \frac{22}{7}$.)



Necessary and Sufficient Conditions for **Deterministic CHAOS** in MAPS:

1. Deterministic equations (iteration: Discrete time)
2. Non-linear equations
3. Bounded
4. Existence of all periods (dense periodic orbits)
5. Existence of non-periodic (wandering) trajectories
6. Sensitive dependence on initial conditions (BUTTERFLY EFFECT) indicated by

   Lyapunov Exponent > 0
7. Topological Transitivity (related to Ergodicity)

Lyapunov Exponent

- It is an indicator of CHAOS
- Let x_1 be the initial condition and let $\{x_1, x_2, \ldots, x_n, \ldots\}$ be a trajectory on the map $x_{n+1} = f(x_n)$.
- Lyapunov Exponent of this trajectory is defined as:

$$\lambda(\mathbf{x_1}) = \lim_{n \to \infty} (1/n)[\ln|f'(x_1)| + \cdots + \ln|f'(x_n)|],$$

$$\lambda(\mathbf{x_1}) = \lim_{n \to \infty} \frac{1}{n} \sum_{i=1}^{n} \ln|f'(x_i)|$$

- The Decimal Map has a Lyapunov Exponent = $\ln(10) > 0$ => CHAOS.

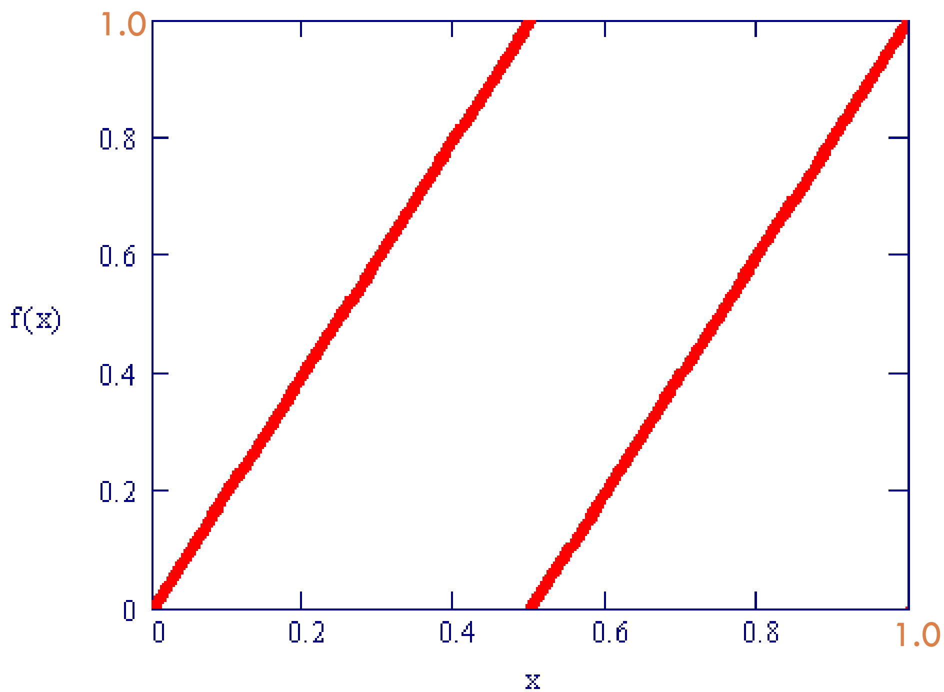
Symbolic Dynamics

Decimal map is a shift map

Symbolic Dynamics

- Owing to measurement problems, we only record in which **interval** the iterate lies.

- Consider the Binary Map: $f(x) = 2x - [2x]$



Lyapunov Exponent
= ln(2) = 0.6931
>0
=> CHAOS

Symbolic Sequence

- Consider an initial condition, say **x = 0.123**. Iterate the map and track where it falls: [0, 0.5) or [0.5, 1).

- Use **L** and **R** to code the trajectory.

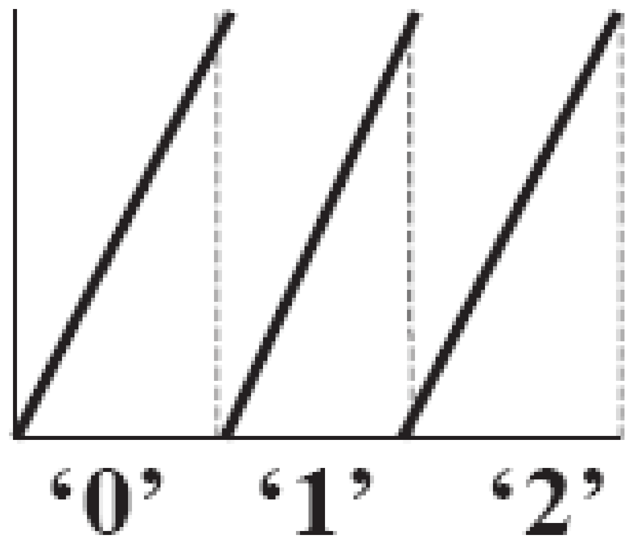
- The sequence thus generated is known as the ***symbolic sequence*** corresponding to the initial condition.

- For the example, 0.123 (L) -> 0.246 (L) -> 0.492 (L) -> 0.984 (R) -> 0.968 (R) -> 0.936 (R) -> 0.872 (R) -> 0.744 (R) -> 0.488 (L) and so on…

- Hence the symbolic sequence for x =0.123 is **LLLRRRRRL… which is actually BINARY EXPANSION.**

What about Ternary Expansion?

- The Ternary map (Base-3 representation of real numbers)



- Lyapunov Exponent of Ternary map is ln(3) > 0
  => Chaos

N-ary expansions….

- All N-ary expansions are ***symbolic sequences*** on 1D chaotic maps

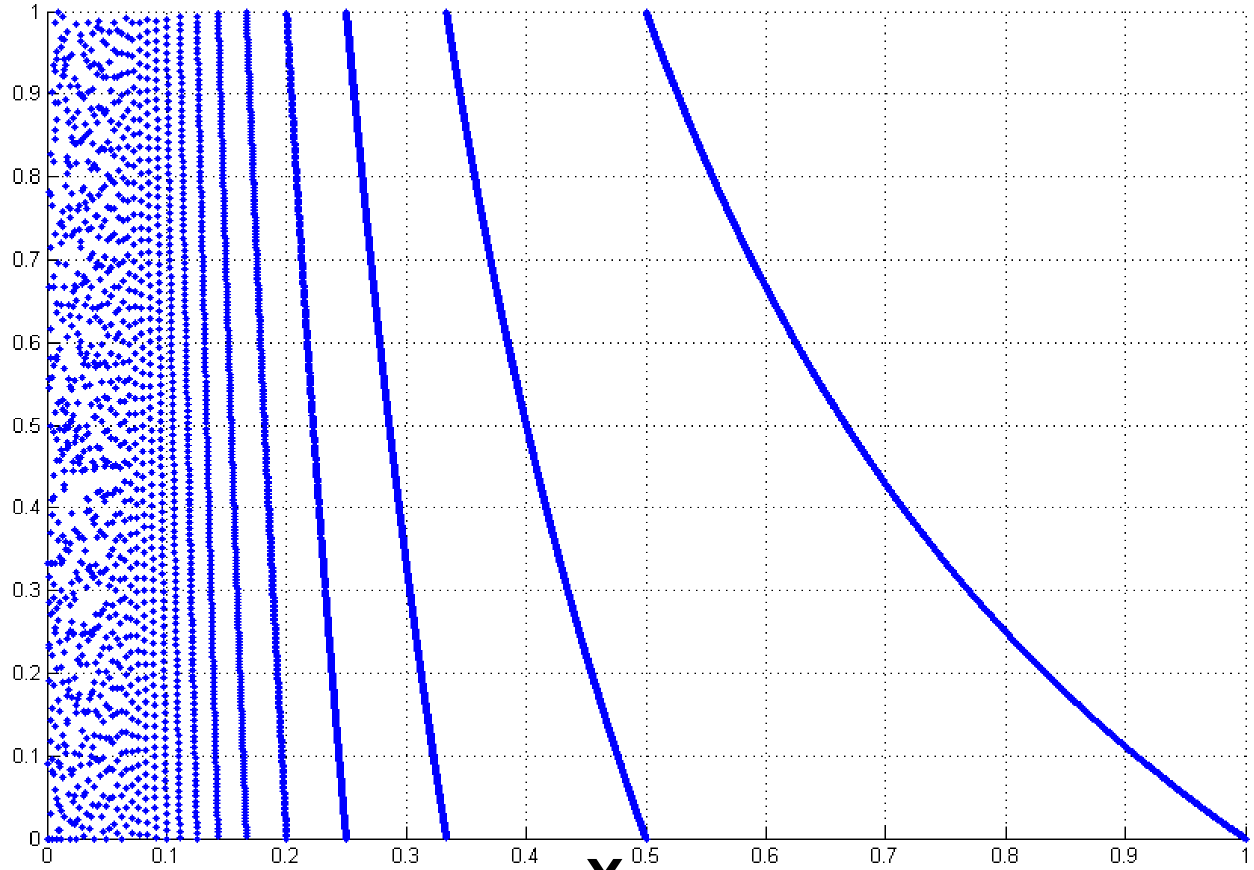
- Lyapunov Exponent = ln(N) > 0

  => Chaos

Another example: Gauss Map

$$X_{n+1} = 1/X_n - [1/X_n]$$

$$X_i \in (0, 1]$$



X_{n+1}

X_n

Tryst with Number Theory…

- Symbolic sequence of the Gauss Map yields ***Continued Fractions***.

$$a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{\ddots + \cfrac{1}{a_n}}}}$$

$$\pi = 3 + \cfrac{1^2}{6 + \cfrac{3^2}{6 + \cfrac{5^2}{6 + \ddots}}}$$

- Lyapunov Exponent of the Gauss Map is also > 0
  => Chaos

- A rich interplay between ***Dynamical Systems*** and ***Number Theory***…

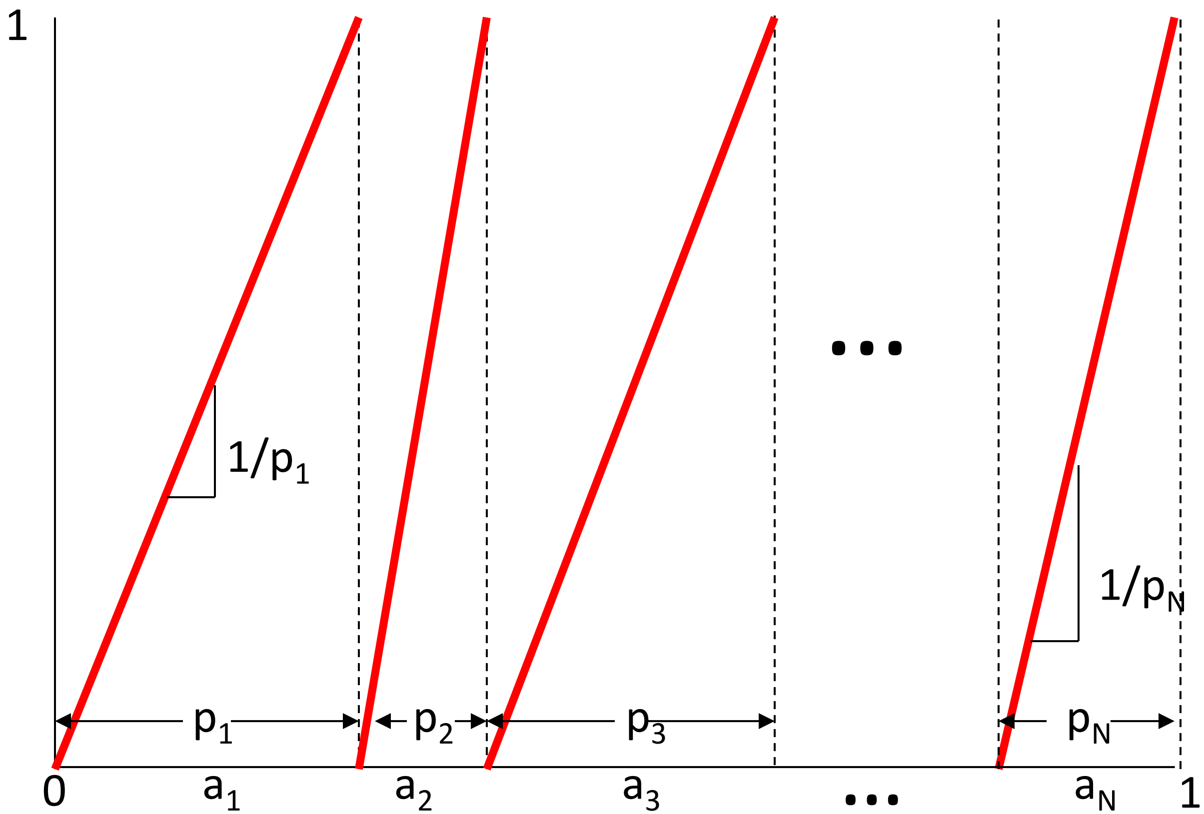
Generalized Luröth Series (GLS)

Generalized Number Systems

or

Generalized Number Expansions

GLS: Luröth’s paper in 1883, Cantor’s work in 1869



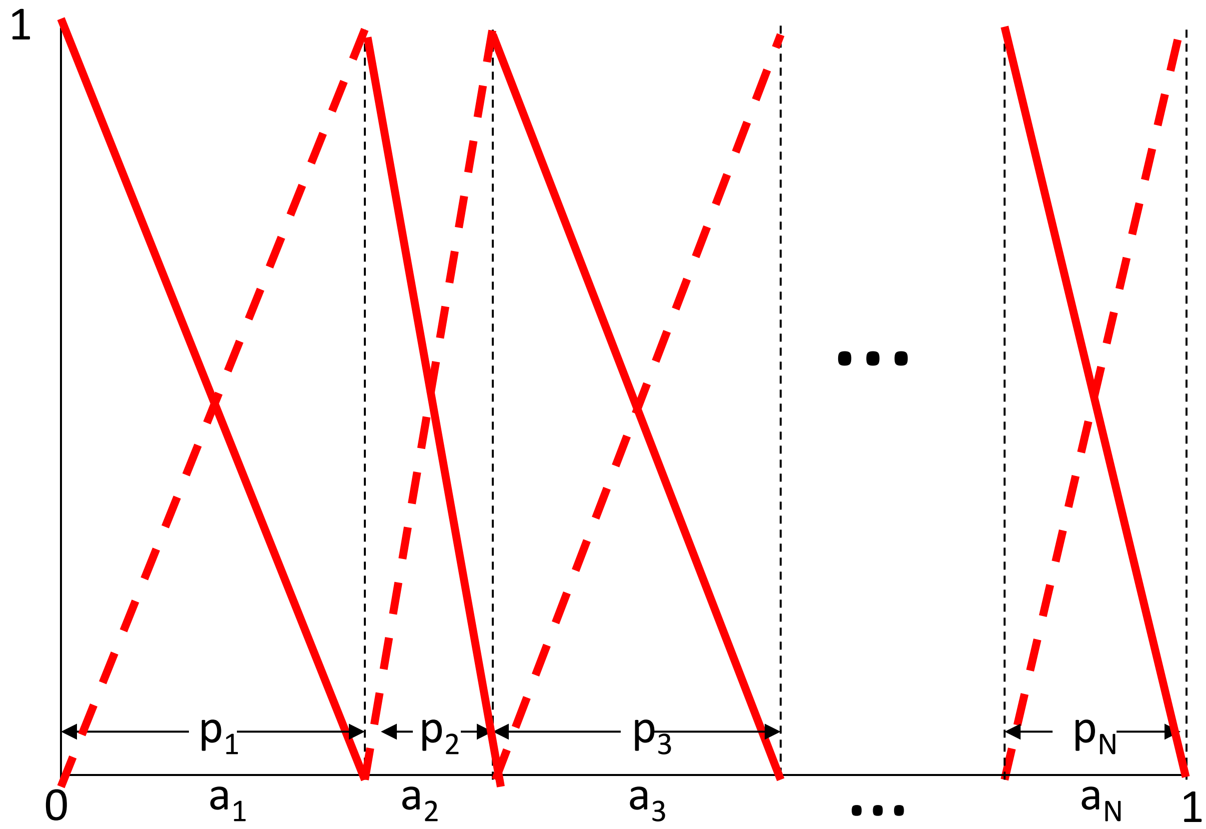
Cantor

Properties of GLS

- Shares the properties of Tent map, Decimal map, Binary map, N-ary map…
- GLS is a **Chaotic** map (**Lyapunov Exp. > 0**)
- GLS finds applications in Number Theory

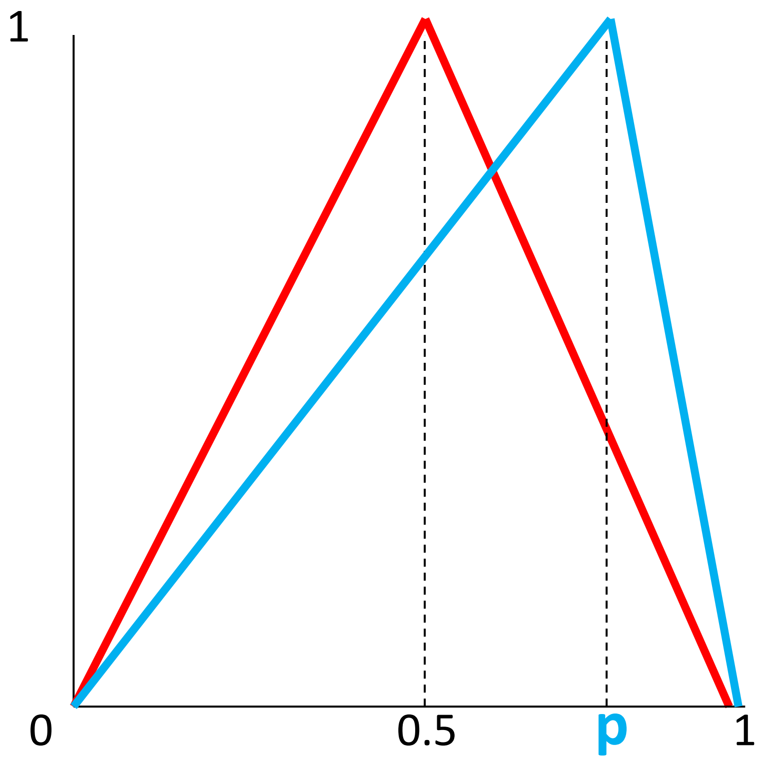
Ref: K Dajani, C Kraaikamp, ***Ergodic Theory of Numbers***, The Mathematical Association of America, 2002.

Modes of GLS

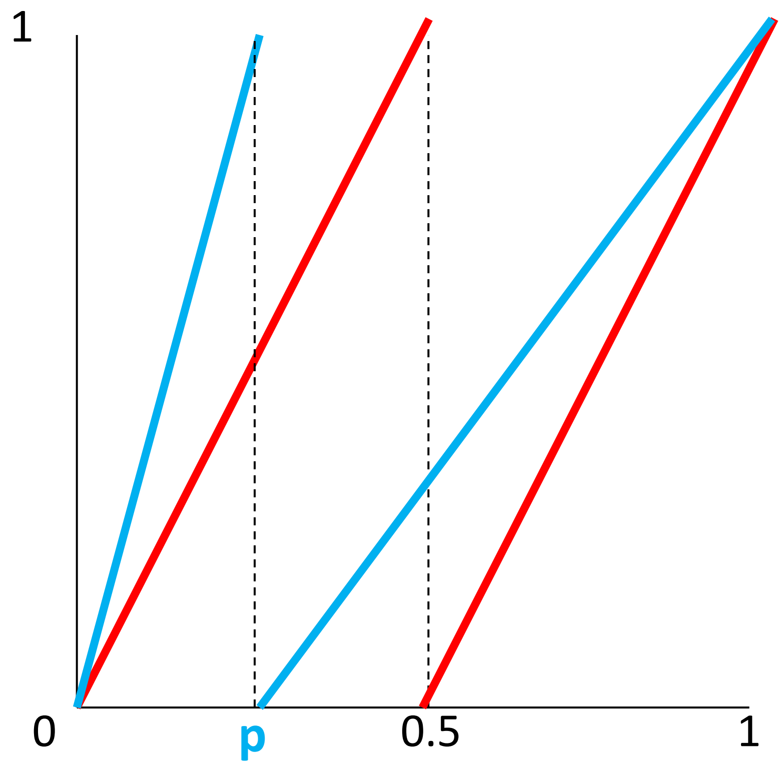


2^N modes (up to permutations)

Well known example of GLS (2 symbols)



Tent Map
(also Skew)



Binary Map
(also Skew)

GLS Maps are Special

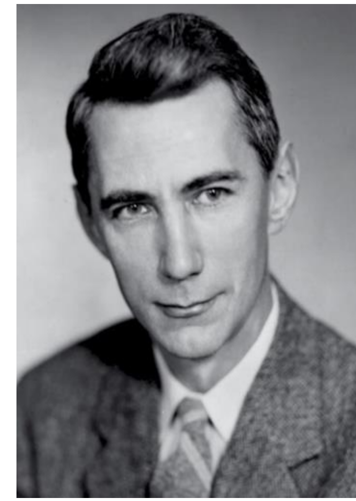
Aleksandr
Lyapunov
(1857-1918)

$$\lambda = H$$

Lyapunov Exponent = Shannon Entropy

$$\lambda = - \sum_{i=1, p_i \neq 0}^{i=N} p_i \, log_2(p_i).$$

bits/iteration



Claude
Shannon
(1916-2001)
Father of
Information Theory

Information theory meets Chaos!!!

$$\lambda = H(\text{Symbolic Seq.}) \text{ bits/iter}$$

- We interpret Lyapunov exponent as the ***amount of information about the initial value revealed by the symbolic sequence of the dynamical system at every iteration***.

End-Note:

- 1D maps such as Decimal Map (long division), Binary, Ternary, N-ary, Tent Map – all exhibit CHAOS
- These piecewise linear maps are part of a family of maps known as GLS maps
- Symbolic Sequence on GLS maps has connections with Information Theory ($H = \lambda$)
- Relation with Markov Chains!

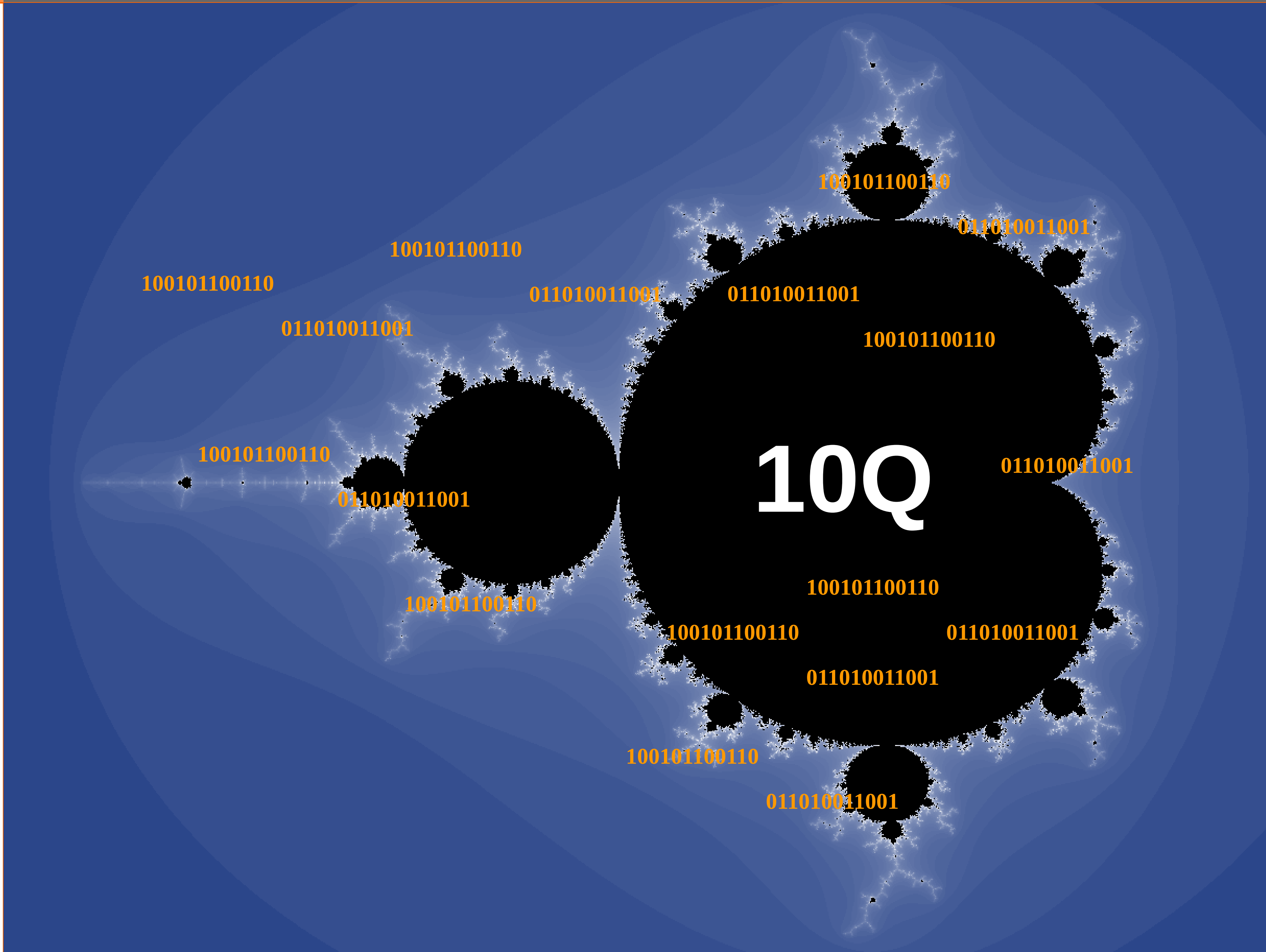
Forthcoming Attractions

- **Talk #2 (Nov. 15th):**

Neurochaos Learning + Monsoon prediction

Chaos + Noise = Learning

We shall build a Neural Network with 1D GLS Maps and use it for classification & prediction tasks.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