Generalization of the extended T-systems via strong duality data

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Combinatorics, Geometry, and Representation Theory

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Plan

1. Mukhin–Young's extended T-systems

   (which is what we will generalize)

2. Strong duality data and affine cuspidal modules

   (fundamental notions to state the main thm)

3. Main Theorem & sketch of proof

Notations

$\mathfrak{g}$: complex affine Lie algebra (e.g. $\mathfrak{g} = \mathfrak{g}_0 \otimes \mathbb{C}[t^{\pm 1}] \oplus \mathbb{C}K$)

$U'_q(\mathfrak{g})$: **quantum affine alg.** with indices $[0, n]$, $q \in \mathbb{C}^\times$ not root of 1

(associative algebra over $\mathbb{C}$ such that "$\lim_{q \to 1} U'_q(\mathfrak{g}) = U(\mathfrak{g})$")

$\mathscr{C}_\mathfrak{g}$: the cat. of **finite-dimensional** $U'_q(\mathfrak{g})$-mod. (of type 1)

- $\mathscr{C}_\mathfrak{g}$ is a monoidal category with $\otimes$ and the trivial module $\mathbf{1}$
  $\Rightarrow K(\mathscr{C}_\mathfrak{g})$ has a ring structure (**Grothendieck ring**)
- Each $M \in \mathscr{C}_\mathfrak{g}$ has the right dual $\mathscr{D}(M)$ & the left dual $\mathscr{D}^{-1}(M)$

It is an important and difficult problem to study the structures (e.g. JH-series) of tensor products $M \otimes N$ for simples $M, N \in \mathscr{C}_\mathfrak{g}$.

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$$\rightsquigarrow L(Y_{\mathbf{1},\mathbf{-2}} Y_{\mathbf{1},\mathbf{2}} Y_{\mathbf{2},\mathbf{3}} Y_{\mathbf{2},\mathbf{7}}^2)$$

A simple module $L(Y_{i,k})$ is called a **fundamental module**.

T-systems

The T-**systems** are certain relations in $K(\mathscr{C}_{\mathfrak{g}})$ for the tensor prod. of Kirillov-Reshetikhin (KR) modules $L(\prod_{k=1}^{p} Y_{i,r+2d_i k})$ for general $\mathfrak{g}$:

Ex. (T-systems for untwisted, simply-laced $\mathfrak{g}$)

$$\left[L\Big(\prod_{k=1}^{p-1} Y_{i,r+2k}\Big) \otimes L\Big(\prod_{k=2}^{p} Y_{i,r+2k}\Big)\right]$$

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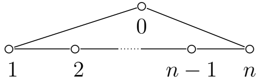
Snake modules in type $A_n^{(1)}$

Assume $\mathfrak{g}$ is of type $A_n^{(1)}$:



Set $J_n := \{(i,k) \mid k \equiv i \pmod 2\} \subseteq [1,n] \times \mathbb{Z}$

$(n = 5)$

$(i \setminus k)$	$\cdots$	0	1	2	3	4	5	6	7	$\cdots$
1			$\circ$		$\circ$		$\circ$		$\circ$	
2		$\circ$		$\circ$		$\circ$		$\circ$		
3			$\circ$		$\circ$		$\circ$		$\circ$	
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Definition

A sequence $\xi = ((i_1,k_1),\dots,(i_p,k_p)) \in J_n^p$ is a **snake**

$\overset{\text{def}}{\Leftrightarrow}$ for $1 \leq {}^\forall r < p$, setting $(i,k) = (i_r,k_r)$ and $(i',k') = (i_{r+1},k_{r+1})$

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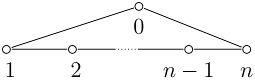

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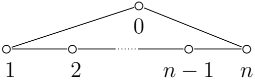

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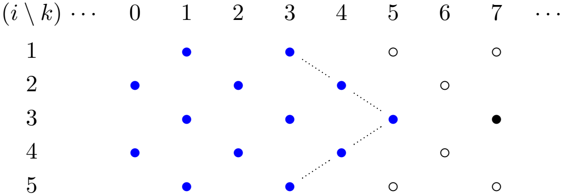

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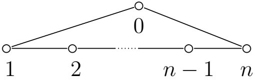

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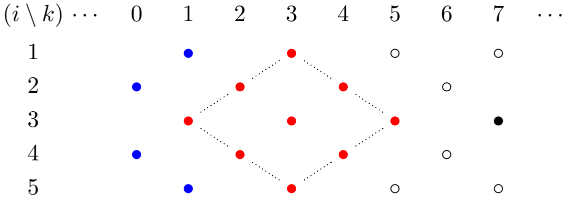

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2	●		●		○		○		○		○		○		
3		○		○		●		○		○		○		○	
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$\boldsymbol{\xi} = \big((i_1,k_1),\dots,(i_p,k_p)\big)$: snake $\Rightarrow L(\boldsymbol{\xi}) = L(\prod_{r=1}^{p} Y_{i_r,k_r})$: **snake mod.**

prime snake $\xi \rightsquigarrow$ two **neighboring snakes** ξ_{H} $(\star)$, ξ_{L} $(\star)$

Theorem (MY12)

- ξ: prime $\Leftrightarrow L(\xi)$: prime (i.e. $L(\xi) \cong M \otimes N \Rightarrow M \cong \mathbf{1}$ or $N \cong \mathbf{1}$)
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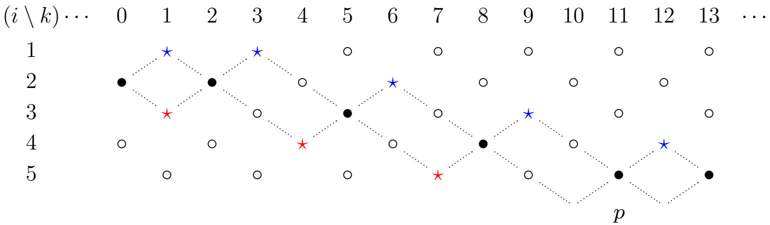

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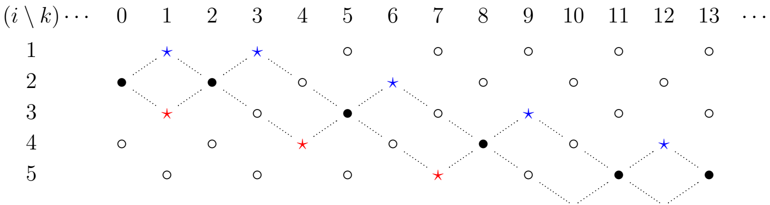

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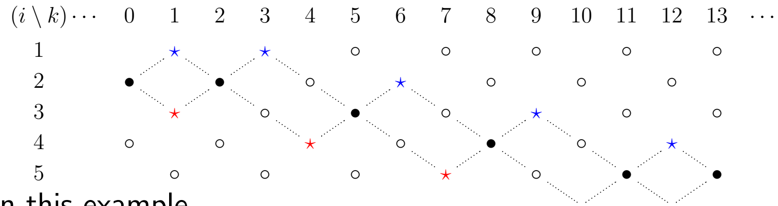

In this example,

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Rem. • straight snake • • • $\cdots$ $\Leftrightarrow$ T-systems

• $B_n^{(1)}$ case is similar but slightly more complicated (omit).

• For the proof, Mukhin-Young used the theory of q-characters.

• It is expected that these are mutations of cluster alg., though it is to be proved.

$$
\begin{array}{c|cccccccccccccccc}
(i \setminus k) \cdots & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & \cdots \\
1 & & \color{blue}{\star} & & \color{blue}{\star} & & \circ & & \circ & & \circ & & \circ & & \circ & \\
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Extended T-systems

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Goal

Our main theorem says that the same rel. also hold if we replace the snake mod. with simple modules of the form $\mathrm{hd}\big(S_{j_1} \otimes \cdots \otimes S_{j_p}\big)$.

($\mathrm{hd}\, M :=$ the maximal semisimple quotient of M)

Here the modules S_j are **affine cuspidal modules**, which are defined from a **strong duality datum** (recalled next).

These notions are introduced by Kashiwara–Kim–Oh–Park (KKOP).

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For simple mod. $M, N \in \mathscr{C}_{\mathfrak{g}}$, an isom. $M \otimes N \xrightarrow{\sim} N \otimes M$ doesn't necessarily exist, but there does exist an isom. **(normalized R-matrix)**

$$R^{\mathrm{norm}}_{M,N} \colon \mathbb{C}(z) \otimes_{\mathbb{C}[z^{\pm 1}]} (M \otimes N[z^{\pm 1}]) \xrightarrow{\sim} \mathbb{C}(z) \otimes_{\mathbb{C}[z^{\pm 1}]} (N[z^{\pm 1}] \otimes M)$$

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- $\mathfrak{d}(M, N) = 0 \iff M \otimes N$: simple and $M \otimes N \cong N \otimes M$.
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strong duality data

Fix a Cartan matrix $C = (c_{ij})_{i,j \in I}$ of finite ADE type (irrelevant to the type of $\mathfrak{g}$)

Definition (KKOP,'23)

A family of real simple mod. $\mathcal{D} = \{L_i\}_{i \in I} \subseteq \mathscr{C}_{\mathfrak{g}}$ is called a **strong duality datum** (associated with C) if

1. $\mathfrak{d}(L_i, \mathscr{D}^k L_i) = \delta_{k,1} + \delta_{k,-1} \quad ({}^\forall i \in I, {}^\forall k \in \mathbb{Z})$,
2. $\mathfrak{d}(L_i, \mathscr{D}^k L_j) = -c_{ij}\delta_{k,0} \quad (i \neq j, {}^\forall k \in \mathbb{Z})$.

Proposition (KKOP,'23)

$\mathcal{D} = \{L_i\}_{i \in I}$: a strong duality datum associated with C

$\Rightarrow$ $^{\exists!}\mathbb{Z}$-alg. hom. $\Phi_{\mathcal{D}} \colon U_q^-(\mathfrak{g}_C)^\vee_{\mathbb{Z}} \to K(\mathscr{C}_{\mathfrak{g}})$ s.t. $\Phi_{\mathcal{D}}(f_i) = [L_i]$, $\Phi_{\mathcal{D}}(q) = 1$.

Moreover, $\Phi_{\mathcal{D}}$ induces an inj. map from the **upper global basis** $\mathbf{B}^{\mathrm{up}} \subseteq U_q^-(\mathfrak{g}_C)^\vee_{\mathbb{Z}}$ to the isom. classes of simple modules in $\mathscr{C}_{\mathfrak{g}}$:

$$\mathbf{B}^{\mathrm{up}} \ni b \mapsto L_b \in (\text{simples in } \mathscr{C}_{\mathfrak{g}}) \qquad (\textbf{not surjective!})$$

Rem. $\Phi_{\mathcal{D}}$ is defined by the composition of the following two hom.:

- $U_q^-(\mathfrak{g}_C)^\vee_{\mathbb{Z}} \xrightarrow{\sim} K(R^C\text{–gmod})$ (R^C: **quiver Hecke algebra**) [Khovanov–Lauda, Rouquier]
- $K(R^C\text{–gmod}) \to K(\mathscr{C}_{\mathfrak{g}})$, which is induced from the **quantum affine Schur–Weyl duality functor** $R^C\text{–gmod} \to \mathscr{C}_{\mathfrak{g}}$.

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affine cuspidal modules

$\mathcal{D}$: strong duality datum assoociated with C

$$\rightsquigarrow \Phi_{\mathcal{D}} \colon \begin{array}{ccc} U_q^-(\mathfrak{g}_C)^\vee_{\mathbb{Z}} & \to & K(\mathscr{C}_{\mathfrak{g}}) \\ \cup & & \cup \\ \mathbf{B}^{\mathrm{up}} & & \{\text{simple mod.}\}/_{\cong} \end{array}$$

Fix a red. word $\boldsymbol{i} = (i_1, \ldots, i_N)$ of longest $w_0 \in W_C$. For $1 \le j \le N$,

$U_q^-(\mathfrak{g}_C)^\vee \ni f_{\beta_j} := T_{i_1} \cdots T_{i_{j-1}}(f_{i_j}) \xrightarrow{\text{normalize}} f_{\beta_j}^\vee \in \mathbf{B}^{\mathrm{up}}$: **dual root vector**

(T_i: Lusztig's braid group action)

Definition (KKOP,'23)

Define the **affine cuspidal modules** $\{S_j = S_j^{\mathcal{D},\boldsymbol{i}} \mid j \in \mathbb{Z}\}$ as follows:

(i) if $1 \le j \le N$, S_j is the image of $f_{\beta_j}^\vee$ under $\Phi_{\mathcal{D}}$, and

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Examples

Assume $\mathfrak{g} = \widehat{\mathfrak{sl}}_{n+1}$ (type $A_n^{(1)}$).

(1) $\mathcal{D}_0 := \{L(Y_{1,-2j+1}) \mid 1 \leq j \leq n\} \subseteq \mathscr{C}_{\widehat{\mathfrak{sl}}_{n+1}}$: SDD of type A_n

$\boldsymbol{i}_0 := (1, \dots, n/1, \dots, n-1/\dots/1, 2/1)$

$\rightsquigarrow$ ..., $S_1 = \Phi_{\mathcal{D}_0}(f_1) = L(Y_{1,-1})$, $S_2 = \Phi_{\mathcal{D}_0}(f^{\vee}_{\alpha_1+\alpha_2}) = L(Y_{2,-2})$, ...,

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$\rightsquigarrow$ $\{S_j^{\mathcal{D}_0,\boldsymbol{i}_0} | j \in \mathbb{Z}\} = \{L(Y_{i,k}) \mid k \equiv i \pmod 2\}$

For later use, define a bij. $\phi\colon J_n := \{(i,k) \mid k \equiv i \pmod 2\} \xrightarrow{\sim} \mathbb{Z}$ by

$$L(Y_{i,k}) = S^{\mathcal{D}_0,\boldsymbol{i}_0}_{\phi(i,k)}.$$

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Main Theorem

Setting $\mathcal{D} \in \mathscr{C}_{\mathfrak{g}}$: a strong duality datum of type A_n.

We consider **only the special** reduced word $\boldsymbol{i}_0 := (1, \dots, n/ \dots /1, 2/1)$

$\rightsquigarrow$ affine cuspidal modules $S_j^{\mathcal{D}, \boldsymbol{i}_0}$ $(j \in \mathbb{Z})$

Set $J_n = \{(i,k) \in [1,n] \times \mathbb{Z} \mid i \equiv k \bmod 2\}$, and define

$$S_{i,k} := S_{\phi(i,k)}^{\mathcal{D}, \boldsymbol{i}_0} \quad \text{for } (i,k) \in J_n.$$

Recall $\phi \colon J_n \xrightarrow{\sim} \mathbb{Z}$ was defined by $L(Y_{i,k}) = S_{\phi(i,k)}^{\mathcal{D}_0, \boldsymbol{i}_0}$ using the special SDD $\mathcal{D}_0 = \{L(Y_{1,-2j+1}) \mid 1 \le j \le n\}$.

Definition (Snake module associated with $\mathcal{D}$)

$\mathbb{S}^{\mathcal{D}}(\boldsymbol{\xi}) := \mathrm{hd}(S_{i_1,k_1} \otimes \cdots \otimes S_{i_p,k_p})$ for a snake $\boldsymbol{\xi} = ((i_1,k_1), \dots, (i_p.k_p)) \in J_n^p$.

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Rem. When $\mathcal{D} = \mathcal{D}_0$,

$$\mathbb{S}^{\mathcal{D}_0}(\boldsymbol{\xi}) = \mathrm{hd}\big(L(Y_{i_1,k_1}) \otimes \cdots \otimes L(Y_{i_p,k_p})\big) \cong L\Big(\prod_{r=1}^{p} Y_{i_r,k_r}\Big) = L(\boldsymbol{\xi})$$ (MY snake mod.)

Theorem (N)

For a prime snake $\xi \in J_n^p$,

$$0 \to \mathbb{S}^{\mathcal{D}}(\xi_{\mathrm{H}}) \otimes \mathbb{S}^{\mathcal{D}}(\xi_{\mathrm{L}}) \to \mathbb{S}^{\mathcal{D}}(\xi_{[1,p-1]}) \otimes \mathbb{S}^{\mathcal{D}}(\xi_{[2,p]}) \to \mathbb{S}^{\mathcal{D}}(\xi) \otimes \mathbb{S}^{\mathcal{D}}(\xi_{[2,p-1]}) \to 0$$

- The first and the third terms are simple.

$\rightsquigarrow [L(\xi_{[1,p-1]}) \otimes L(\xi_{[2,p]})] = [L(\xi) \otimes L(\xi_{[2,p-1]})] + [L(\xi_{\mathrm{H}}) \otimes L(\xi_{\mathrm{L}})]$ when $\mathcal{D} = \mathcal{D}_0$.

Definition (Snake module associated with $\mathcal{D}$)

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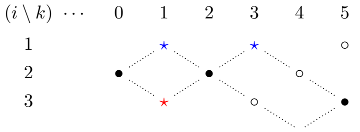
Ex. type A_3, $\boldsymbol{i}_0 = (1,2,3,1,2,1)$



$\rightsquigarrow$

$$0 \to \mathbb{S}^{\mathcal{D}}\big((1,3),(1,1)\big) \otimes \mathbb{S}^{\mathcal{D}}\big((3,1)\big)$$
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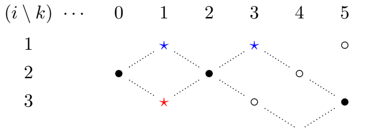

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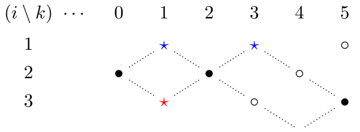

$$\rightsquigarrow \quad \begin{aligned} 0 &\to \mathbb{S}^{\mathcal{D}}\big((1,3),(1,1)\big) \otimes \mathbb{S}^{\mathcal{D}}\big((3,1)\big) \\ &\to \mathbb{S}^{\mathcal{D}}\big((3,5),(2,2)\big) \otimes \mathbb{S}^{\mathcal{D}}\big((2,2),(2,0)\big) \\ &\to \mathbb{S}^{\mathcal{D}}\big((3,5),(2,2),(2,0)\big) \otimes \mathbb{S}^{\mathcal{D}}\big((2,2)\big) \to 0 \end{aligned}$$

(i) $L_i = L(Y_{1,-2i+1}) \rightsquigarrow S_{i,k} = L(Y_{i,k}) \rightsquigarrow \mathbb{S}^{\mathcal{D}}\big((i_1,k_1),\cdots,(i_p,k_p)\big) = L(\prod_{r=1}^{p} Y_{i_r,k_r})$

$$L(Y_{1,3}Y_{1,1}) \otimes L(Y_{3,1}) \to L(Y_{3,5}Y_{2,2}) \otimes Y(Y_{2,2}Y_{2,0}) \to L(Y_{3,5}Y_{2,2}Y_{2,0}) \otimes L(Y_{2,2})$$

(ii) $L_i = L(Y_{1,2i-7}) \rightsquigarrow S_{3,5} = L(Y_{1,7}Y_{1,5}Y_{1,3})$, $S_{2,2} = L(Y_{3,1}Y_{3,-1})$,

$S_{2,0} = L(Y_{3,3}Y_{3,1})$, $S_{1,3} = L(Y_{1,7})$, …, etc.

$$\rightsquigarrow \mathbb{S}^{\mathcal{D}}\big((3,5),(2,2),(2,0)\big) = \mathrm{hd}\big(L(Y_{1,7}Y_{1,5}Y_{1,3}) \otimes L(Y_{3,1}Y_{3,-1}) \otimes L(Y_{3,3}Y_{3,1})\big)$$

$$= L(Y_{1,7}Y_{1,5}Y_{1,3}Y_{2,2}Y_{2,0}), \text{ etc.}$$

$$L(Y_{1,7}Y_{3,3}Y_{3,1}Y_{3,-1}) \otimes L(Y_{1,-3}) \to L(Y_{1,7}Y_{1,5}Y_{1,3}Y_{3,1}Y_{3,-1}) \otimes L(Y_{2,2}Y_{2,0})$$

$$\to L(Y_{1,7}Y_{1,5}Y_{1,3}Y_{2,2}Y_{2,0}) \otimes L(Y_{3,1}Y_{3-1})$$

Remarks on the Main Theorem

Theorem (N)

For a prime snake $\boldsymbol{\xi} \in J_n^p$,

$$0 \to \mathbb{S}^{\mathcal{D}}(\boldsymbol{\xi}_{\mathrm{H}}) \otimes \mathbb{S}^{\mathcal{D}}(\boldsymbol{\xi}_{\mathrm{L}}) \to \mathbb{S}^{\mathcal{D}}(\boldsymbol{\xi}_{[1,p-1]}) \otimes \mathbb{S}^{\mathcal{D}}(\boldsymbol{\xi}_{[2,p]}) \to \mathbb{S}^{\mathcal{D}}(\boldsymbol{\xi}) \otimes \mathbb{S}^{\mathcal{D}}(\boldsymbol{\xi}_{[2,p-1]}) \to 0$$

- The first and the third terms are simple.

Rem.

- In the previous work in [KKOP, '24], Kashiwara–Kim–Oh–Park proved exact seq. coming from the T-systems in general types for simple modules constructed from a strong duality datum. This strongly motivated our work.
- We also generalized MY extended T-systems of type $B_n^{(1)}$.

Key Lemma

$\mathcal{D}, \boldsymbol{i}$: arbitrary $\rightsquigarrow \{S_j = S_j^{\mathcal{D},\boldsymbol{i}} \mid j \in \mathbb{Z}\}$: affine cuspidal modules

Lemma (KKOP, N)

For a sequence $\boldsymbol{k} = (k_1 < k_2 < \cdots < k_p) \in \mathbb{Z}^p$, denote by

$$\mathbb{S}_{\boldsymbol{k}}[s,t] = \mathrm{hd}(S_{k_s} \otimes S_{k_{s+1}} \otimes \cdots \otimes S_{k_t}) \quad (1 \le s \le t \le p).$$

Assume that

(i) $\mathfrak{d}\big(S_{k_s}, \mathbb{S}_{\boldsymbol{k}}[s+1,t]\big) = 1$ for all $1 \le s < t \le p$,

(ii) $\mathfrak{d}\big(\mathbb{S}_{\boldsymbol{k}}[s,t-1], S_{k_t}\big) = 1$ for all $1 \le s < t \le p$.

Then we have

$$0 \to \mathrm{hd}\Big(\bigotimes_{r=1}^{p-1} \mathrm{hd}(S_{k_{r+1}} \otimes S_{k_r})\Big) \to \mathbb{S}_{\boldsymbol{k}}[1,p-1] \otimes \mathbb{S}_{\boldsymbol{k}}[2,p] \to \mathbb{S}_{\boldsymbol{k}}[1,p] \otimes \mathbb{S}_{\boldsymbol{k}}[2,p-1] \to 0.$$

Moreover, the first and the third terms are both simple.

Q. How to calculate the values of $\mathfrak{d}$?

Recall $\Phi_{\mathcal{D}} : U_q^-(\mathfrak{g}_C)^\vee_{\mathbb{Z}} \to K(\mathscr{C}_{\mathfrak{g}})$

$$\begin{array}{ccc} U_q^-(\mathfrak{g}_C)^\vee_{\mathbb{Z}} & \xrightarrow{\Phi_{\mathcal{D}}} & K(\mathscr{C}_{\mathfrak{g}}) \\ \cup & & \cup \\ \mathbf{B}^{\mathrm{up}} & & \{\text{simple mod.}\}/_{\cong} \end{array}$$

Rem. $\mathbf{B}^{\mathrm{up}}$ has a Kashiwara (bi-)crystal structure.

Proposition (KKOP)

For a simple module $S = \mathrm{hd}(S_1^{\otimes a_1} \otimes \cdots \otimes S_N^{\otimes a_N})$,

1. $\mathfrak{d}(\mathscr{D}L_i, S) = \varepsilon_i\big(\Phi_{\mathcal{D}}^{-1}(S)\big)$,
2. $\mathfrak{d}(S, \mathscr{D}^{-1}L_i) = \varepsilon_i^*\big(\Phi_{\mathcal{D}}^{-1}(S)\big)$,

where $\varepsilon_i(b) = \max\{r \in \mathbb{Z}_{\geq 0} \mid \tilde{e}_i^r(b) \neq 0\}$, and ε_i^* is defined similarly.

When S is a snake module, we can calculate $\varepsilon_i\big(\Phi_{\mathcal{D}}^{-1}(S)\big)$ and $\varepsilon_i^*\big(\Phi_{\mathcal{D}}^{-1}(S)\big)$ using the **Reineke's algorithm**, and show that they satisfy the conditions of the previos lemma.

Thank you for your attention

Recall $\Phi_{\mathcal{D}}\colon U_q^-(\mathfrak{g}_C)_{\mathbb{Z}}^{\vee} \to K(\mathscr{C}_{\mathfrak{g}})$, with $\mathbf{B}^{\mathrm{up}} \subset U_q^-(\mathfrak{g}_C)_{\mathbb{Z}}^{\vee}$ and $\{\text{simple mod.}\}/_{\cong} \subset K(\mathscr{C}_{\mathfrak{g}})$.

Rem. $\mathbf{B}^{\mathrm{up}}$ has a Kashiwara (bi-)crystal structure.

Proposition (KKOP)

For a simple module $S = \mathrm{hd}(S_1^{\otimes a_1} \otimes \cdots \otimes S_N^{\otimes a_N})$,

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Thank you for your attention

Recall

$$\begin{array}{ccc} \Phi_{\mathcal{D}} : U_q^-(\mathfrak{g}_C)^\vee_{\mathbb{Z}} & \to & K(\mathscr{C}_\mathfrak{g}) \\ \cup & & \cup \\ \mathbf{B}^{\mathrm{up}} & & \{\text{simple mod.}\}/_{\cong} \end{array}$$

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When S is a snake module, we can calculate $\varepsilon_i\big(\Phi_{\mathcal{D}}^{-1}(S)\big)$ and $\varepsilon_i^*\big(\Phi_{\mathcal{D}}^{-1}(S)\big)$ using the **Reineke's algorithm**, and show that they satisfy the conditions of the previos lemma.

Thank you for your attention