Crossing Probabilities for 2D Lattice Models

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Outline

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Ising Model

Ising Model [Lenz 1920]

A model for ferromagnet, to understand the phase transition.

- $G = (V, E)$ a finite graph
- $\sigma \in \{\ominus, \oplus\}^V$
- $H(\sigma) = -\sum_{x \sim y} \sigma_x \sigma_y$

Ising model is the probability measure of inverse temperature $\beta > 0$:

$$\mu_{\beta,G}[\sigma] \propto \exp(-\beta H(\sigma))$$

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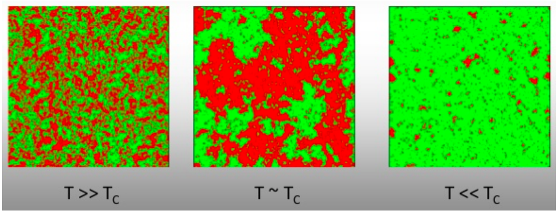

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- $\beta \approx \beta_c$: critical
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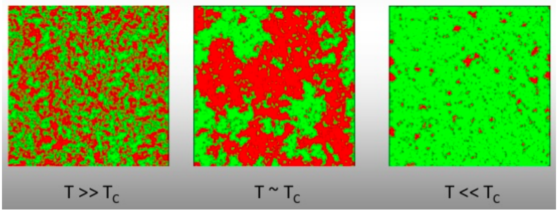

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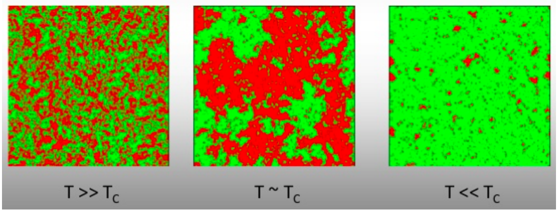

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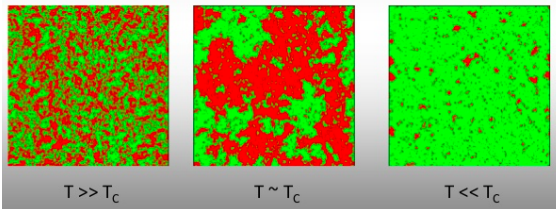

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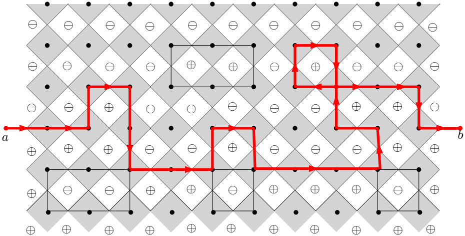
Correlation function

$\mu[\sigma_{z_1} \cdots \sigma_{z_n}] \to \phi(z_1, \ldots, z_n)$.

Schramm Loewner Evolution (SLE)

The law of interfaces is conformally invariant.

Conformal Invariance of Interfaces



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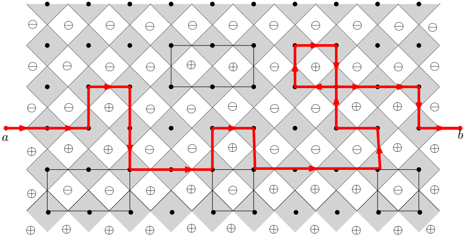

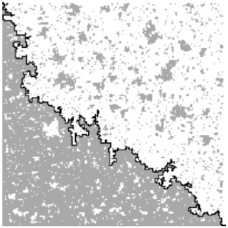

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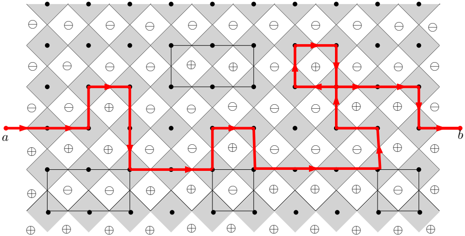
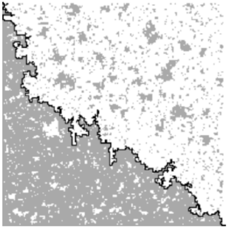
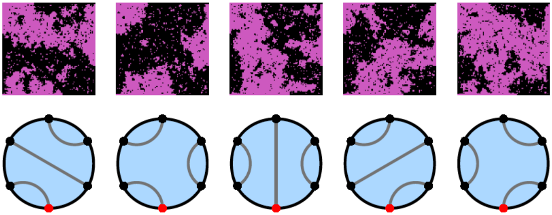




Theorem [Chelkak-Smirnov et al. Invent. '12]

The interface in critical Ising model on $\mathbb{Z}^2$ with Dobrushin boundary conditions converges weakly to SLE_3.

Crossing Probabilities for Ising Model



Theorem [Peltola-W. AAP23+]

The connection of Ising interfaces forms a planar link pattern $\mathcal{A}_\delta$.

$$\lim_{\delta\to 0}\mathbb{P}[\mathcal{A}_\delta=\alpha]=\frac{\mathcal{Z}_\alpha(\Omega;x_1,\ldots,x_{2N})}{\mathcal{Z}_{Ising}(\Omega;x_1,\ldots,x_{2N})},\quad \mathcal{Z}_{Ising}=\sum_{\alpha\in\mathrm{LP}_N}\mathcal{Z}_\alpha,$$

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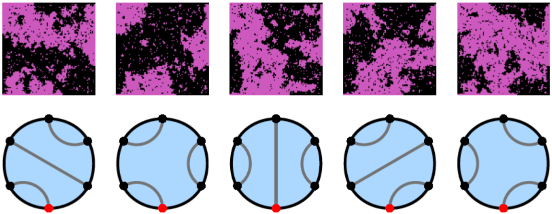

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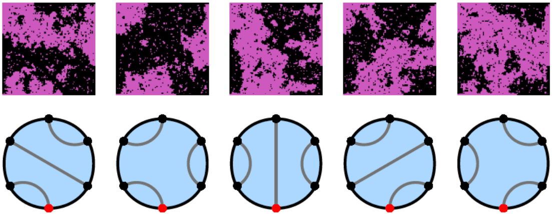

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- Related to correlation functions in CFT.

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$\{\mathcal{Z}_\alpha : \alpha \in \mathrm{LP}\}$ is a collection of smooth functions satisfying PDE, COV, ASY.

PDE : $\left[\frac{\kappa}{2}\partial_i^2 + \sum_{j\neq i}\left(\frac{2}{x_j - x_i}\partial_j - \frac{2h}{(x_j-x_i)^2}\right)\right]\mathcal{Z}(x_1, \ldots, x_{2N}) = 0$, where $h = (6-\kappa)/2\kappa$.

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Questions

Existence and uniqueness ?

Pure partition functions

Uniqueness [Flores-Kleban, CMP'15]

Fix $\kappa \in (0, 8)$. If there exist collections of smooth functions satisfying PDE, COV and ASY, they are (essentially) unique.

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Existence

- $\kappa \in (0, 8) \setminus \mathbb{Q}$ [Kytölä-Peltola, CMP'16]
- $\kappa \in (0, 4]$ [Peltola-W. CMP'19, Beffara-Peltola-W. AOP'21]
- $\kappa \in (0, 6]$ [W. CMP'20]

- Coulumb gas techniques
- Global multiple SLEs
- Hypergeometric SLE

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Theorem [W. CMP'20]

Fix $\kappa \in (0, 6]$. The pure partition functions are the recursive collection $\{\mathcal{Z}_\alpha : \alpha \in \cup_N \mathrm{LP}_N\}$ of smooth functions $\mathcal{Z}_\alpha : \mathfrak{X}_{2N} \to \mathbb{R}$ uniquely determined by the following properties :

PDE, COV, ASY as well as **PLB** :

$$0 < \mathcal{Z}_\alpha(x_1, \ldots, x_{2N}) \le \prod_{\{a,b\} \in \alpha} |x_b - x_a|^{-2h}, \quad \forall (x_1, \ldots, x_{2N}) \in \mathfrak{X}_{2N}.$$

$\{\mathcal{Z}_\alpha : \alpha \in \mathrm{LP}_N\}$ is linearly independent and forms a basis for the solution space.

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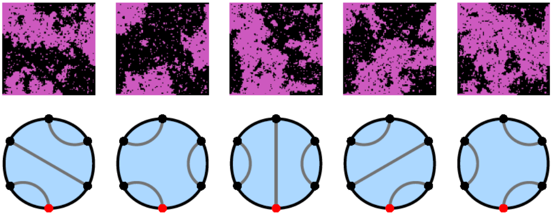

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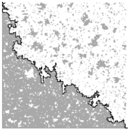
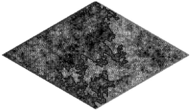
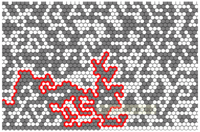
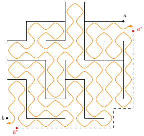

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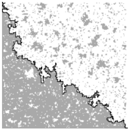
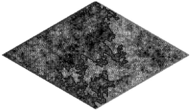
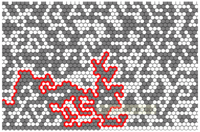
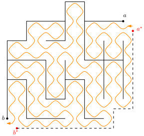

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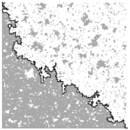

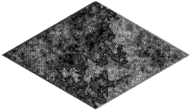

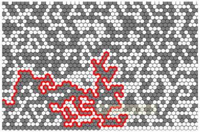

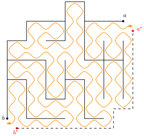

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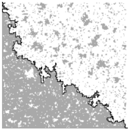
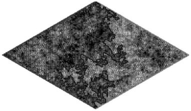
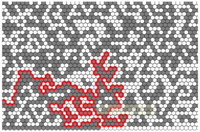
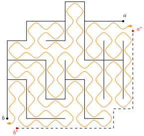

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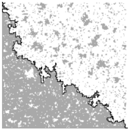

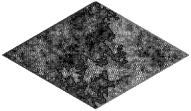

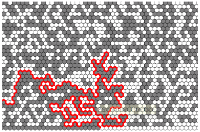

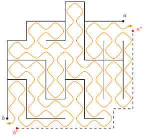

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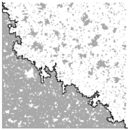
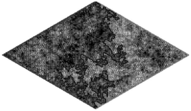
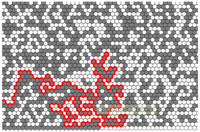
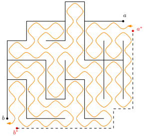
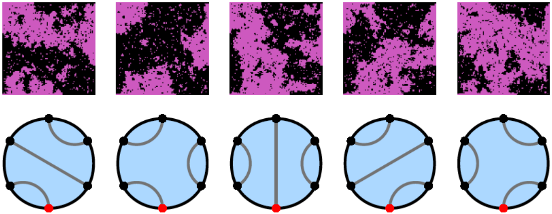
- Percolation : $\kappa = 6$ [Smirnov'01]

Conformal Invariance in 2D Lattice Model

- Loop-erased random walk (LERW) : $\kappa = 2$ [Lawler-Schramm-Werner, AOP'04]
- Ising model : $\kappa = 3$ [Chelkak-Smirnov et al.'12]
- Level lines of GFF : $\kappa = 4$ [Schramm-Sheffield, ACTA'09]
- FK-Ising model : $\kappa = 16/3$ [Chelkak-Smirnov et al.'12]
- Percolation : $\kappa = 6$ [Smirnov'01]
- Uniform spanning tree (UST) : $\kappa = 8$ [Lawler-Schramm-Werner, AOP'04]

Crossing Probabilities



- Multiple LERWs in UST : $\kappa = 2$. [Karrila-Kytölä-Peltola, CMP'19]

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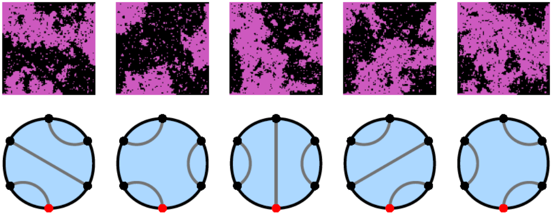

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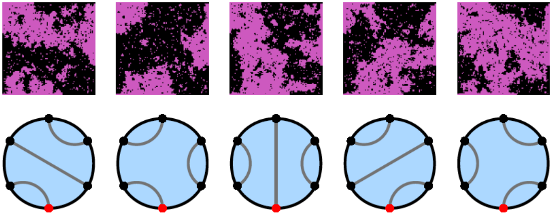

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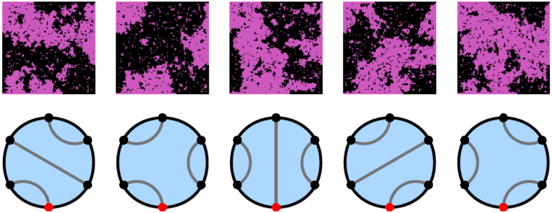

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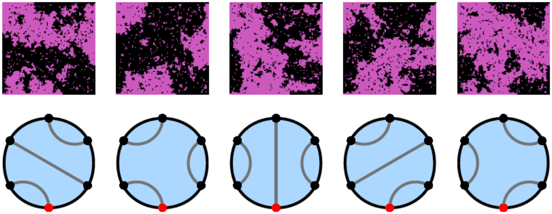
Crossing Probabilities



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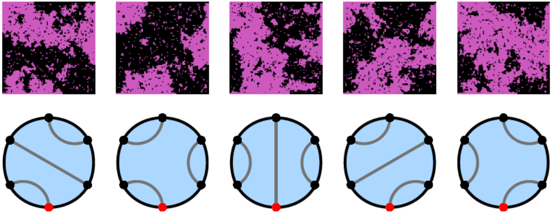

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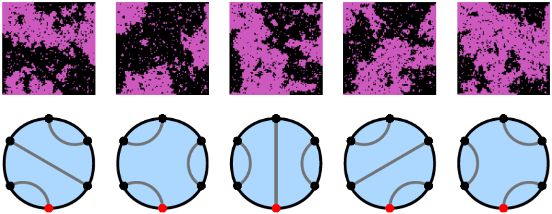
Crossing Probabilities



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1. [Peltola-W. CMP'19] Global and local multiple SLEs for $\kappa \leq 4$ and connection probabilities for level lines of GFF. *Comm. Math. Phys.* 366(2) : 469-536, 2019.
2. [W. CMP'20] Hypergeometric SLE : conformal Markov characterization and applications *Comm. Math. Phys.* 374(2) : 433-484, 2020.
3. [Beffara-Peltola-W. AOP'21] On the uniqueness of global multiple SLEs *Ann. Probab.* 49(1) : 400-434, 2021.
4. [Liu-W. EJP'21] Scaling limits of crossing probabilities in metric graph GFF *Electron. J. Probab.* 26 : article no. 37, 1-46, 2021.
5. [Ding-Wirth-W. AIHP'22] Crossing estimates from metric graph and discrete GFF *Ann. Inst. H. Poincaré Probab. Statist.* 58(3) :1740-1774, 2022.
6. [Liu-W. Bernoulli'23] Loop-erased random walk branch of uniform spanning tree in topological polygons. *Bernoulli.* 29(2) : 1555-1577, 2023.
7. [Peltola-W. AAP'23+] Crossing probabilities of multiple Ising interfaces *Ann. Appl. Probab.* to appear. 2023+
8. [Han-Liu-W.'20] Hypergeometric SLE with $\kappa = 8$: convergence of UST and LERW in topological rectangles. arXiv :2008.00403 (submitted). 2020.
9. [Liu-Peltola-W.'21] Uniform spanning tree in topological polygons, partition functions for SLE(8), and correlations in $c = -2$ logarithm CFT. arXiv :2108.04421 (submitted). 2021.
10. [Feng-Peltola-W.'22] Connection probabilities of multiple FK-Ising interfaces. arXiv :2205.08800 (submitted). 2022.