Cosmological perturbation theory

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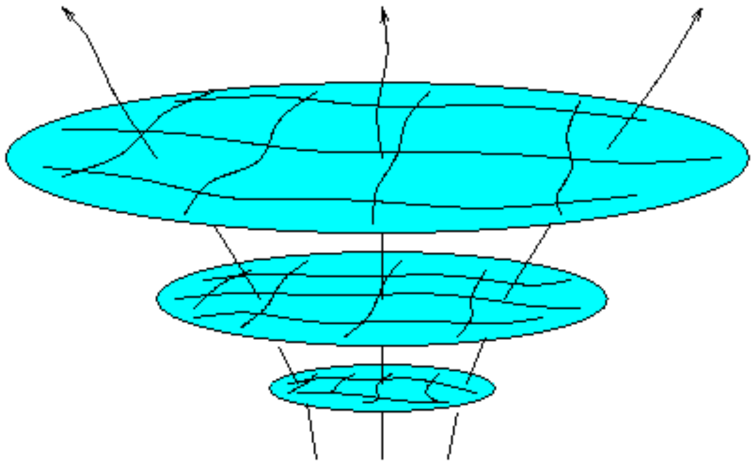
Lecture 2

- Perturbation theory building-blocks
  - Perturbative expansion
  - Scalar/Vector/Tensor decomposition
  - Fourier transform
  - Statistical properties

- Metric perturbations
- Gauge dependence
- Einstein equations

breaking spatial symmetry

inhomogeneous pressure waves in the primordial plasma
or clustering of matter in the late-time universe
break the symmetry of the spatially homogeneous cosmology



full nonlinear numerical solutions have limited dynamic range

perturbative approach valid while inhomogeneities are small
(typically this applies at early times and/or large scales)

Metric perturbations

- Split metric into background and perturbations (10 dof):

$$g_{\mu\nu} = \bar{g}_{\mu\nu} + \delta g_{\mu\nu} \,.$$

- Spatially-flat FLRW background:

$$\bar{g}_{00} = a^2 \,, \quad \bar{g}_{0i} = 0 \,, \quad \bar{g}_{ij} = a^2 \delta_{ij}$$

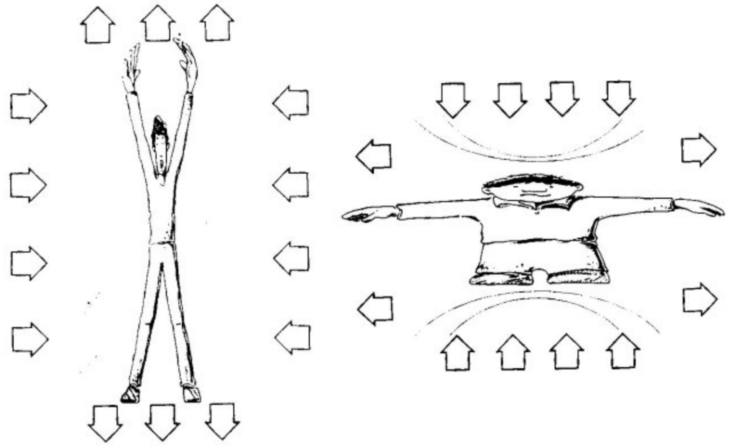
- split perturbations into scalars, vectors and tensors:

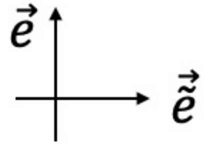
$$\begin{aligned}
\delta g_{00} &= 2a^2 A \\
\delta g_{0i} &= a^2 \left(\nabla_i B - S_i \right) \\
\delta g_{ij} &= a^2 \left(2C\delta_{ij} + 2\nabla_i \nabla_j E + \nabla_i F_j + \nabla_j F_i + h_{ij} \right)
\end{aligned}$$

  - 4 scalars: A, B, C, E — 4 dof
  - 2 vectors: S_i, F_i (2 polarisations each, e_i and $\tilde{e}_i$) — 4 dof
  - 1 tensor: h_{ij} (2 polarisations, $q_{ij}^{(+)}$ and $q_{ij}^{(\times)}$) — 2 dof
- scalar/vector/tensor evolve independently for linear perturbns

Gravitational waves

e.g., $\delta g_{ij} \propto q^{(+)}_{\vec{k}\,ij} = \frac{1}{\sqrt{2}} \left(e_{\vec{k}\,i} e_{\vec{k}\,j} - \tilde{e}_{\vec{k}\,i} \tilde{e}_{\vec{k}\,j} \right)$





- Transverse+tracefree metric perturbations:

$$h_{ij}(t,\vec{x}) = \int \frac{d^3k}{(2\pi)^3} \left\{ h^{(+)}_{\vec{k}}(t) q^{(+)}_{\vec{k}\,ij} + h^{(\times)}_{\vec{k}}(t) q^{(\times)}_{\vec{k}\,ij} \right\} e^{i\vec{k}.\vec{x}}$$

- linearised G_{ij} Einstein equations yield wave equation:

$$\ddot{h}_{\vec{k}} + 3H\dot{h}_{\vec{k}} + \frac{k^2}{a^2} h_{\vec{k}} = 8\pi G \Pi^{(t)}$$

(no constraints $\Rightarrow$ free gravitational waves)

Vector perturbations

Transverse velocity and metric perturbations:

- linearised G_{0i} Einstein momentum constraint:

$$\nabla^2 \left(F_i' + S_i\right) = 16\pi G a^2 \left(\rho + P\right) \left(V_i^{(v)} - S_i\right) .$$

  - relates vector metric perturbations to fluid vorticity

- linearised G_{ij} Einstein evolution equation:

$$F_i'' + S_i' + 6\mathcal{H} \left(F_i' + S_i\right) = 16\pi G a^2 \Pi_i^{(v)} .$$

  - transverse vectors decay in absence of anisotropic stress

$$F_i' + S_i \propto a^{-6}$$

Scalar metric perturbations

- Perturbed line-element including only scalar perturbations:

$$ds^2 = a^2(\tau)\left\{-(1+2A)d\tau^2 + 2(\partial_i B)dx^i d\tau + \left[(1+2C)\delta_{ij} + 2(\partial_{ij}E)\right]\right\} dx^i dx^j$$

  where four scalar perturbations are
  - A = lapse perturbation
  - $\partial_i B = \partial B/\partial x^i$ = shift perturbation
  - C = spatial curvature perturbation
  - $\partial_{ij} E = \partial^2 E/\partial x^i \partial x^j$ = off-diagonal spatial perturbation
- Coupled to scalar density and velocity perturbations via Einstein energy+momentum constraints

Geometrical interpretation

- Temporal gauge (time-slicing) in 4D spacetime defines a hypersurface orthogonal 4-vector field:

$$N_\mu \propto \frac{\partial \tau}{\partial x^\mu}$$

normalise such that $N_\mu N^\mu = -1$.

  - intrinsic curvature of constant τ hypersurfaces:

$${}^{(3)}R = -\frac{4}{a^2}\nabla^2 C$$

  - expansion of constant τ hypersurfaces:

$$\theta = \frac{3}{a}\left(\frac{a'}{a}(1-A) + C' + \frac{1}{3}\nabla^2\sigma\right)$$

  - shear:

$$\sigma_{ij} = \left(\nabla_i\nabla_j - \frac{1}{3}\nabla^2\right)\sigma\,, \qquad \sigma = E' - B$$

  - acceleration:

$$a_i = \nabla_i A$$

Gauge dependence

- Scalar quantity, e.g., density, $\rho|_P$, at given point P is invariant

$$\rho(\tau,\vec{x})|_P = \tilde{\rho}(\tilde{\tau},\tilde{\vec{x}})|_P$$

under first-order change of coordinates:

$$\begin{aligned} \tilde{\tau} &= \tau + \delta\tau(\tau,\vec{x}) \\ \tilde{\vec{x}} &= \vec{x} + \delta\vec{x}(\tau,\vec{x}) \end{aligned}$$

- but background-perturbation split is gauge-dependent

$$\begin{aligned} \rho_0(\tau) + \delta\rho|_P &= \rho_0(\tilde{\tau}) + \widetilde{\delta\rho}|_P \\ \Rightarrow \widetilde{\delta\rho}|_P &= \delta\rho|_P + \rho_0(\tau) - \rho_0(\tilde{\tau}) \\ &= \delta\rho|_P - \rho_0'\delta\tau \end{aligned}$$

Gauge dependence

- Scalar quantity, e.g., density, $\rho|_P$, at given point P is invariant

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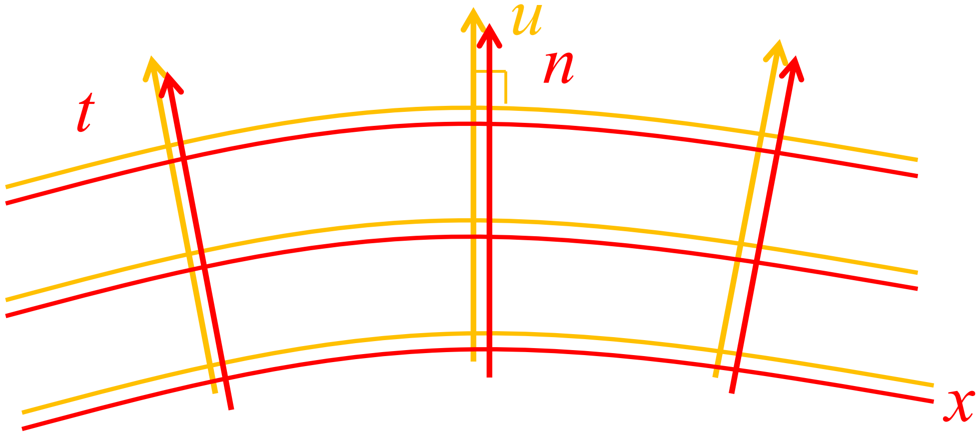
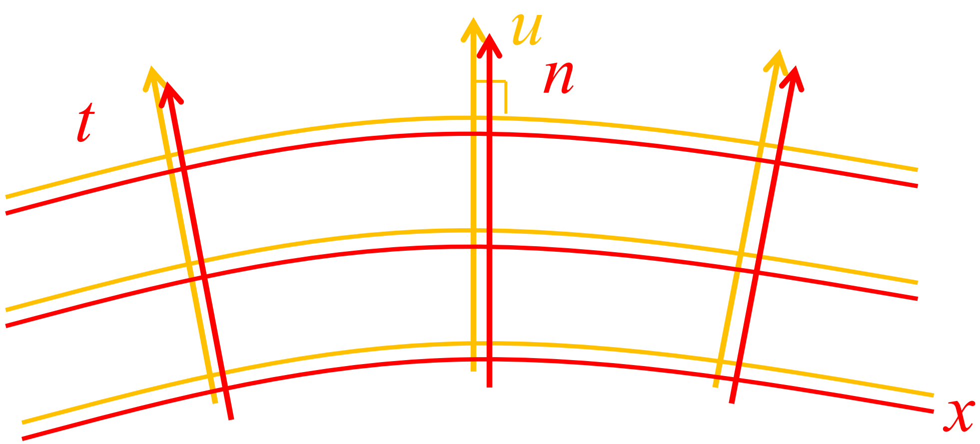
- *it's a feature, not a bug!* GR is a covariant theory



FLRW cosmology

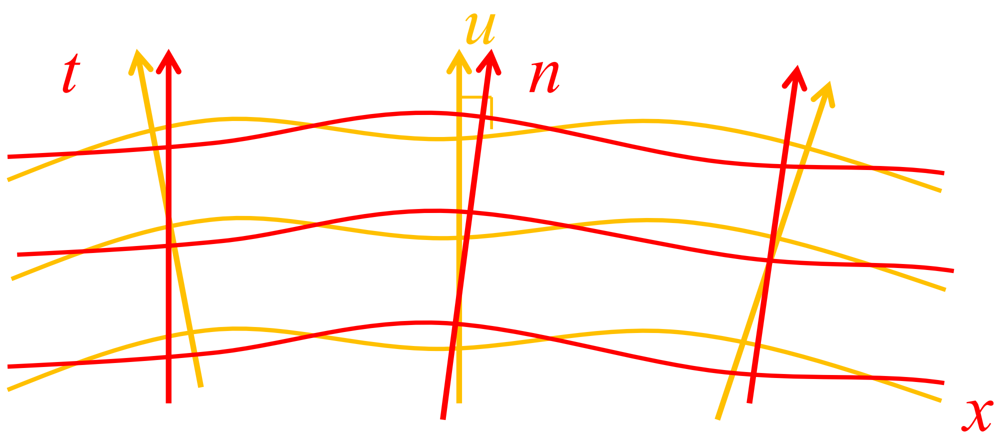
preferred coordinates for homogeneous and isotropic space

preferred space+time split in FRW cosmology
breaks symmetry of Einstein's theory



FLRW cosmology

no unique choice of time (slicing) and space coordinates (threading) in an inhomogeneous universe



FLRW cosmology
+ perturbations

arbitrary gauge *(t,x)*

gauge problem: find different perturbations in different gauges

Linear gauge transformation rules

- Scalar coordinate change:

$$\begin{aligned} \text{time-slicing:} \quad \tilde{\tau} &\to \tau + \delta\tau(\tau, \vec{x}) \\ \text{spatial-threading:} \quad \tilde{\vec{x}} &\to \vec{x} + \vec{\nabla}\delta x(\tau, \vec{x}) \end{aligned}$$

- Scalar gauge transformations:

$$\begin{aligned} \text{density:} \quad \widetilde{\delta\rho} &= \delta\rho - \rho'\delta\tau \\ \text{pressure:} \quad \widetilde{\delta P} &= \delta P - P'\delta\tau \\ \text{velocity:} \quad \tilde{v}^i &= v^i + \nabla^i \delta x' \end{aligned}$$

geometric perturbations dependent on time-slicing:

$$\begin{aligned} \text{lapse:} \quad \tilde{A} &= A - \frac{a'}{a}\delta\tau - \delta\tau' \\ \text{curvature:} \quad \tilde{C} &= C - \frac{a'}{a}\delta\tau \\ \text{shear:} \quad \tilde{\sigma} = \tilde{E}' - \tilde{B} &= \sigma - \delta\tau \end{aligned}$$

plus spatial metric dependent on threading $\tilde{E} = E - \delta x$

Linear gauge transformation rules

- Vector coordinate change:

$$\text{time-slicing:} \quad \tilde{\tau} \rightarrow \tau$$
$$\text{spatial-threading:} \quad \tilde{\vec{x}} \rightarrow \vec{x} + \delta\vec{x}^{(v)}(\tau, \vec{x})$$

- Vector gauge transformations:

$$\text{density:} \quad \widetilde{\delta\rho} = \delta\rho$$
$$\text{pressure:} \quad \widetilde{\delta P} = \delta P$$
$$\text{velocity:} \quad \tilde{v}^i = v^i + \delta\vec{x}^{(v)i\prime}$$

vector metric perturbations dependent on spatial-threading:

$$\tilde{S}_i = S_i + \delta\vec{x}_i^{(v)\prime}$$
$$\tilde{F}_i = F_i - \delta\vec{x}_i^{(v)} \qquad (2)$$

Linear gauge transformation rules

- There is *no tensor coordinate change*:

$$\begin{aligned} \text{time-slicing:} \quad \tilde{\tau} &\rightarrow \tau \\ \text{spatial-threading:} \quad \tilde{\vec{x}} &\rightarrow \vec{x} \end{aligned}$$

- No tensor gauge transformations:

$$\begin{aligned} \text{density:} \quad \widetilde{\delta\rho} &= \delta\rho \\ \text{pressure:} \quad \widetilde{\delta P} &= \delta P \\ \text{velocity:} \quad \tilde{v}^i &= v^i \end{aligned}$$

tensor perturbations are automatically gauge-independent (at first order):

$$\tilde{h}_{ij} = h_{ij}$$

Gauge-invariant variables

- gauge-independent variables
  - some quantities are automatically independent of gauge
    - tensor perturbations at first order
    - quantities that are constant in background spacetime, e.g., non-adiabatic pressure perturbation at first order
- gauge-fixed definitions of gauge-dependent quantities
  - pick a gauge to completely fix the coordinates (i.e., eliminate gauge freedom)

Longitudinal gauge/Conformal Newtonian

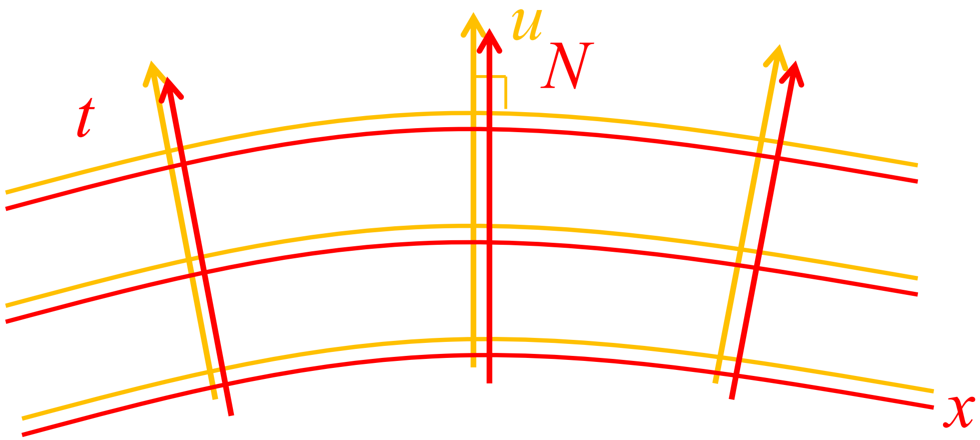
- pick a gauge to completely fix the coordinates
- for example: *longitudinal gauge (zero-shear time-slices)*:
  - set $\sigma \to \tilde{\sigma} = 0$
    which requires a transform $\delta\tau = \sigma$
  - we then have

$$\text{density:} \quad \delta \equiv \frac{\delta\rho}{\rho} \quad \to \quad \widetilde{\delta} = \delta - \frac{\rho'}{\rho}\sigma$$

including two gauge-invariant metric perturbations:

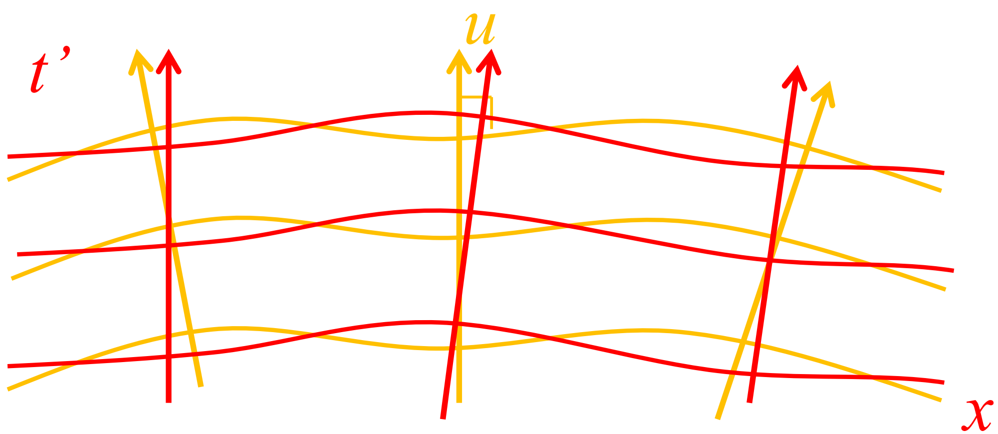
$$\text{lapse:} \quad A \quad \to \quad \Psi \equiv A - \frac{a'}{a}\sigma - \sigma'$$

$$\text{curvature:} \quad C \quad \to \quad \Phi \equiv C - \frac{a'}{a}\sigma$$



FLRW cosmology

longitudinal = conformal Newtonian = Poisson gauge
*hypersurface-orthogonal 4-vector field **N** is shear-free*



FLRW cosmology
+ perturbations

longitudinal gauge coordinates *(t',x)*

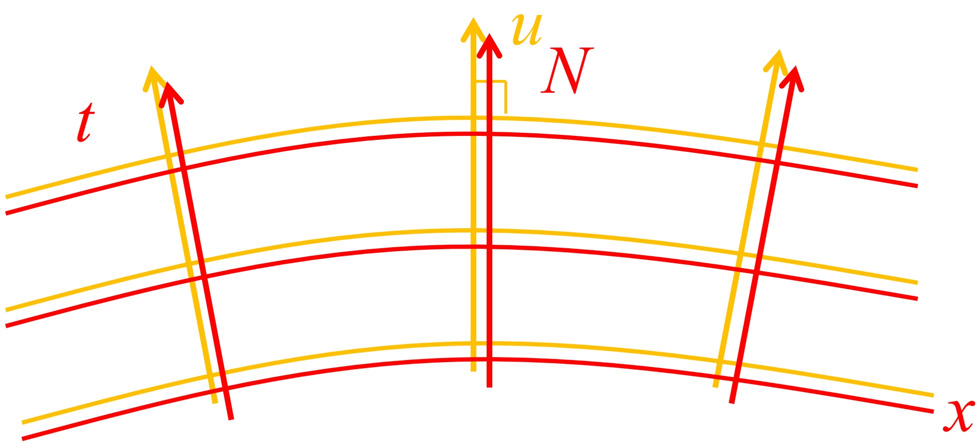
Comoving-orthogonal gauge

- pick a gauge to completely fix the coordinates
- for example: *comoving-orthogonal gauge*:
  - choose
    - comoving spatial-threading, velocity $v \to \tilde{v} = 0$
    - orthogonal time-slices, shift $B \to \tilde{B} = 0$
  - which requires $\delta\tau = -(v + B)$ and $\delta x' = -v$
  - gauge-invariant/comoving density perturbation:

$$\delta\rho \to \tilde{\delta\rho} = \delta\rho + \rho'(v + B)$$

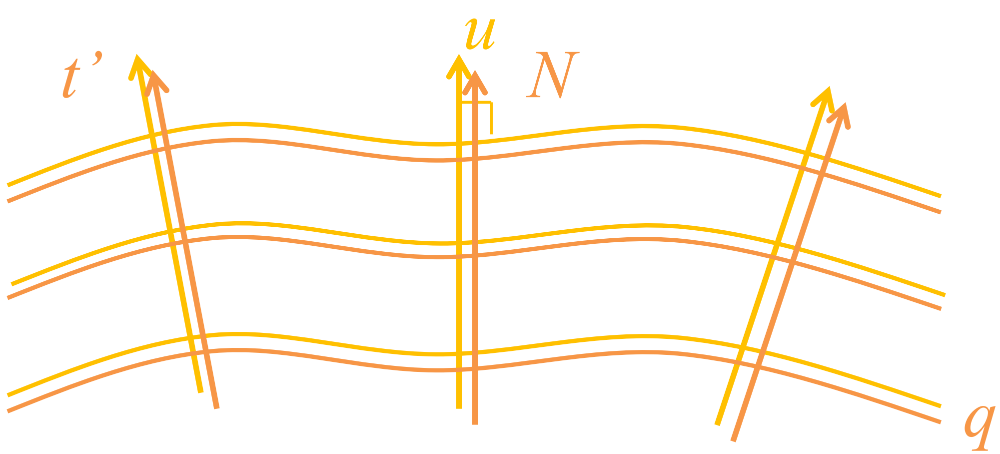
  - gauge-invariant/comoving curvature perturbation:

$$C \to \mathcal{R} \equiv C + \frac{a'}{a}(v + B)$$



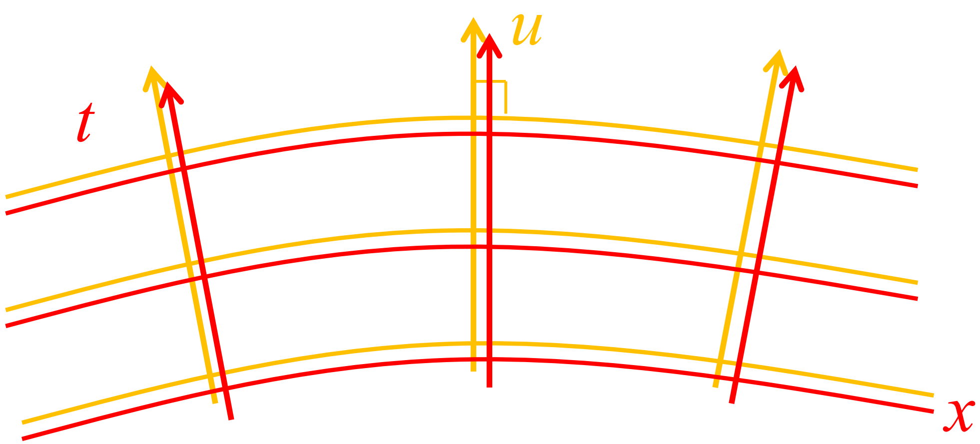
FLRW cosmology

synchronous+comoving with pressureless cold dark matter
time-slicing orthogonal to comoving worldlines



FLRW cosmology
+ perturbations

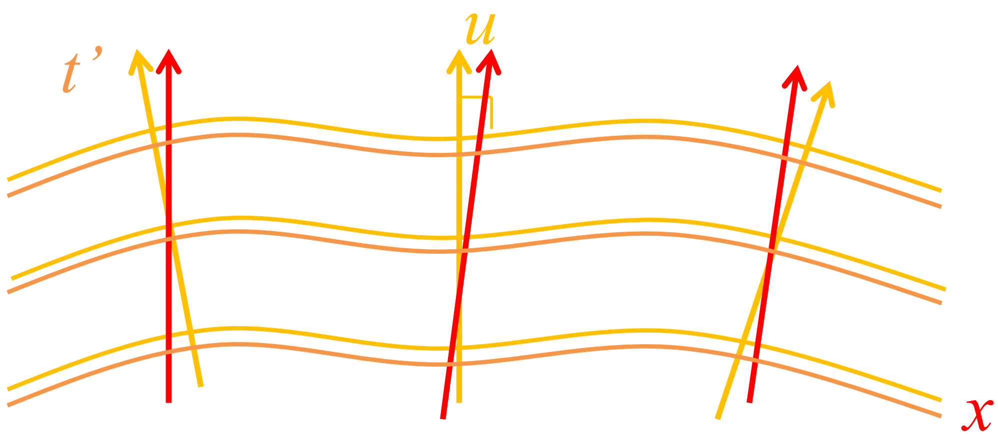
comoving-Lagrangian coordinates ***(t,q)***



FLRW cosmology

time-slicing orthogonal to comoving worldlines

spatial threading is same as longitudinal (Eulerian, not Lagrangian)



FLRW cosmology
+ perturbations

total-matter coordinates *(t',x)*

Uniform-density gauge

- pick a gauge to completely fix the coordinates
- for example: *uniform-density time-slices*:
  - set $\delta\rho \to \widetilde{\delta\rho} = 0$
    which requires a transform $\delta\tau = \delta\rho/\rho'$
  - we then have

$$\begin{array}{rcl} \text{density:} \quad \delta\rho & \to & \widetilde{\delta\rho} = 0 \\ \text{pressure:} \quad \delta P & \to & \widetilde{\delta P} = \delta P - c_s^2\delta\rho \equiv \delta P_{\rm nad} \end{array}$$

    where $c_s^2 = P'/\rho'$ = adiabatic sound speed.
  - gauge-invariant/uniform-density curvature perturbation:

$$C \to \zeta \equiv C - \frac{a'}{a}\frac{\delta\rho}{\rho'}$$

Uniform-α-density gauge

- more generally, for any fluid with density $\rho_\alpha(\tau, \vec{x})$, we can identify the curvature perturbation on uniform-α-density time-slices:

$$\zeta_\alpha \equiv C - \frac{a'}{a}\frac{\delta\rho_\alpha}{\rho'_\alpha} = C + \frac{1}{3(1+w_\alpha)}\delta_\alpha$$

where $\rho'_\alpha = -3(1+w_\alpha)(a'/a)\rho_\alpha$ and $\delta_\alpha \equiv \delta\rho_\alpha/\rho_\alpha$.

Spatially-flat gauge

- pick a gauge to completely fix the coordinates
- for example: *zero-curvature time-slices*:
  - set $C \to \widetilde{C} = 0$
    which requires a transform $\delta\tau = C/\mathcal{H}$
  - we then have

$$\delta\rho_\alpha \quad \to \quad \widetilde{\delta\rho}_\alpha = \delta\rho_\alpha - \frac{\rho'_\alpha}{\mathcal{H}} C$$

  - spatially-flat gauge commonly used for gauge-invariant scalar field fluctuations (Sasaki-Mukhanov variable) during inflation

$$\delta\varphi \quad \to \quad Q \equiv \delta\varphi - \frac{\varphi'}{\mathcal{H}} C$$

Relating gauge-invariant variables

- Gauge-invariant variables are *not unique*, and they are *not independent*
  - the curvature in the uniform-density gauge can be written in terms of the longitudinal gauge metric potential and density perturbation:

$$\zeta_\alpha \equiv \Phi + \frac{1}{3(1+w_\alpha)}\delta_\alpha$$

for example, for radiation and non-relativistic matter:

$$\zeta_\gamma \equiv \Phi + \frac{1}{4}\delta_\gamma \,, \qquad \zeta_m \equiv \Phi + \frac{1}{3}\delta_m$$