Day 2 Geometry of the global nilpotent cone

2.1 Donagi-Pantev's programme

Geometric Langlands X R.S. $g \geq 2$

$\mathbb{C}^\times$ Rk ~~one~~ n local system on X

$GL(n,\mathbb{C})$ (L, ∇) ~~extends~~ yields a D-module uniquely

to $Jac^0(X)$

$N_X(n,0) = \check{N}_X \; (GL(n,\mathbb{C}))$ ✓

$$\begin{tikzcd}
\pi_1(X) \arrow[rr, "\rho"] \arrow[dr] & & \mathbb{C}^\times \\
& \pi_1(X)_{ab} = \pi_1(Jac^0(X)) \arrow[ur, "\rho_{Jac}"] &
\end{tikzcd}$$

$\pi_1(X)_{ab} = H_1(X, \mathbb{Z})$

So ρ_{Jac} corresponds to $(\mathbb{L}, \nabla) \to Jac$

Recall from Day 1 that $\exists$

$$r : M_X(n,0) \dashrightarrow N_X(n,0)$$

Through the Hitchin system we may dominate $N_X(n,d)$ by $\mathrm{Jac}^0(X_b)$ (see § 2.2)

$$r_b : \mathrm{Jac}^0(X_b) \dashrightarrow N_X(n,d)$$

$$\begin{array}{ll}
(\mathcal{L}, \nabla) \longrightarrow X \quad \text{of rk } n & \\
\updownarrow \text{NAHC} & \\
(E,\varphi) \in h^{-1}(b) \rightsquigarrow \mathcal{L} \in \mathrm{Jac}^0(X_b) & \\
\qquad\qquad\qquad \downarrow \mathrm{GLC}_{\mathbb{C}^\times} \text{ (+ mirro... symmetr...)} & \\
\qquad\qquad\quad \check{\mathcal{L}} \longrightarrow \mathrm{Jac}^0(X_b) &
\end{array}$$

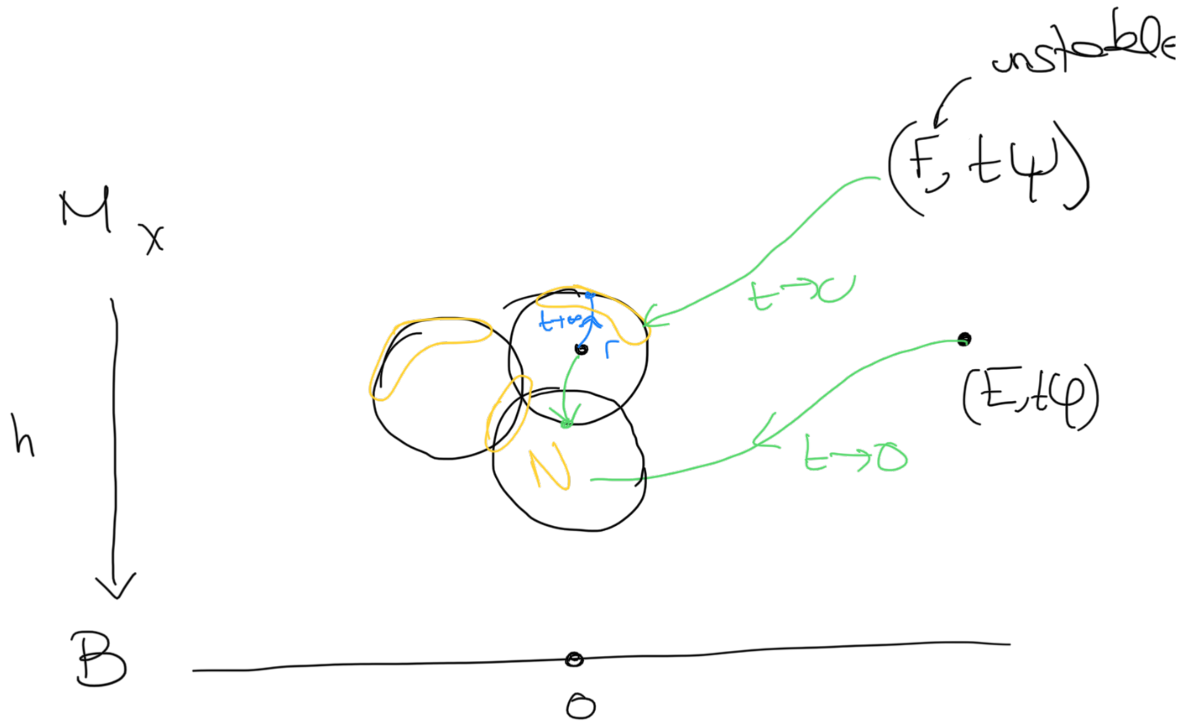
- Identify $\mathrm{Jac}^0(X_b)$ w/ spectral variety for $N_X(n,d)$
- Resolve $r_b \rightsquigarrow r_{b*}\check{\mathcal{L}}$ Higgs bund... w/ extra (parabolic) structure
- Mochizuki D-module on $N_X(n,d$...

Step 0 : understand r & r_b & in particular, their locus of indefinition

2.2. The nilpotent cone in all this

Definition : the nilpotent cone is $h^{-1}(0)$.

Rk : (E,φ) is nilpotent $\Leftrightarrow$ $h(E,\varphi)=0$



$\to h$ is $\mathbb{C}^\times$ equivariant for the action $t\cdot(E,\varphi)=(E,t\varphi)$

$t \cdot (b_1, \dots, b_n) = (t b_1, \dots, t^n b_n)$

$\to h$ is proper

$$\lim_{t \to 0} h(E, t\varphi) \in h(0)$$

$h^{-1}(0)$ not smooth

Lagrangian
of pure dimension $\frac{1}{2}$

<u>Rk</u> $N_X(n,d) \subset h^{-1}(0)^{\mathbb{C}^\times}$

$$E \mapsto (E, 0)$$

$h^{-1}(0)$ complete

<u>Def</u> (Drinfeld) a vector bundle E is
very stable if $\nexists\ \varphi \in H^0(\mathrm{End}\, E \otimes K)$
nilpotent $\neq 0$.

<u>Lauman</u>: very stable bundles
are stable and generic inside N_X^s

<u>"Proof"</u>

φ nilpotent $\rightsquigarrow$ iterated kernel filtration

$$0 = E_{-1} \subsetneq E_0 \subsetneq E_1 \subsetneq \cdots \subsetneq E_\ell = E$$

$$F_i = E_i / E_{i-1} \qquad d_i = \deg F_i \qquad n_i = \deg F_i$$

Def the type of (E, φ) is the 2n uple $(\bar{n}, \bar{d})$

Thus: $\mathcal{C}_{\bar{n}, \bar{d}} = \{ (E, \varphi) \in h^{-1}(0) \mid \text{type } (E, \varphi) = (\bar{d}, \bar{n}) \}$

is of pure dimension $\frac{1}{2} \dim M_X$, locally closed

$$\mathcal{C}_{n,0} \overset{\text{v.s}}{=} N_X$$

LL

More generally $\overline{\mathcal{C}_{\bar{n}, \bar{d}}} = \bigcup$ irreducible components

(follows from flatness of h Ginzburg)

Work by Ginzburg, Laumon, Bradlow Heinloth, García-Prada, Gothen, Bozec

$\rightsquigarrow \overline{\mathcal{C}_{\bar{n}, \bar{d}}}$ are the components

v.s.

Def: $W = N_X \setminus N_X^*$

Drinfeld announced that it is of pure codimension one.

Conjecture (Donagi–Pantev) resolve r/r_b by a finite number of blowups along $Un/Un_b = \left\{ (E,\varphi) \in M/h^{-1}(b) \,\middle|\, E \text{ is not ss} \right\}$

Then the image of the exceptional divisor is the wobbly locus.

Rk: a priori no need for the nilpotent con to define wobbly bundles but they are intimately related. Indeed

$$W^s = \bigcap_{(\bar{n},\bar{d}) \neq (n,d)} \overline{C_{\bar{n},\bar{d}}} \cap \overline{C_{n,d}}$$

Counter example: the bundle

$$E = \mathcal{O} \oplus \mathcal{O} \in N_X(2,0)$$

is ss $\Rightarrow$ NOT very stable. But

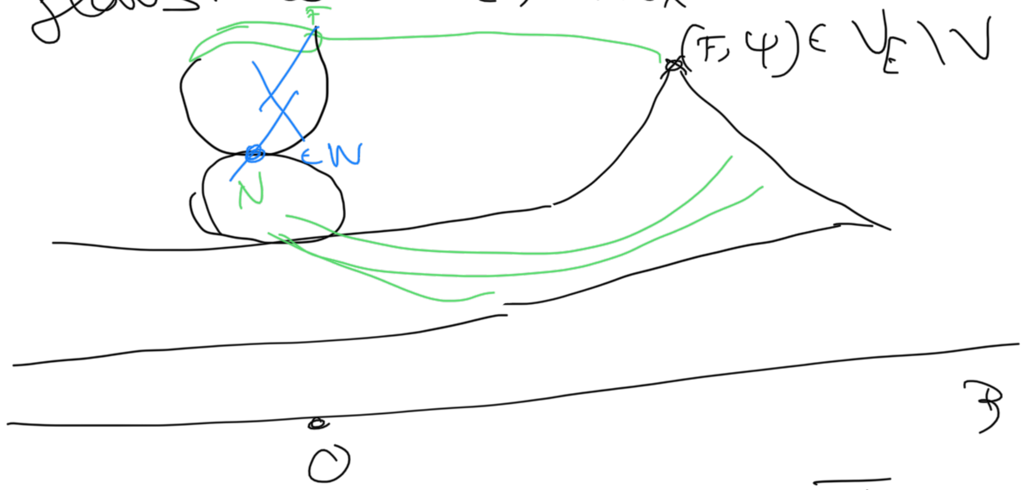
$\forall\ \varphi \in H^0(\mathrm{End}E \otimes K) \cap h^{-1}(0)$

$$(E,\varphi) \sim_s (\mathcal{O} \oplus \mathcal{O},\ 0)$$

Thm (Pauly – P.N., Zelaci) $E \in \mathcal{N}_X^s$ is very stable $\Leftrightarrow h_E : H^0(\mathrm{End}\, E \otimes K) \longrightarrow B$ is proper.

$\Leftarrow$) Known. Proper between affines of the same dimension $\Rightarrow$ finite. But E wobbly & stable $\Rightarrow (E, \mathbb{C}\varphi) \subset h_E^{-1}(0)$

$\Rightarrow$) One proves that if $H^0(\mathrm{End}\, E \otimes K)$ not proper then $\overline{V_E} \setminus V_E \subset Un \Rightarrow$ it flows down to $h^{-1}(0) \setminus \mathcal{N}_X$.



In fact, one proves that $V_E \neq \overline{V_E}$ $\Leftrightarrow V_E \cap h^{-1}(0) \neq \overline{V_E} \cap h^{-1}(0)$.

Corollary (BNR) $h^{-1}(b) \dashrightarrow \mathcal{N}_X$ is dominant.

Proof

$\forall E \in N_X^{v.s.}$ $\quad V_E \cap h^{-1}(b)\ \forall b$

(h_E finite $\Rightarrow$ surjective)

<u>Rk</u> BNR prove it otherwise for generic fibers

$$\begin{array}{ccc} T^*N & \xrightarrow{\lambda} & N_X \times B \\ (E,\varphi) & & (E\ ,\ h(E,\varphi)) \end{array}$$

$$\lambda^{-1}(E\ ,0) = (E,0) \qquad \forall E \in N_X^{v.s.}$$

$\Rightarrow$ $h^{-1}(E,\varphi)$ gnically finite (dimension thm)

+ diff is an iso

(rk is maximal for v.s.)

2.3 Proof of Donagi-Pantev's conjecture

Let $(n,d)=1$

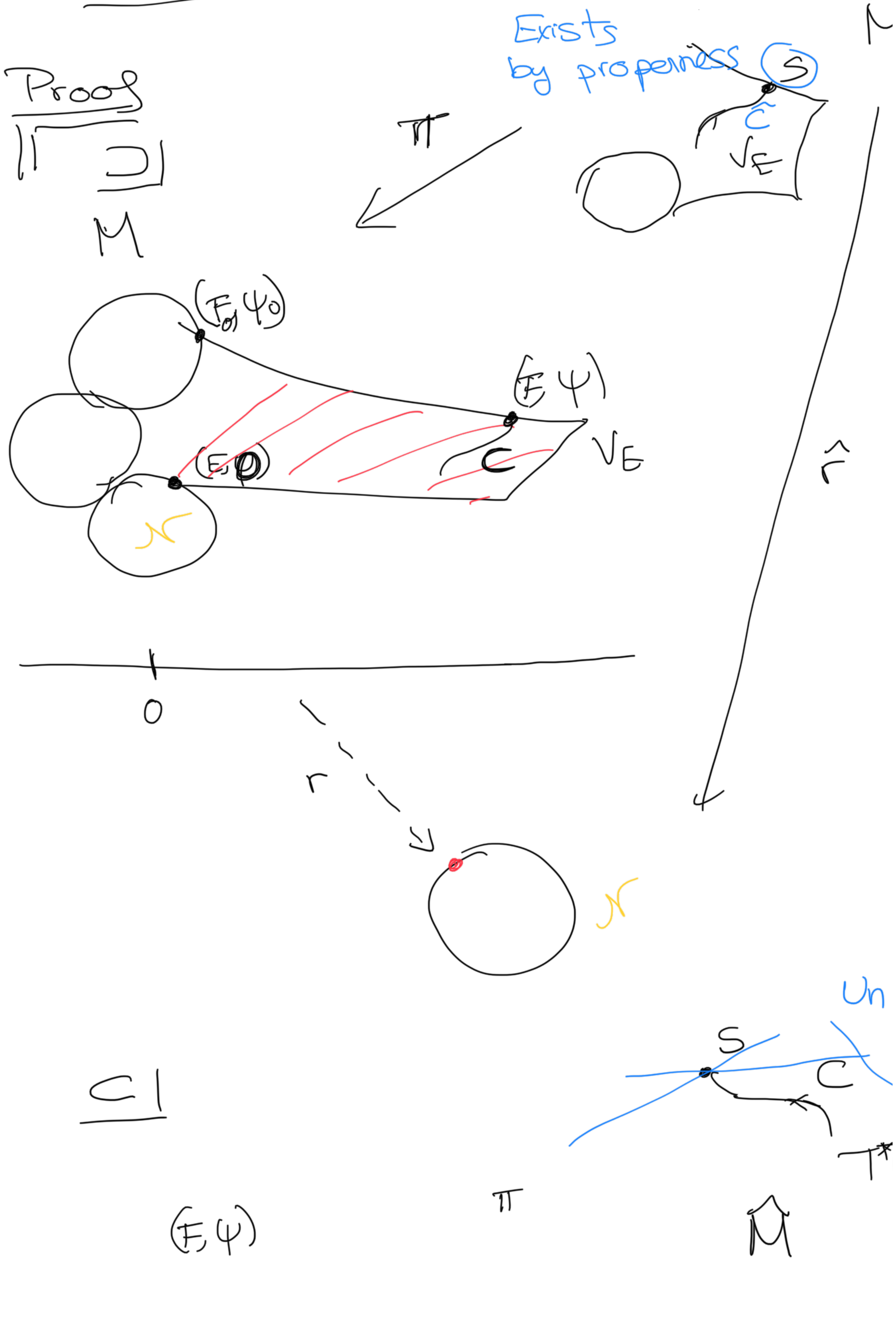
$$\begin{tikzcd} U_n \subset M_X(n,d) \arrow[rr, dashed, "r"] & & N_X(n,d) \\ \cdots \arrow[u] \arrow[urr, "\tilde{r}"'] & & \end{tikzcd}$$

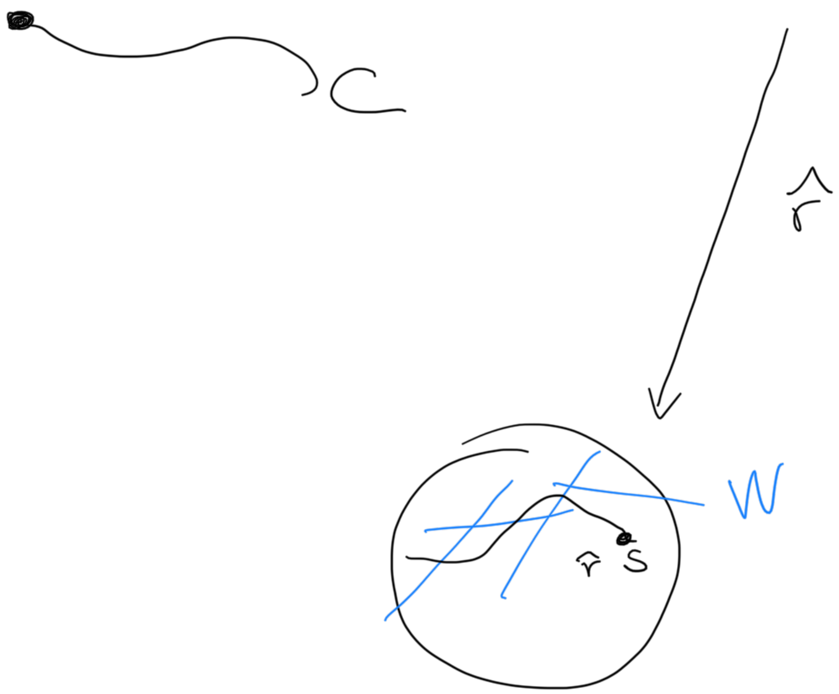
$Bl_{U_n}(M) = M_X(n,d)$

≥ 1

<u>Theorem</u> (PN'20) $\hat{r}(\hat{U}_n) = W$.

<u>Proof</u>





$\hat{r}(S)$ v.s. $\Longrightarrow$ $\hat{r}(C)^{v.s.} \subset C$ dense

Flow $\pi(C)$ to the nilpotent cone to find that $s \in N \cap C_{\bar{n},\bar{d}}$ $(\bar{n},\bar{d}) \neq (n,d)$

$\blacksquare$