An Introduction to Percolation Theory -I

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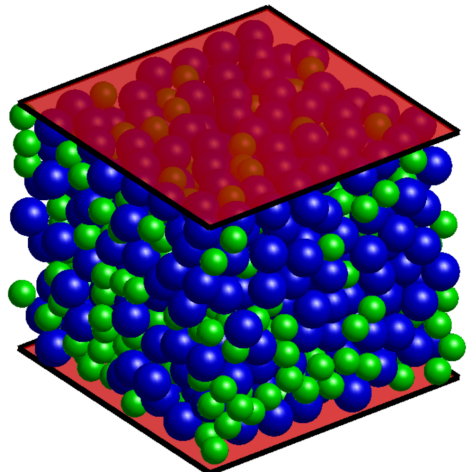
The basic phenomenology of percolation theory is intuitively obvious. Consider a box in which we pour in a mixture of equal-sized spherical balls made of wood and aluminum. After pouring, we attach a conducting sheet at the top and bottom, and see if the mixture allows an electric current to flow from one sheet to another.

It is clear that if the fraction of metal spheres is low, it will not, and will if the concentration is high. Also, the conductivity can only increase if a wooden ball is changed to metal anywhere in the sample. Thus there is a phase transition from an insulating phase to a conducting phase, as the fraction of metal balls is increased.



A randomly poured assembly of two types of spheres. Picture taken from the website physics.emory.edu.

This is called a geometric phase transition. To study this, we only need to have notions of probability and geometry. More advanced concepts like energy or temperature is not required.

Actually, this phase transition shares many common features with other phase transitions, and can act as a stepping stone for these studies.

"Quite apart from the fact that percolation theory had its origin in an honest applied problem (see Hammersley and Welsh (1980)), it is a source of fascinating problems of the best kind a mathematician can wish for: problems which are easy to state with a minimum of preparation, but whose solutions are (apparently) difficult and require new methods. At the same time many of the problems are of interest to or proposed by statistical physicists and not dreamt up merely to demonstrate ingenuity."

Taken from the preface of Kesten's book 'Percolation theory for Mathematicians' (1982).

But there is more to percolation than study of phase transitions. The list of applications is large:

1. Gas masks
2. Disease propagation in apple orchards
3. Properties of alloys, steel...
4. Oil recovery from porous oil-bearing rock
5. Modelling of epidemic spread with moving agents
6. Connectivity in social networks, mobile phone service networks,

...

and many others...

Basic models

We consider a lattice, say hypercubical lattice in d dimensions).

At each site $\mathbf{x}$, there is a random variable $\eta(\mathbf{x})$ taking values 1 or 0, with probabilities p_s and $(1 - p_s)$ respectively, independently of other sites.

A set of occupied sites, connected by nearest neighbor bonds is called a connected cluster.

The questions we study are related to the statistical properties of connected clusters.

The model we defined is usually called Benoulli site percolation model. In this case, all 'nearest neighbor bonds' are assumed present, but sites may be occupied or unoccupied.

One also defines a bond- percolation model, in which all sites are present, but the connecting bonds are prent with probability p_b, or bond-site percolation models, with both parameters p_s and p_b needed to specify the model.

Random resistor networks: One considers a resistor at each bond, and the resistance of the bonds are i.i.d. random variables, from a specified distribution $g(\rho)$.

The typical question of interst is the bulk conductance σ of a large sample, and how it changes of the distribution is changed.

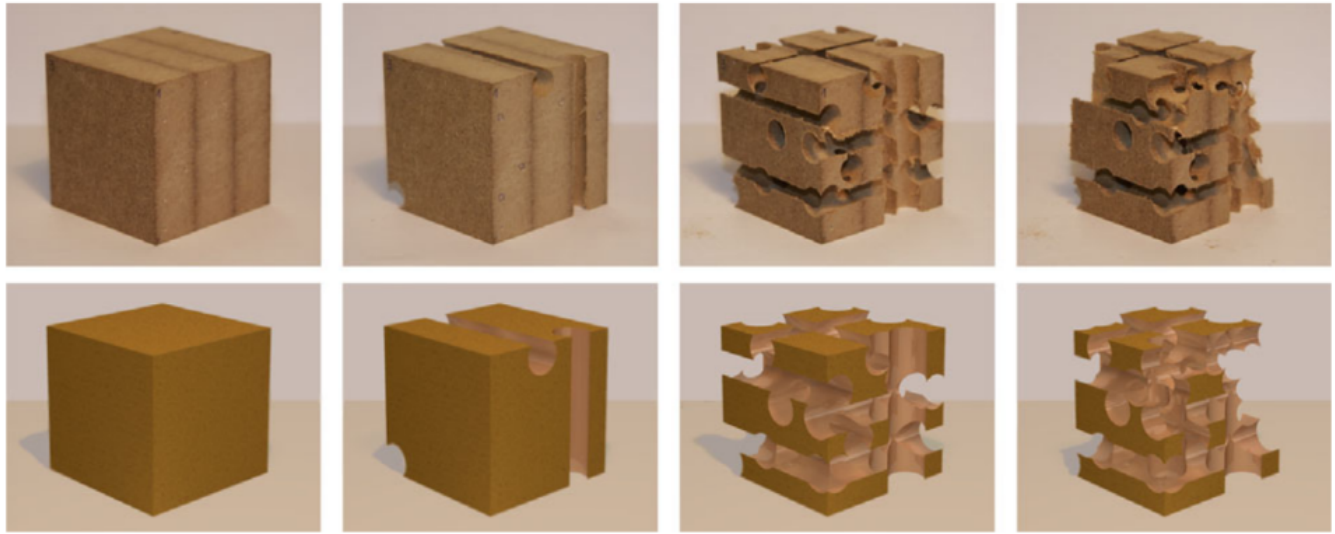
First passage percolation: In this case, we consider spread of a fluid, injected say from apoint source at the origin. The time the fluid takes to traverse a given bond B is random variable τ_B, where the τ_b's are i.i.d. random variables, with a common probability distribution $g(\tau)$.

An example of the type of questions one studies is the avearge behavior of the minimum time the the fluid will take between to specified points at a large distance R from each other.

There are many variations of these basic models. But let me mention

Drilling percolation: One takes a large $L \times L \times L$ cube. On each face, one selects a fraction p of the sites, and drills a hole perpendicular to the face, all the way to the opposite face.

Only if p is small enough, some parts of the cube will remain as a three-dimensional structure. Else, it will crumble into small disconnected parts.



(a) (b) (c) (d)

FIG. 1. Cube drilling. The upper row shows photos of the experimental setup; the lower panels are the corresponding numerical results. The faces of the cube are divided with a square-lattice mesh of linear size $L = 6$, such that each face can be drilled $L^2 = 36$ times. From left to right, (a)–(d), the number of drilled holes per face are (a) 0, (b) 1, (c) 6, and (d) 8. From our

Picture taken from Schrenk et al, Phys. Rev. Lett. 116, 055701 (2016).

Basic phenomenology

As mentioned earlier, the general behavior of percolation models seems simple to understand.

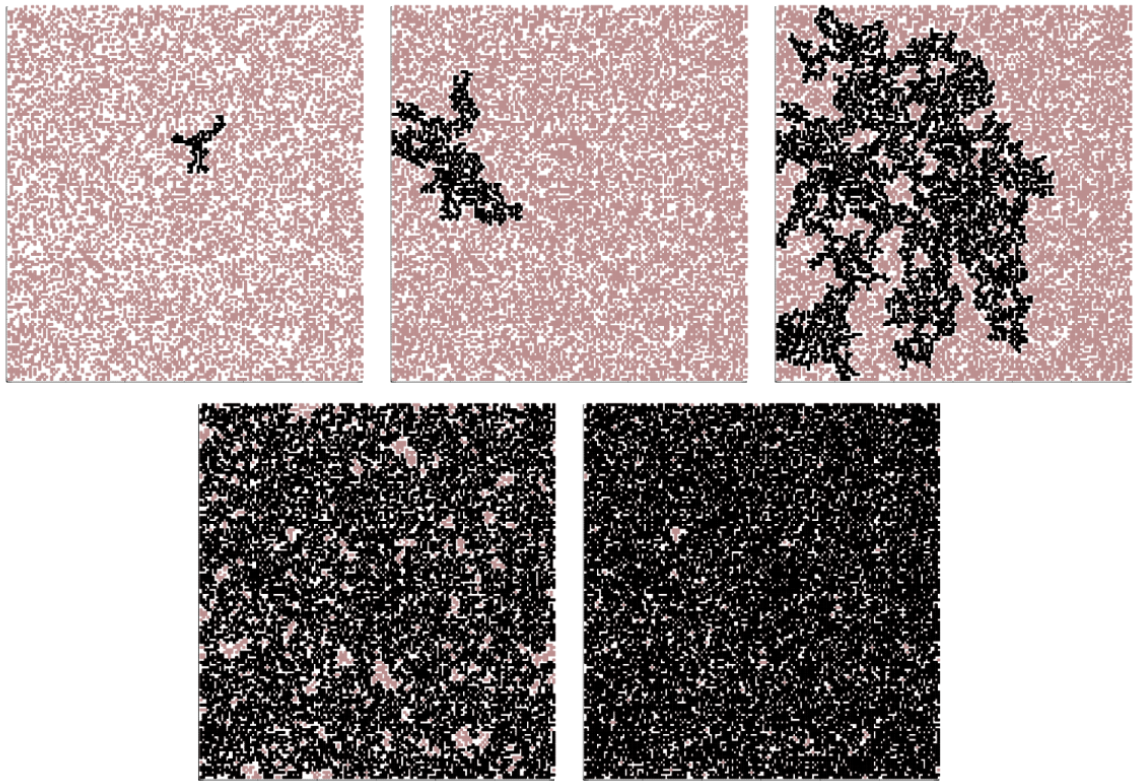


Figure 1.2: Percolation in $2d$ square lattices with system size $L \times L = 150 \times 150$. Occupation probability $p = 0.45, 0.55, 0.59, 0.65,$ and 0.75, respectively. Notice, that the largest cluster *percolates* through the lattice from top to bottom in this example when $p \geq 0.59$.

Figure taken from the lecture notes 'Percolation theory' by K. Christensen from the internet: web.mit.edu

Consider first Site Percolation. As we increase p_s, the typical size of clusters increases, and above the threshold, there is a large cluster containing a finite fraction of the sites.

We can consider several quantities:
The mean cluster size on the cluster containing the origin diverges as a power near p_c : $\langle s \rangle \sim |p_c - p|^{-\gamma}$, for $p \nearrow p_c$.

Mean linear size of the clusters ξ diverges: $\xi \sim |p - p_c|^{-\nu}$.

Fraction of sites in the infinite cluster $P_{infty} \sim [p - p_c]^{\beta}$, for $p \searrow p_c$.

At p_c, the clusters have a non-trivial fractal structure. This may be characterized by the non-trivial value of fractal dimension D: the average mass of the clster within distance R : $M(R) \sim R^D$.

Resistor Networks

Let us consider the simple case where each bond has a resistor with conductance taking values σ_1 and σ_2 with probabilities $(1-p)$ and p, with $\sigma_1 < \sigma_2$.

We consider a hypercube of size L, and measure the conductance C(L) across opposite facesdefine bulk conductivity $\sigma(p, \sigma_1, \sigma_2)$ by the limit
$\sigma(p, \sigma_1, \sigma_2) = \lim_{L\to\infty} C(L)L^{2-d}$.

This conductivity is self-averaging for large L.

Clearly, $\sigma(p, \sigma_1, \sigma_2)$ increases from σ_1 to σ_2 as p increases from 0 to 1.
The special limits where $\sigma_1 = 0, \sigma_2 = 1$ (called Insulator -Resistor percolation), and $\sigma_1 = 1, \sigma_2 = \infty$ (metal-superconductor percolation) are of special interest.

In the first case, $\sigma(p, \sigma_1, \sigma_2) = 0$ for $p < p_c$. And in the second case, the bulk conductivity is infinite for $p > p_c$.

Clearly, $\sigma(p, \Lambda\sigma_1, \Lambda\sigma_2) = \Lambda\sigma(p, \sigma_1, \sigma_2)$.
Hence we may write
$\sigma(p, \sigma_1, \sigma_2) = \sigma_1 f(p, \sigma_1/\sigma_2)$, where f is function only of the ratio σ_1/σ_2, and p.

Here, in the first case, we can define an exponent t_1 as the power with which σ increases from zero value for p just above p_c.
Similarly, we define t_2 as the exponent diverges to infinity for p just below p_c.

For non-zero value σ_1/σ_2, the function f is a continuous function of p even at p_c, but perhaps would be a singular function of p at p_c).

First -passage percolation

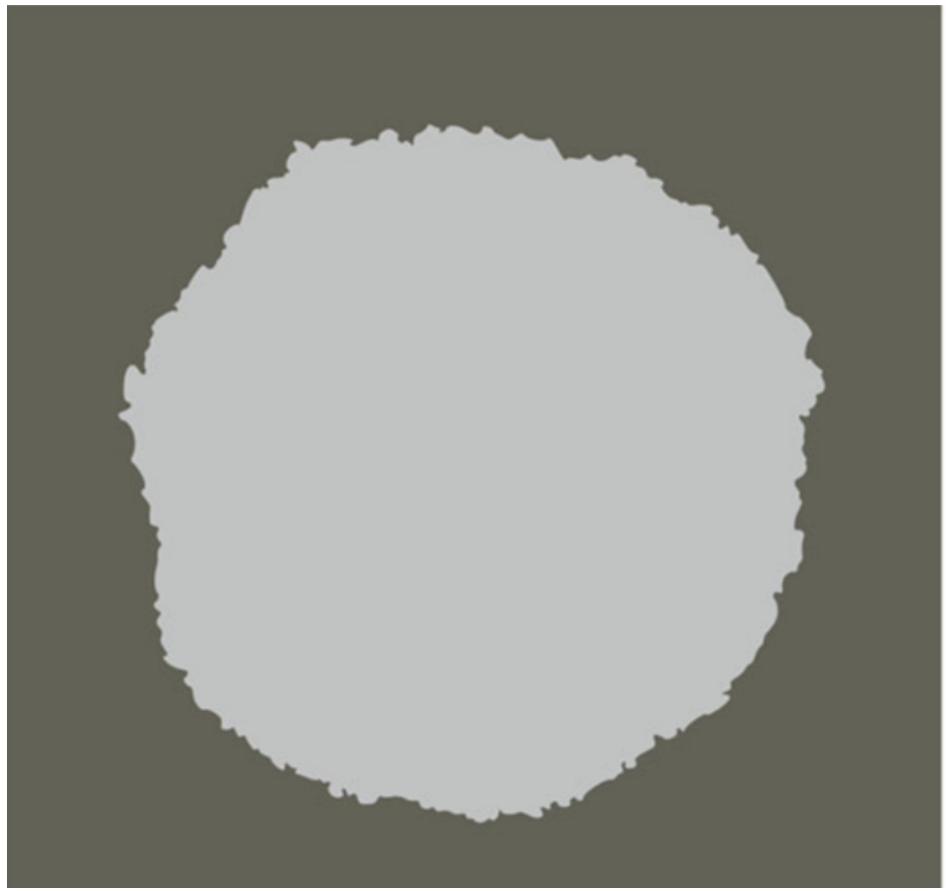
Here also, we can study many different distributions of bond-traversal times. Consider the case where the traversal time for a bond takes only two possible values: τ_1, and τ_2, with probabilities p and $(1-p)$, with $\tau_1 < \tau_2$.
The limiting cases of special interest are $\tau_1 = 0, \tau_2 = 1$, and $\tau_1 = 1, \tau_2 = \infty$. Qualitatitive behavior is similar to the resistor networks case.

Depending on p, in the first case, there are two possible phases, one in which fluid can spread infinitely far in zero time, and one where it spreads with finite velocity. In the second case, for p below percolation threshold, the fluid only spreads to a finite distance from the source.

One can consider a general probability distribution for τ, where τ takes any non-negative real value. In the following, we will assume that the distribution is nonzero only in a bounded interval $0 < \tau \leq t_{max}$.

The basic result in this case is that region that is reached by the fluided up to time T has a linear size that increases linearly with T. In fact, one can define the asymptotic speed $v(\Omega)$ of the advancing front in the direction Ω as the linit of the distance reached up to time T in the direction Ω and T. Exact calculation of $v(\Omega)$ has not been possible in any 'non-trivial' case, including the two-dimensional square lattice, even for the more tractable case where τ has an exponential distribution of mean 1.

A picture of a the wetted cluster in the first passage percolation on the square lattice (taken from the article by F. Manin et al [Discrete and Computational Geometry (2023) 69:771–799]. This model is also called the Eden model in literature. The invaded set has 10^5 sites here.



Exact solution of the percolation problem on a line

It is quite straight forward to calculate exactly different properties in the one-dimensional case.

Consider a infinite line of sites labelled by integer x,
$-\infty < x < +\infty$.

Consider site percolation, with each site occupied independently of others with probability p. We put $q = 1 - p$.
Then, it is easily seen that
(1) Prob. that origin belongs to cluster of size $s = q$, for $s = 0$
$= sq^2p^s$, for $s \geq 1$.
(2) Expected number of sites in the cluster at the origin $= 1/q$.

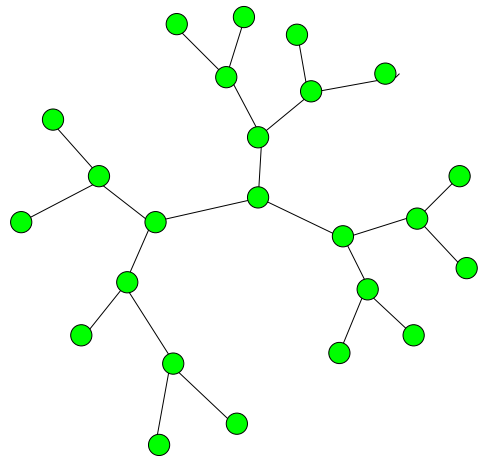
(3) Prob. that two sites at distance r belong to the same cluster = p^r.

Hence we see that p_c for 1-d site percolation =1.

Correlation length diverges as $1/q$ as $p \to 1$.
The solution of the bond percolation problem is similar.

Exact solution on the Bethe lattice

The Bethe lattice is an infinite tree of uniform coordination number (here 3).



Let A_s be the number of distinct clusters of s sites, rooted at the origin.

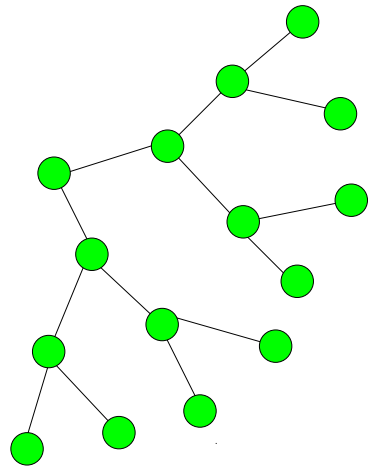
We define the animal numbers generating function $A(x)$ by

$$A(x) = \sum_{s=1}^{\infty} A_s x^s.$$

Then,

$$A(x) = x + 3x^2 + 9x^3 + ...$$

It is convenient to define a generating function for clusters starting at the top site of a subtree, and growing below.



Clearly, $B(x) = x + 2x^2 + 5x^3 + ...$
Then we see that we have

$$A(x) = x[1 + B(x)]^3.$$

and

$$B(x) = x[1 + B(x)]^2.$$

Solving this quadratic equation for $B(x)$, we get

$$B(x) = [1 - 2x - \sqrt{1 - 4x}]/2x.$$

Use binomial theorem to expand this powers of x, we get, for large s,

On the Bethe lattice, each cluster of s sites has external perimeter $s+2$, and hence the probability of the origin belonging to a cluster of s sites ($s \geq 1$) is $A_s p^s q^{s+2}$

This is largest for large s , if $p = 1/2 = p_c$ here. At p_c, $\mathrm{Prob}(s)$ decreases as $s^{-3/2}$ for large s.

For $p > p_c$, let B_∞ denote the probability the given subtree is connected to sites at infinity, then clearly,

$$B_\infty = 2pB_\infty - p^2 B_\infty^2,$$

which has the solution

$$B_\infty = (2p-1)/p^2.$$

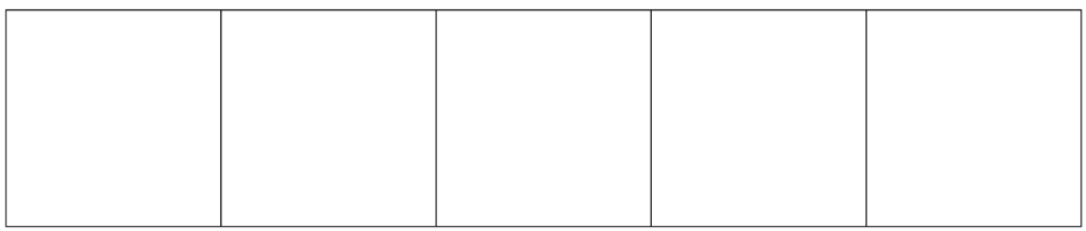
And we have probability that the origin belongs to an infinite cluster

$$P_\infty = p[1-(1-B_\infty)^3].$$

Exercises

1. Can you relate the exponents t_1 and t_2 defined in the random resitor model, using some good behavior condition on the scaling function f?

2. Find the exact formula for the correlation length in the bond-percolation problem on the infinite ladder graph



3. Extend the analysis of Bethe lattice case to other coordination numbers.

Some general references:

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