Spectral form factor in fermionic and bosonic models of many-body quantum chaos

Dibyendu Roy
Theoretical Physics Group
Raman Research Institute, Bangalore

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Abstract

I will discuss quantum chaos and spectral correlations in periodically driven fermionic and bosonic interacting chains in the presence and absence of particle-number conservation. I will show that the spectral form factor precisely follows the prediction of random matrix theory in the regime of long chains, and for timescales that exceed the Thouless time. For long-range interactions, random phase approximation can be used to rewrite the spectral form factor in terms of a bi-stochastic many-body process generated by effective spin or boson Hamiltonians. In the particle-number conserving case, the effective Hamiltonians have $SU(2)$ and $SU(1,1)$ symmetry, respectively for fermions and bosons, resulting in universal quadratic system-size scaling of the Thouless time, irrespective of the particle number. In the absence of particle-number conservation, while we find a nontrivial system-size dependence of the Thouless time for the bosonic model, it is independent of system size for kicked fermionic chains.

Periodically driven (Floquet) interacting lattices

A 1D lattice of interacting spinless fermions or bosons with a time-periodic kicking in the nearest-neighbor coupling (hopping and pairing):

$$\begin{aligned}\hat{H}(t) &= \hat{H}_0 + \hat{H}_1 \sum_{m\in\mathbb{Z}} \delta(t-m),\\ \hat{H}_0 &= \sum_{i=1}^{L} \epsilon_i \hat{n}_i + \sum_{i<j} U_{ij}\hat{n}_i\hat{n}_j,\\ \hat{H}_1 &= \sum_{i=1}^{L}(-J\hat{a}_i^\dagger \hat{a}_{i+1} + \Delta \hat{a}_i^\dagger \hat{a}_{i+1}^\dagger + \text{H.c.}),\end{aligned}$$

Number operator $\hat{n}_i = \hat{a}_i^\dagger \hat{a}_i$; creation operator of a fermion/boson $\hat{a}_i^\dagger$

$U_{ij} = U_0/|i-j|^\alpha$, with $1<\alpha<2$; random onsite energies ϵ_i described as Gaussian *i.i.d.* variables of zero mean and standard deviation $\Delta\epsilon$.

$\Delta = 0$ or $\neq 0$ corresponds respectively to conservation or violation of a total fermion/boson number $\hat{N} = \sum_{i=1}^{L} \hat{n}_i$.

Spectral form factor (SFF) $K(t)$

Fluctuations in the spectral density of energy or quasienergy are often used as the main signatures of quantum chaos and the appropriate random matrix theory (RMT) type is determined solely by the symmetry of the underlying dynamical systems.

Quasienergies of interest are the eigenphases φ_m of a unitary Floquet propagator $\hat{U}$ of evolution after one cycle: $\hat{U} = \mathcal{T}\exp(-i\int_0^1 dt \hat{H}(t))$

$\hat{U}|m\rangle = e^{-i\varphi_m}|m\rangle$ for $m = 1, 2, \ldots, \mathcal{N}$ (dimension of the Hilbert space)

Spectral density $\rho(\varphi) = \frac{2\pi}{\mathcal{N}} \sum_m \delta(\varphi - \varphi_m)$, $\langle \rho(\varphi) \rangle_\varphi \equiv \int_0^{2\pi} \frac{d\varphi}{2\pi} \rho(\varphi) = 1$

Pair correlation function $R(\vartheta) = \langle \rho(\varphi + \vartheta/2)\rho(\varphi - \vartheta/2) \rangle_\varphi - \langle \rho(\varphi) \rangle_\varphi^2$ provides a measure of spectral fluctuations.

$$K(t) = \frac{\mathcal{N}^2}{2\pi} \int_0^{2\pi} d\vartheta \langle R(\vartheta) e^{-i\vartheta t} \rangle = \langle (\mathrm{tr}\hat{U}^t)(\mathrm{tr}\hat{U}^{-t}) \rangle - \mathcal{N}^2 \delta_{t,0}$$

where $\mathrm{tr}\hat{U}^t = \sum_m e^{-i\varphi_m t}$, and $\langle \ldots \rangle$ denotes an average over disorder.

SFF within RPA: bi-stochastic many-body process

Time-evolution operator $\hat{U}$ of each cycle as a two-step Floquet propagator:

$$\hat{U} = \hat{V}\hat{W}, \quad \hat{W} = e^{-i\hat{H}_0} \text{ and } \hat{V} = e^{-i\hat{H}_1}$$

A basis of Fock states $|\underline{n}\rangle \equiv |n_1, n_2, \ldots, n_L\rangle$ (not eigenstate of $\hat{U}$), where $n_j \in \{0,1\}$ ($\{0,1,\ldots,N\}$) represents an occupation number of fermions (bosons) at the lattice site j, and $N \equiv \langle \underline{n}|\hat{N}|\underline{n}\rangle = \sum_{j=1}^{L} n_j$.

Using random phase approximation (RPA) to perform the disorder averaging over different realizations and making further the asymptotic approximation via identity permutations between two replicas, we can write

$$K(t) = 2t\,\mathrm{tr}\mathcal{M}^t,$$

where $\mathcal{M}_{\underline{n},\underline{n}'} = |\langle \underline{n}|\hat{V}|\underline{n}'\rangle|^2 = |\langle \underline{n}|e^{-iH_1}|\underline{n}'\rangle|^2$ is a $\mathcal{N} \times \mathcal{N}$ square matrix.

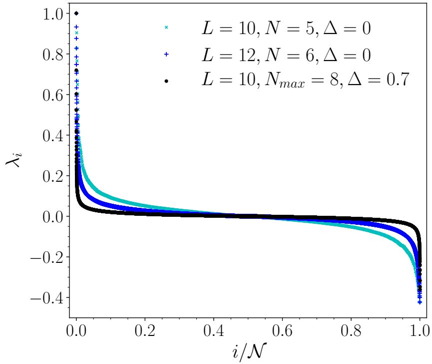
$\sum_{\underline{n}'} \mathcal{M}_{\underline{n},\underline{n}'} = \sum_{\underline{n}'} \langle \underline{n}|\hat{V}|\underline{n}'\rangle\langle \underline{n}'|\hat{V}^\dagger|\underline{n}\rangle = \langle \underline{n}|\hat{V}\hat{V}^\dagger|\underline{n}\rangle = 1$: bi-stochastic matrix

Universal RMT form of $K(t)$

Largest eigenvalue of a bi-stochastic matrix is 1, while the rest of the eigenvalues $(1, \lambda_1, \lambda_2, \ldots$ with $1 \geq |\lambda_j| \geq |\lambda_{j+1}|)$ are real as $\mathcal{M}$ is also real and symmetric. Then, we obtain SFF as a sum over eigenvalues λ_j

$$K(t) = 2t\big(1 + \sum_j \lambda_j^t\big),$$

where $K(t) \simeq 2t$ is a leading order in t/t_H result of RMT of circular orthogonal ensemble: averaged over ensemble!



Thouless time t^* to reach universal $K(t)$

For large enough L, we approximate $K(t)$ at long time t, $1 \ll t \ll t_H$, by truncating it after the second largest eigenvalue λ_1 of $\mathcal{M}$.

Consider λ_1 scales with system size L as $1-1/t^*(L)$ where $t^*(L) \simeq L^\beta/D$:

$$K(t) \simeq 2t(1+\lambda_1^t) \simeq 2t(1+(1-1/t^*(L))^t) \simeq 2t(1+e^{-t/t^*(L)}).$$

L-dependence of λ_1 and t^* is found by (a) numerically diagonalizing $\mathcal{M}$, and (b) mapping $\mathcal{M}$ to an effective Hamiltonian in the continuous-time/Trotter regime, i.e., at small J, Δ.

Expand $\hat{V}$ in the Trotter regime of the Hamiltonian $\hat{H}_1$:

$$\begin{aligned} \mathcal{M} &= e^{-i\hat{H}_1} \bullet e^{i\hat{H}_1} \\ &= (\mathbb{1} - i\hat{H}_1 - \frac{1}{2}\hat{H}_1^2 + \dots) \bullet (\mathbb{1} + i\hat{H}_1 - \frac{1}{2}\hat{H}_1^2 + \dots) \\ &= \mathbb{1} + \hat{H}_1 \bullet \hat{H}_1 - \hat{H}_1^2 \bullet \mathbb{1} + \mathcal{O}(\hat{H}_1^4), \end{aligned}$$

where $\hat{H}_1 \bullet \hat{H}_1$ is an element-wise square of $\hat{H}_1$ in the Fock space basis.

Mapping $\mathcal{M}$ to effective Hamiltonian : Fermions

For $J, \Delta \to 0$, $\mathcal{M}$ can be generated by anisotropic Heisenberg model.

$$\mathcal{M} = (1 - c_x L)\mathbb{1}_{\mathcal{N}} + \sum_{j=1}^{L} \sum_{\nu=x,y,z} c_\nu \sigma_j^\nu \sigma_{j+1}^\nu + \mathcal{O}(J^4, \Delta^4),$$

$c_x = (J^2 + \Delta^2)/2,\ c_y = c_z = (J^2 - \Delta^2)/2$. σ_j^ν: Pauli matrix at site j.

"Ground state" of the generating Hamiltonian with an eigenvalue 1 is a ferromagnet polarized in x-direction.

For $\Delta = 0$, isotropic Heisenberg model ($SU(2)$ symmetry) whose eigenenergy spectrum is gapless for any magnetization (any N). Eigenvalue of first "excited state" $\lambda_1 = 1 - c_1/L^2$ (one x-polarized magnon excitation with momentum $k = 2\pi/L$). $\beta = 2$ and Thouless time, $t^* \simeq L^2/c_1$.

For $\Delta \neq 0$, anisotropic Heisenberg model which has a finite and system-size independent gap in the energy spectrum between the ground and first excited state. $\beta = 0$ and L-independent Thouless time.

Mapping $\mathcal{M}$ to effective Hamiltonian : Bosons

Generating Hamiltonian in the Trotter regime of small J when $\Delta = 0$:

$$\mathcal{M} = \mathbb{1} + \sum_{i=1}^{L} \Big(J^2 (\hat{K}_i^- \hat{K}_{i+1}^+ + \hat{K}_{i+1}^- \hat{K}_i^+) - 2J^2 (\hat{K}_i^0 \hat{K}_{i+1}^0 - \frac{1}{4}) \Big) + \mathcal{O}(J^4)$$

in terms of $\hat{K}_i^0 = -(\hat{n}_i + 1/2)$, $\hat{K}_i^+ = \hat{a}_i \sqrt{\hat{n}_i}$, $\hat{K}_i^- = \sqrt{\hat{n}_i} \hat{a}_i^\dagger$, which satisfy the commutation relations of $SU(1,1)$ algebra

$$[\hat{K}_i^+, \hat{K}_j^-] = -2\hat{K}_i^0 \delta_{ij}, \; [\hat{K}_i^0, \hat{K}_j^\pm] = \pm \hat{K}_i^\pm \delta_{ij}.$$

We have $[\hat{K}^\alpha, \mathcal{M}] = 0$, where $\hat{K}^\alpha = \sum_{i=1}^{L} \hat{K}_i^\alpha$, $\alpha \in \{+, -, 0\}$ satisfy $SU(1,1)$ algebra.

Generating Hamiltonian of the Markov matrix $\mathcal{M}$ has $SU(1,1)$ symmetry in the particle-number conserving case.

Numerics shows $\mathcal{M}$ has $SU(1,1)$ symmetry for arbitrary values of J

L-dependence of Thouless time ($\Delta = 0$): Bosons

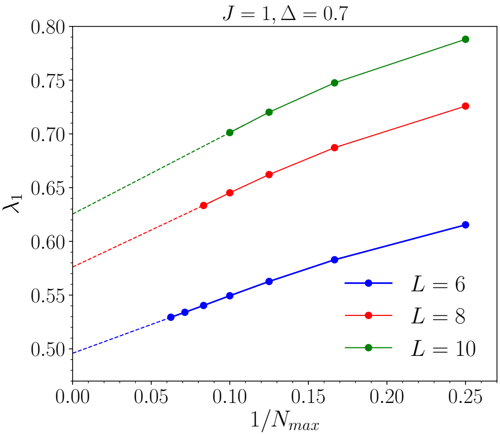
Due to $SU(1,1)$ symmetry of the generating Hamiltonian, its lowest excited states can be obtained as degenerate descendants of the single-particle $(N = 1)$ states, i.e., by applying the operator $\hat{K}^-$. Therefore, the L-dependence of λ_1 is independent of N when $\Delta = 0$.

$$\mathcal{M}|_{\Delta=0}^{N=1} = (\mathbb{1} - 2J^2) + \sum_{i=1}^{L} J^2(\hat{a}_i^\dagger \hat{a}_{i+1} + \hat{a}_{i+1}^\dagger \hat{a}_i) + \mathcal{O}(J^4).$$

"Ground state" of $\mathcal{M}|_{\Delta=0}^{N=1}$ is a state with eigenvalue 1 and with zero momentum. Eigenenergy spectrum is gapless, and first "excited state" (with momentum $k = 2\pi/L$) goes as $\lambda_1 = 1 - c_2/L^2$. Thouless time, $t^* \simeq L^2/c_2$, for single boson and, due to $SU(1,1)$ symmetry, for any number of bosons.

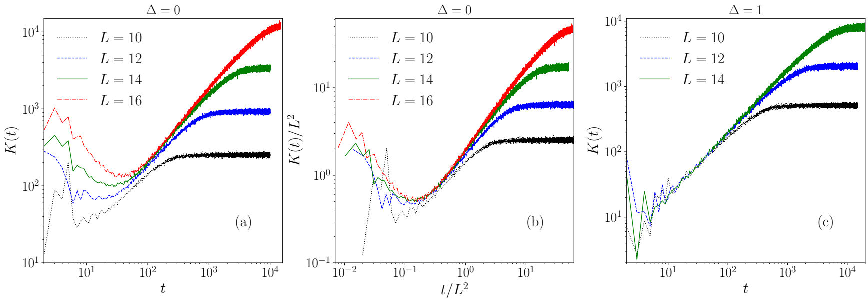
Generating Hamiltonian of $\mathcal{M}$ lacks $SU(1,1)$ symmetry when $\Delta \neq 0$. Consequently, λ_1 changes with N or $N_{\max}$ for a fixed L.

L-dependence of Thouless time ($\Delta \neq 0$): Bosons



Dashed lines indicate a linear extrapolation of the last few large $N_{\max}$ points. These linear extrapolations give $\lambda_1 \sim 1 - 1.43/L^{0.58}$ or $e^{-2.89/L^{0.79}}$ at $1/N_{\max} \to 0$, which predicts a finite system-size dependence of the Thouless time (e.g., $t^* = \mathcal{O}(L^\gamma)$, $\gamma = 0.7 \pm 0.1$ when $J = 1, \Delta = 0.7$)

Exact numerically computed $K(t)$: fermions



Spectral form factor $K(t)$ for different system sizes L of the kicked spinless fermion chain with $(\Delta = 0)$ (a,b) and without $(\Delta = 1)$ (c) particle-number conservation. Here, $J = 1, U_0 = 15, \alpha = 1.5, \Delta\epsilon = 0.3$ and $N/L = 1/2$ for $\Delta = 0$. An averaging over 10^3 realizations of disorder is performed. In (b) we show data collapse in scaled time t/L^2.

Temporal growth of $K(t)$ for $\Delta = 1$ at $t \ll t_H$ is independent of L which confirms our analytical prediction based on the RPA.

For $\Delta = 0$, we find a nice data collapse for various L and $t < t_H$ which confirms our above predicted L-dependence of $K(t)$ using the RPA.

References

Random matrix spectral form factor in kicked interacting fermionic chains, Dibyendu Roy & Tomaž Prosen, *Phys. Rev. E* **102**, *060202(R) (2020)*

Spectral form factor in a minimal bosonic model of many-body quantum chaos, Dibyendu Roy, Divij Mishra & Tomaž Prosen, *arXiv:2203.05439 (2022)*

Derivation of SFF: Fock space basis

Time-evolution operator $\hat{U}$ of each cycle can be written as a two-step unitary Floquet propagator:

$$\hat{U} = \hat{V}\hat{W}, \quad \hat{W} = e^{-i\hat{H}_0} \text{ and } \hat{V} = e^{-i\hat{H}_1}$$

We consider a basis of Fock states $|\underline{n}\rangle \equiv |n_1, n_2, \ldots, n_L\rangle$, where $n_j \in \{0,1\}$ ($\{0,1,\ldots,N\}$) represents an occupation number of spinless fermions (bosons) at the lattice site j, and $N \equiv \langle \underline{n}|\hat{N}|\underline{n}\rangle = \sum_{j=1}^{L} n_j$.

For $\Delta = 0$, $[\hat{U}, \hat{N}] = 0$; we take only those basis states $\mathcal{N} = \frac{L!}{N!(L-N)!}$ $(\frac{(N+L-1)!}{N!(L-1)!})$, which have total N fermions (bosons)

For $\Delta \neq 0$, we take $\mathcal{N} = 2^{L-1}$ $(\sum_{N=0,2,..}^{N_{\max}} \frac{(N+L-1)!}{N!(L-1)!})$ states $|\underline{n}\rangle$ with all allowed either even or odd N fermions (bosons); $N_{\max}$: cutoff for bosons

$|\underline{n}\rangle$ are not eigenstates of $\hat{U}$ but eigenstates of $\hat{W}$

$$\hat{W}|\underline{n}\rangle = e^{-i\theta_{\underline{n}}}|\underline{n}\rangle, \ \theta_{\underline{n}} = \sum_{i=1}^{L} \epsilon_i n_i + \sum_{i<j} U_{ij} n_i n_j$$

Random phase approximation (RPA)

Floquet propagator for t time steps, $\mathrm{tr}\hat{U}^t$, can be evaluated by inserting $\mathbb{1}_{\mathcal{N}} = \sum_{\underline{n}_\tau} |\underline{n}_\tau\rangle\langle\underline{n}_\tau|$ at different time steps $\tau = 1, 2, \ldots, t$.

$$\mathrm{tr}\hat{U}^t = \sum_{\underline{n}_1,\ldots,\underline{n}_t} \langle\underline{n}_1|\hat{V}\hat{W}|\underline{n}_2\rangle\langle\underline{n}_2|\hat{V}\hat{W}\ldots|\underline{n}_t\rangle\langle\underline{n}_t|\hat{V}\hat{W}|\underline{n}_1\rangle$$

$$= \sum_{\underline{n}_1,\ldots,\underline{n}_t} e^{-i\sum_{\tau=1}^{t}\theta_{\underline{n}_\tau}} \prod_{\tau=1}^{t} V_{\underline{n}_\tau,\underline{n}_{\tau+1}}, \quad V_{\underline{n}_\tau,\underline{n}_{\tau+1}} = \langle\underline{n}_\tau|\hat{V}|\underline{n}_{\tau+1}\rangle,$$

where we use PBC in time $|\underline{n}_{t+1}\rangle \equiv |\underline{n}_1\rangle$.

$$K(t) = \sum_{\underline{n}_1,\ldots,\underline{n}_t} \sum_{\underline{n}'_1,\ldots,\underline{n}'_t} \langle e^{-i\sum_{\tau=1}^{t}(\theta_{\underline{n}_\tau} - \theta_{\underline{n}'_\tau})}\rangle \prod_{\tau=1}^{t} V_{\underline{n}_\tau,\underline{n}_{\tau+1}} V^*_{\underline{n}'_\tau,\underline{n}'_{\tau+1}}$$

The phases $\theta_{\underline{n}_\tau}$ for different $|\underline{n}_\tau\rangle$ (modulo 2π) are assumed to be independent random numbers for $\hat{H}_0$ with random onsite energies and long-range interaction. Within RPA, averaging over the disorder realizations gives:

$$\langle e^{-i\sum_{\tau=1}^{t}(\theta_{\underline{n}_\tau} - \theta_{\underline{n}'_\tau})}\rangle \simeq \delta_{\{\underline{n}_1,\ldots,\underline{n}_t\},\{\underline{n}'_1,\ldots,\underline{n}'_t\}},$$

Bi-stochastic (Markov) matrix $\mathcal{M}$

For times $t \ll t_H = \mathcal{N}$ in a large enough system, the probability that two configurations repeat in time t is proportional to t/t_H, and all configurations $\underline{n}_\tau$ in the string $\{\underline{n}_1, \underline{n}_2, \ldots, \underline{n}_t\}$ can be assumed different.

RPA implies that there is a permutation $\pi \in S_t$ so that we can relate $\underline{n}'_\tau = \underline{n}_{\pi(\tau)}$. There are t cyclic permutations and t anti-cyclic permutations for which the matrices $V_{\underline{n}_\tau, \underline{n}_{\tau+1}}$ in a string $\{\underline{n}_1, \underline{n}_2, \ldots, \underline{n}_t\}$ are the same as $V_{\underline{n}'_\tau, \underline{n}'_{\tau+1}}$ in a string $\{\underline{n}'_1, \underline{n}'_2, \ldots, \underline{n}'_t\}$. Thus, we get in the leading order:

$$
\begin{aligned}
K(t) &= \sum_{\underline{n}_1, \ldots, \underline{n}_t} \prod_{\tau=1}^{t} V_{\underline{n}_\tau, \underline{n}_{\tau+1}} V^*_{\underline{n}_{\pi(\tau)}, \underline{n}_{\pi(\tau+1)}} \\
&= 2t \sum_{\underline{n}_1, \ldots, \underline{n}_t} \prod_{\tau=1}^{t} |V_{\underline{n}_\tau, \underline{n}_{\tau+1}}|^2 = 2t \, \mathrm{tr} \mathcal{M}^t,
\end{aligned}
$$

where $\mathcal{M}_{\underline{n}, \underline{n}'} = |\langle \underline{n} | \hat{V} | \underline{n}' \rangle|^2 = |\langle \underline{n} | e^{-iH_1} | \underline{n}' \rangle|^2$ is a $\mathcal{N} \times \mathcal{N}$ square matrix.

$$
\sum_{\underline{n}'} \mathcal{M}_{\underline{n}, \underline{n}'} = \sum_{\underline{n}'} \langle \underline{n} | \hat{V} | \underline{n}' \rangle \langle \underline{n}' | \hat{V}^\dagger | \underline{n} \rangle = \langle \underline{n} | \hat{V} \hat{V}^\dagger | \underline{n} \rangle = 1
$$

L-dependence of Thouless time : Fermions

L-dependence of first excited states (single magnon states) in different magnetization sector is the same due to the $SU(2)$ symmetry of isotropic Heisenberg model, and it gives the quadratic L-dependence of t^* for all filling fractions N/L (including $N = 1$) in the presence of $U(1)$ symmetry.

Numerics shows $\mathcal{M}$ has $SU(2)$ symmetry for arbitrary J when $\Delta = 0$.

$J = 1, \Delta = 0$				$J = 1, \Delta = 0.3$			
L	λ_1	λ_2	λ_3	L	λ_1	λ_2	λ_3
10	0.653	0.5023	0.4275	10	0.7594	0.6139	0.5288
12	0.7495	0.6087	0.5263	12	0.75938	0.6079	0.5677
14	0.8115	0.6892	0.6098	14	0.75938	0.6152	0.6041
16	0.8535	0.7493	0.6939	15	0.75938	0.6328	0.6026

$J = 1, \Delta = 0$, $\beta = 1.86$ (using $L = 12, 14, 16$) if we are fitting to $\lambda_1 = 1 - 1/t^*$ and $\beta = 2.08$ if fitting to $\lambda_1 = e^{-1/t^*}$. When $J = 1, \Delta = 0.3$, we find consistently $\beta = 0$.

L-dependence of Thouless time ($\Delta = 0$): Bosons

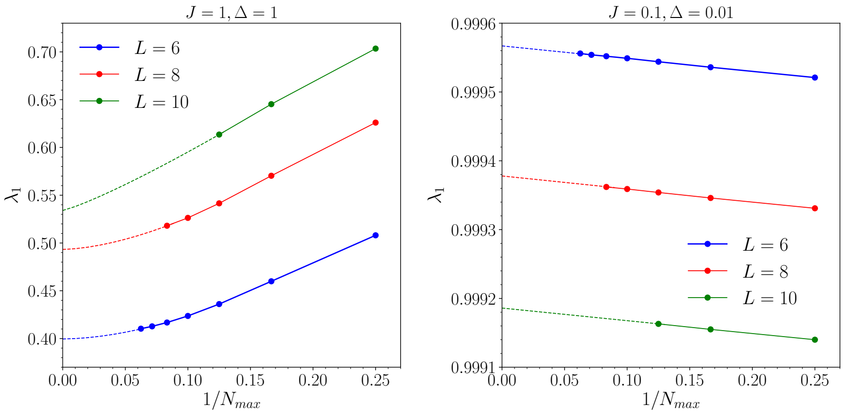
$J=1,\Delta=0,N/L=1/2$				$J=1,\Delta=0,N/L=1/4$			
L	λ_1	λ_2	λ_3	L	λ_1	λ_2	λ_3
8	0.8526	0.7486	0.6680	8	0.8526	0.7486	0.4847
10	0.9042	0.8283	0.7658	12	0.9329	0.8764	0.8278
12	0.9329	0.8764	0.8278	16	0.9619	0.9278	0.8970
14	0.9504	0.9071	0.8688	20	0.9755	0.9529	0.9320

$J=1,\Delta=0$: $\lambda_1 \sim 1-8.29/L^{1.94}$ (or $\lambda_1 \sim e^{-11.4/L^{2.05}}$) for $N/L=1/2$ (using $L=10,12,14$), and $\lambda_1 \sim 1-9.0/L^{1.97}$ (or $\lambda_1 \sim e^{-10.5/L^{2.02}}$) for $N/L=1/4$ (using $L=12,16,20$).

The above exponents for two different finite size fittings of λ_1 show a clear trend towards $\mathcal{O}(L^2)$ scaling of t^* in the bosonic chain when $\Delta=0$.

The generating Hamiltonian of $\mathcal{M}$ lacks $SU(1,1)$ symmetry when $\Delta \neq 0$. Consequently, the second largest eigenvalue λ_1 changes with N or $N_{\max}$ for a fixed L.

L-dependence of Thouless time ($\Delta \neq 0$): Bosons



Second largest eigenvalue λ_1 of $\mathcal{M}$ with $1/N_{\rm max}$, for three lengths L and two sets of J, Δ in the absence of $U(1)$ symmetry. The dashed lines denote algebraic and linear extrapolation of the last few large $N_{\rm max}$ points for $J, \Delta = 1$ and $J = 0.1, \Delta = 0.01$, respectively. The two plots at different J, Δ display an opposite trend in the L-dependence of λ_1 and Thouless time for a small and a large J, Δ.