Arithmetic Quantum Field Theory

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0. Arithmetic Geometry

Slogan:

Geometry has always involved a duality between states and observables

States: Points of tangent or cotangent bundles of manifold, points of projective spaces, linear functionals, objects in n-categories.

Observables: Functions, operators, objects in n-categories.

Arithmetic Geometry

is geometry based on *absolutely finitely-generated number systems*:

$$\mathbb{Z}, \quad \mathbb{Z}[\sqrt{2}], \quad \mathbb{Z}[\sqrt{2}, \sqrt{-1}, \exp(2\pi i/7)],$$
$$\mathbb{Z}[\sqrt{2}, \sqrt{-1}, \exp(2\pi i/7), \pi, e, \Gamma(5)]$$

In each of these examples, every element can be generated by performing addition, subtraction, and multiplication repeatedly starting from a finite collection of 'seed' elements (generators).

These are rings R that are isomorphic to quotients of $\mathbb{Z}[x_1, x_2, \ldots, x_n]$ for some n:

$$R \simeq \mathbb{Z}[x_1, x_2, \ldots, x_n]/I.$$

Arithmetic Geometry

To such a ring, we associate its *spectrum*

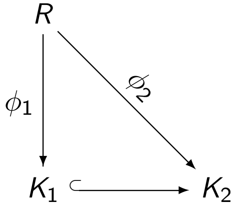
$$\mathrm{Spec}(R),$$

the set of all *pure states* on R, ring homomorphisms

$$R \longrightarrow K$$

to a field, subject to an equivalence relation generated by increasing the codomain.

We have the generating equivalence $\phi_1 \sim \phi_2$ if we have a commutative diagram



Arithmetic Geometry

Examples:

Spec($\mathbb{Z}$) consists of

$$\phi_p : \mathbb{Z} \longrightarrow \mathbb{F}_p$$

for each p and

$$\phi_{gen} : \mathbb{Z} \hookrightarrow \mathbb{Q}.$$

Spec($\mathbb{C}[x]$) consists of

$$\phi_a : f(x) \longrightarrow f(a) \in \mathbb{C}$$

for each $a \in \mathbb{C}$ and

$$\phi_{gen} : \mathbb{C}[x] \hookrightarrow \mathbb{C}(x)$$

Arithmetic Geometry

$\mathrm{Spec}(\mathbb{C}[x_1, x_2])$ consists of:

(0)

$$\phi_{(a_1,a_2)} : f(x_1, x_2) \mapsto f(a_1, a_2) \in \mathbb{C}$$

for each $(a_1, a_2) \in \mathbb{C}^2$.

(2)

$$\phi_{gen} : \mathbb{C}[x_1, x_2] \hookrightarrow \mathbb{C}(x_1, x_2)$$

(1)

$$\phi_D : \mathbb{C}[x_1, x_2] \longrightarrow \mathbb{C}[D] \hookrightarrow \mathbb{C}(D),$$

for each irreducible curve $D \subset \mathbb{C}^2$.

Arithmetic Geometry

An *arithmetic scheme* X is glued out of finitely many copies of finitely-generated rings:

$$X = \cup \mathrm{Spec}(R_i),$$

and is the fundamental object of study in arithmetic geometry.

For example,

$$\mathrm{Proj}(\mathbb{Z}[z_0, z_1, z_2]/(z_0 z_2^2 - z_1^3 + z_0^2 z_1)))$$

$$= E_0 \cup E_1 \cup E_2$$

where

$$E_0 = \mathrm{Spec}(\mathbb{Z}[x, y]/(y^2 - x^3 + x)),$$

etc.

Idle speculation on arithmetic geometry:

Perhaps arithmetic geometry, wherein observables are intrinsically discrete, are relevant to quantum mechanics.

Arithmetic Geometry: Topology

Maps between arithmetic schemes are built up from ring homomorphisms

$$A \longrightarrow B$$

which is equivalent to

$$\mathrm{Spec}(B) \longrightarrow \mathrm{Spec}(A).$$

Using this, one can choose various categories of maps

$$\coprod U_i \longrightarrow X$$

to be open covers, which gives rise to a Cech nerve

$$\cdots \longrightarrow \coprod U_i \times_X U_j \times U_k \longrightarrow \coprod U_i \times_X U_j$$

$$\longrightarrow \coprod U_i \longrightarrow X,$$

from which we can extract various notions of topology.

Arithmetic Geometry and Algebraic Number Theory

An algebraic number field is a field

$$F \supset \mathbb{Q},$$

which is finite-dimensional as a $\mathbb{Q}$-vector space.

Thus, F is isomorphic to $\mathbb{Q}^d$ equipped with a bilinear map

$$\mathbb{Q}^d \times \mathbb{Q}^d \longrightarrow \mathbb{Q}^d,$$

that makes it into a field.

It turns out that

$$F \simeq \mathbb{Q}[x]/(f(x)),$$

where $f(x) \in \mathbb{Q}[x]$ is an irreducible polynomial.

By choosing a complex root α of $f(x)$, one also has

$$F \simeq \mathbb{Q}[\alpha] \subset \mathbb{C}.$$

Arithmetic Geometry and Algebraic Number Theory

$F = \mathbb{Q}^2$ might be considered with the multiplication

$$(a, b) \cdot (c, d) = (ac - bd, ad + bc)$$

which is $\mathbb{Q}[\sqrt{-1}] \simeq \mathbb{Q}[x]/(x^2 + 1)$,

or with the multiplication

$$(a, b) \cdot (c, d) = (ac + 5bd, ad + bc)$$

which is $\mathbb{Q}[\sqrt{5}] \simeq \mathbb{Q}[x]/(x^2 - 5)$.

The triples

$$F = \mathbb{Q}^3$$

with

$$(a, b, c) \cdot (a', b', c') = (aa' + 2bc' + 2cb', ab' + ba' + 2cc', ac' + bb' + ca')$$

is $\mathbb{Q}[\sqrt[3]{2}] \simeq \mathbb{Q}[x]/(x^3 - 2)$.

Arithmetic Geometry and Algebraic Number Theory

The ring of algebraic integers $\mathscr{O}_F \subset F \simeq \mathbb{Q}^d$ is the maximal subring of F isomorphic to $\mathbb{Z}^d$.

$$\mathbb{Z}[i] \subset \mathbb{Q}[i]$$

$$\mathbb{Z}[\frac{1+\sqrt{5}}{2}] \subset \mathbb{Q}[\sqrt{5}]$$

$$\mathbb{Z}[\sqrt[3]{2}] \subset \mathbb{Q}[\sqrt[3]{2}]$$

$$\mathbb{Z}[\sqrt{-5}] \subset \mathbb{Q}[\sqrt{-5}]$$

Picture

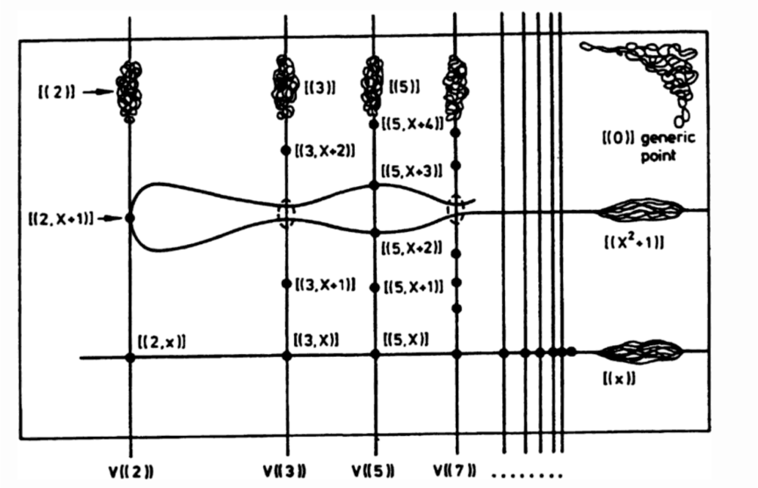


Figure: Mumford's picture of Spec($\mathbb{Z}[x]$)

Algebraic Number Theory and Arithmetic Topology

Arithmetic topology starts from the observation that the arithmetic scheme

$$\mathrm{Spec}(\mathscr{O}_F)$$

has many of the topological features of a closed 3-manifold.

If we consider F_v, the completion of F with respect to a non-Archimedean absolute value v,

$$\mathrm{Spec}(F_v)$$

has many features of a closed 2 manifold.

Where we are going

A *3d topological quantum field theory* Z:

Closed 3-manifold $M_3 \longrightarrow Z(M_3) \in \mathbb{C}$

Closed 2-manifold $M_2 \longrightarrow Z(M_2) \in \mathrm{Vect}_{\mathbb{C}}$

Closed 1-manifold $S^1 \longrightarrow Z(S^1) \in \mathrm{Cat}_{\mathbb{C}}$

Closed zero-manifold pt $\longrightarrow Z(\mathrm{pt}) \in 2\text{-}\mathrm{Cat}_{\mathbb{C}}$

In short,

$$Z(\cdot)$$

is a complicated function of manifolds with natural properties reflecting the construction of manifolds.

Where we are going

We will try to extend such a theory to an *arithmetic topological quantum field theory* Z:

$$Z(\mathrm{Spec}(\mathscr{O}_F)) \in \mathbb{C}$$

$$Z(\mathrm{Spec}(F_v)) \in \mathrm{Vect}_{\mathbb{C}}$$

$$Z(\mathrm{Spec}(\mathbb{F}_q)) \in \mathrm{Cat}_{\mathbb{C}}$$

Would like to do this via *quantisation of spaces of arithmetic fields*.

The basic technique used so far is to imitate the *Feynman path integral* used by physicists.

I. A Tour of Physics

Classical Systems

Space of all possible states: *symplectic manifold* $(\mathbb{S}, \omega)$, where $\omega \in \Omega^2(\mathbb{S})$ is non-degenerate and closed.

Examples:

$$\mathbb{R}^3 \times \mathbb{R}^3 = \{(p, q)\}$$

states of a single point particle in Euclidean space. Symplectic form

$$\sum_{i=1}^{3} dp_i \wedge dq_i = d(\sum_i p_i dq_i)$$

Generalises to

$$T^*X,$$

where X is a manifold: $\theta := \sum_i p_i dq_i$ is invariantly defined. Put $\omega = d\theta$.

Classical Systems

Examples (continued):

Smooth complex projective varieties. Symplectic form is associated to Kaehler metric pulled back from projective space:

$$\omega = g(J\cdot, \cdot)$$

Most importantly for us, the space of solutions to the equations of motion for a field theory on a spacetime manifold of the form

$$M = N \times \mathbb{R}$$

has the structure of a symplectic manifold.

Classical Systems

In the C^∞-case, every symplectic structure locally looks like

$$\sum_{i=1}^{n} dp_i \wedge dq_i = d\Big(\sum_i p_i dq_i\Big)$$

for a suitable choice of coordinates p_i, q_i.

Thus, a symplectic structure has no local invariants, leading to a topological nature in its classification.

Algebraic symplectic structures on algebraic varieties are hard to construct, but T^*X gives a good collection of examples.

Classical Systems: Dynamics

On $\mathbb{R}^n \times \mathbb{R}^n$, dynamics of a classical system are described by Hamilton's equations:

$$\frac{\partial q_i}{\partial t} = \frac{\partial h}{\partial p_i}$$
$$\frac{\partial p_i}{\partial t} = -\frac{\partial h}{\partial q_i}$$

where h is a function representing energy, e.g., $h = p^2/2m + kq^2$.

This can be written invariantly as the vector field X_h associated to dh:

$$\omega(X_h, \cdot) = dh.$$

The equation

$$\gamma'(t) = X_h(\gamma(t))$$

locally looks like Hamilton's equations. (Here as in the following, may be many sign errors.)

Classical Systems: Dynamics

More generally, associate vector field X_f to function f via formula

$$\omega(X_f, \cdot) = df.$$

Symplectic structure is also used to define Poisson bracket of two smooth functions:

$$\{f, g\} = \omega(X_f, X_g)$$

Locally looks like

$$\sum(\frac{\partial f}{\partial p_i}\frac{\partial g}{\partial q_i} - \frac{\partial f}{\partial q_i}\frac{\partial g}{\partial p_i}).$$

Note that

$$\{p_i, q_j\} = \delta_{ij}$$

expressing a classical shadow of Heisenberg's commutation relations.

Classical Systems: Dynamics

Hamilton's equations for functions: Given f, can consider $f \circ \phi_{-t}$, where ϕ_t is the one-parameter family of diffeomorphisms induced by X_h.

Then Hamilton's equations imply

$$\frac{df}{dt} = \{f, h\}$$

In particular, $df/dt = 0$ iff $\{f, h\} = 0$. In this case, we say f is a *conserved quantity*.

The correspondence

$$\{f\}/\text{constants} \leftrightarrow \{X_f\}$$

is behind *Noether's theorem* relating symmetries and conserved quantities.

Quantum Systems

Space of states is a Hilbert space $\mathbb{H} = \{\psi\}$.

Role of functions played by self-adjoint operators

$$O : \mathbb{H} \longrightarrow \mathbb{H}.$$

Evaluating a function f at a point gets replaced by

$$\psi \mapsto \frac{\langle \psi, O\psi \rangle}{\langle \psi, \psi \rangle},$$

the expectation value of O in the state ψ.

Quantum Systems

Time evolution given by Schroedinger's equation:

$$\frac{d\psi}{dt} = \frac{1}{i\hbar} H\psi$$

for an operator H representing energy.

For observables O, if

$$O(t) := \exp(t\frac{i}{\hbar}H)O\exp(-t\frac{i}{\hbar}H),$$

then

$$\frac{dO}{dt} = \frac{i}{\hbar}[H, O].$$

Quantum Systems

Quantisation refers to a process

$$(M, \omega, h) \longrightarrow (\mathbb{H}, H).$$

It should come with a process for converting functions to operators:

$$a \mapsto \hat{a}$$

so that physical quantities like energy, momentum, position have quantum mechanical expectation values. Energy is especially important in practice.

Example: $(\mathbb{R} \times \mathbb{R}, \omega)$, $h = \frac{p^2}{2m} + kq^2$ is quantised via

$$\mathbb{H} = L^2(\mathbb{R}, \mathbb{C}),$$

$q \longrightarrow$ multiplication operator;
$p \longrightarrow \frac{-id}{dq}$;

$$h \longrightarrow H = \frac{1}{2m}\nabla^2 + kq^2.$$

Quantum Systems

Poisson brackets should go to commutators (have started to ignore constants and signs now):

$$\widehat{\{a, b\}} = i[\hat{a}, \hat{b}].$$

This is motivated by the Heisenberg commutation relations

$$[P_i, Q_j] = i\delta_{ij}$$

as well as the comparison between Hamilton's equation and Schroedinger's equation:

$$\frac{df}{dt} = \{f, h\}$$

$$\frac{d\hat{f}}{dt} = [\hat{f}, H]$$

Quantum Systems

More generally, $(\mathbb{R}^n \times \mathbb{R}^n = \{(q,p)\}, \omega)$ goes to $L^2(\mathbb{R}^n)$ where q_i goes to the multiplication operator and p_i goes to $i\frac{\partial}{\partial q_i}$. Quantise many h of the form

$$h = \frac{p^2}{2m} + V(q),$$

where $V(q)$ is interpreted as a potential energy.

However, this does *not* extend to

$$f \mapsto \hat{f}.$$

Works for linear and quadratic functions.

Quantum Systems

More precisely, finite-dimensional symplectic vector space (V, ω) can be quantised 'to first order'.

There is a Heisenberg Lie algebra

$$Heis(V, \omega) = \mathbb{R} \rtimes V$$

with Lie bracket

$$[(a, v), (b, w)] = (0, \omega(v, w)).$$

For any constant c (e.g., $i\hbar$), it has a unique represention

$$\mathbb{H}_c$$

on which $x \in \mathbb{R}$ acts via xc. This allows us to quantise linear functions and simple functions of the linear functions that often arise as Hamiltonians, e.g.,

$$h(p, q) = \frac{p^2}{2m} + V(q).$$

Quantum Systems

Two other constructions:

1. Can replace $L^2(\mathbb{R}^n)$ by $L^2(L)$, where $L \subset \mathbb{R}^n \times \mathbb{R}^n$ is any Lagrangian subspace.

Given such an L, almost canonical isomorphism

$$\mathscr{F}_L : L^2(L) \simeq L^2(\mathbb{R}^n)$$

2. Instead can use $L^2_{hol}(\mathbb{C}^n, \mu)$, where μ is a Gaussian measure.

This is naturally thought of as a completion of

$$\mathrm{Sym}(\mathbb{C}^n)$$

with respect to

$$\int |f(z)|^2 \exp(-|z|^2) dz d\bar{z}.$$

Then z_i acts by multiplication while $\bar{z}_i$ acts by d/dz_i with

$$z_i = p_i + iq_i, \bar{z}_i = p_i - iq_i.$$

Quantum Systems

Several advantages, including the fact that quantum states can be evaluated at a point on the classical state space.

This extends to the idea of Kaehler geometric quantisation: Given the symplectic $(\mathbb{S}, \omega)$, put on it a Kaehler structure. Construct a holomorphic line bundle $\mathscr{L} \longrightarrow \mathbb{S}$ with connection such that $c_1(\mathscr{L}, \nabla) = \omega$.

Note that $1 \in L^2_{hol}(\mathbb{C}^n, \mu)$ spans the unique line killed by the $\bar{z}_i$. Sometimes has the interpretation of a *vacuum* state. Thus, there is such a line spanned by $v_0 \in L^2(\mathbb{R}^n)$. In fact, for any Lagrangian subspace L, there is a line spanned by $v_L \in L^2(\mathbb{R}^n)$.

Quantum Systems

This is believed to work quite generally: When $(\mathbb{S}, \omega)$ is quantised to $\mathbb{H}$, there should be something like a cycle map

$$L \mapsto v_L \in \mathbb{H}$$

from Lagrangian submanifolds to lines in $\mathbb{H}$.

Essentially, if $\dim \mathbb{S} = 2n$ and

$$L = \{f_1 = 0, f_2 = 0, \cdots, f_n = 0\},$$

then v_L is defined by

$$\hat{f}_i v_L = 0.$$

II. Fields

Fields

Roughly speaking, to a physicist, everything is a field.

However, more precisely, there is a stack

$$\mathscr{S} \longrightarrow M$$

over the spacetime manifold and fields are its sections:

$$\mathscr{F} := \Gamma(M, \mathscr{S})$$

These comprise the so-called kinematics of a theory.

The 'dynamics' are usually formulated in terms of an *action*

$$L : \mathscr{F} \longrightarrow \mathbb{C}$$

The solutions of the Euler-Lagrange equation

$$EL(\mathscr{F}) := \{\phi \in \mathscr{F} \mid dL(\phi) = 0\}$$

make up the *classical state space*.

Fields

Example:

$$\mathscr{S} = T^*M \longrightarrow M$$

$$\mathscr{F} = \Omega^1(M)$$

$$L(A) = Max(A) := \int_M \|dA\|^2 dvol_M$$

In this case, the E-L equation amounts to the equation

$$*dA = 0.$$

Together with $d(dA) = 0$, we get Maxwell's equations (for the six components of dA).

Can generalise this to $\mathscr{F} = Conn(\mathscr{L})$, the space of connections on a line bundle $\mathscr{L}$.

Fields

Also of interest is

$$\mathscr{S} = (TM)^{\otimes n} \otimes (T^*M)^{\otimes m}$$

or natural subquotients.

For example, when $\mathscr{S} = Met(M) \subset (T^*M)^{\otimes 2}$,

$$EH(g) = \int_M R(g) dvol_g,$$

where $R(g)$ is the scalar curvature of g, is the *Einstein-Hilbert action*.

The E-L equation

$$dEH(g) = 0$$

is the vacuum Einstein equation.

Fields

Another important example is

$$\mathscr{F} = M \times \Sigma,$$

where Σ is another manifold.

This kind of theory is called a *sigma model*. In that case, fields are identified with maps

$$\phi : M \longrightarrow \Sigma.$$

If Σ is equipped with a metric, then

$$L(\phi) = \int_M \|d\phi\|^2 dvol_M$$

defines an action, whose critical points are called the harmonic maps from M to Σ. It's often the case that Σ is equipped with other fields that are used to define the action.

Fields

Why manifolds?

Any natural field theory defined on Minkowski space should be defined on manifolds, possibly with some extra structure (e.g., metrics, spin, complex).

Sometimes, no extra structure: *topological field theories*

Field theories typically come with a dimension, that of the spacetime manifolds on whose fields the theory can be defined. For example, most kinds of string theories are two-dimensional, while the original Chern-Simons theory is three-dimensional.

Fields

The action is typically a global integral of local functions of the fields:

$$\int_M \langle D\phi, D\phi \rangle + \text{h.o.t.}$$

But there are other important functions in field theory that are supported on subspaces $N \subset M$.

For example, if $\mathscr{A}(P)$ is the space of connections on a principal G-bundle P, then a map $K : S^1 \longrightarrow M$ together with a representation V of G determines a function

$$Wil(K, V) : \mathscr{A} \longrightarrow \mathbb{C},$$

$$Wil(K, V)(\nabla) = Tr(Hol_K(f^*(\nabla))|V)$$

called the Wilson loop function.

Fields: Quantisation

Given a d-dimensional field theory, can consider

$$EL(\mathscr{F}_{N\times\mathbb{R}})$$

where $\dim N = d - 1$. This typically has the structure of a symplectic manifold Σ given by considering initial conditions on N.

Thus, can quantise to a Hilbert space

$$Z(N)$$

viewed as a function of N:

$$N \mapsto EL(N \times \mathbb{R}) \mapsto Z(N).$$

Fields: Quantisation

Map is monoidal:

$$N \coprod N' \mapsto Z(N) \otimes Z(N').$$

$$N \coprod N' \mapsto EL([N \times \mathbb{R}] \coprod [N' \times \mathbb{R}])$$
$$= EL(N \times \mathbb{R}) \times EL(N' \times \mathbb{R}) \mapsto Z(N) \otimes Z(N').$$

Also,

$$Z(N^-) = Z(N)^*$$

(Change of orientation reverses sign of symplectic form, which quantises to the dual.)

Fields: Quantisation

If M is a d-manifold such that

$$\partial M = N_1^+ \coprod N_2^-,$$

i.e., a bordism from N_1 to N_2, then

$$EL(M) \longrightarrow EL(N_1 \times \mathbb{R}) \times EL(N_2 \times \mathbb{R})$$

is a Lagrangian correspondence.

Thus, get a vector

$$Z(M) \in Z(N_1)^* \otimes Z(N_2) = \mathrm{Hom}(Z(N_1), Z(N_2)).$$

Concise expression:

$$Z(M) \in Z(\partial M)$$

Fields: Quantisation

This represents the time evolution in the theory from a quantum state in $Z(N_1)$ to one in $Z(N_2)$.

Compatible with composition of bordisms: If

$$M = M \cup_{N_2} M',$$

where

$$\partial M = N_1^+ \coprod N_2^-, \quad \partial M' = N_2^+ \coprod N_2^-$$

then

$$Z(M) = Z(M_2) \circ Z(M_1).$$

Fields: Quantisation

We have

$$Z(\phi) = \mathbb{C}.$$

Thus, for a closed manifold M, get

$$Z(M) \in \mathbb{C}^* \otimes \mathbb{C} = \mathbb{C}.$$

Fields: Path Integrals

In reality, very difficult to define the $Z(M) \longrightarrow$ perturbation theory, renormalisation, etc.
However, there is an interpretation in terms of integrals like

$$Z(M) = \int_{\mathscr{F}} e^{-\frac{i}{\hbar}L(\phi)} d\phi$$

or

$$\langle f_1(\phi) f_2(\phi) \cdots f_k(\phi) \rangle$$
$$= \int_{\mathscr{F}} f_1(\phi) f_2(\phi) \cdots f_k(\phi) e^{-\frac{i}{\hbar}L(\phi)} d\phi$$

These are also typically ill-defined (modern Zeno's paradox), but tremendously useful guides for plausible computations and formulation of conjectures.

Fields: Path Integrals

For example, when M is compact Riemannian, $H^1(M) = 0$, $\mathscr{F} = \Omega^1_M$, and

$$L(\phi) = \int_M \|\phi\|^2 dvol_M = \int_M \langle \phi, \Delta\phi \rangle dvol_M,$$

then

$$\int_{\mathscr{F}} \exp(-\pi L(\phi)) d\phi = \frac{1}{\sqrt{\det \Delta_{1,M}}}$$

Another realm in which $Z(M)$, $Z(N)$, etc have been defined successfully is *topological quantum field theory*.

III. Arithmetic Topology

Arithmetic Topology

Let $\mathscr{O}_F$ be the ring of algebraic integers in a number field F and let

$$X := \mathrm{Spec}(\mathscr{O}_F).$$

It has many properties of a compact closed three-manifold.

If v is a maximal ideal in $\mathscr{O}_F$, then $k_v = \mathscr{O}_F/v$ is a finite field and the inclusion

$$\mathrm{Spec}(k_v) \hookrightarrow X$$

is analogous to the inclusion of a knot. The main reason is

$$\pi_1(\mathrm{Spec}(k_v)) = \widehat{\mathbb{Z}}.$$

The completion $\mathrm{Spec}(\mathscr{O}_{F,v})$ is like the tubular neighbourhood of the knot.

Arithmetic Topology

The completion F_v of F is the fraction field of $\mathscr{O}_{F,v}$, so that

$$\mathrm{Spec}(F_v) = \mathrm{Spec}(\mathscr{O}_{F,v}) \setminus v$$

is like the tubular neighbourhood with the knot deleted, which should be homotopic to a torus.

If B is a finite set of primes and $\mathscr{O}_{F,B}$ is the set of B-integers, then

$$X_B := \mathrm{Spec}(\mathscr{O}_{F,B}) = \mathrm{Spec}(\mathscr{O}_F) \setminus B$$

is like a three-manifold with boundary, the boundary having one torus component $\mathrm{Spec}(F_v)$ for each prime in B.

$$\partial X_B = \coprod_{v \in B} \mathrm{Spec}(F_v) \longrightarrow X_B \hookrightarrow X.$$

Arithmetic topology: Arithmetic Fields

Instead of the spaces themselves, can focus on moduli spaces

$$\mathscr{M}(X_B, R) := \{\rho : \pi_1(X_B) \longrightarrow R\} /\!\!/ R$$

for a p-adic Lie group R.

When $B = \phi$, should get $Z(X)$ via quantization of $\mathscr{M}(X, R)$.

In general, should get

$$Z(X_B) \in \otimes_{v \in B} Z(\mathrm{Spec}(F_v))$$

Arithmetic topology: Functions

A pair (x, V), where $x \in X_B$ and V is a finite-dimensional representation of R, defines a function

$$\rho \mapsto Tr(\rho(Fr_x)|V)$$

on $\mathscr{M}(X_B, R)$, an *arithmetic Wilson loop*.

For an AQFT, there should be an *arithmetic action* underlying all constructions.

Arithmetic Topology and TQFT?

A 3d *arithmetic TQFT* will naturally assign a number

$$Z(X)$$

to X.

A vector space

$$Z(F_v)$$

to F_v: functions on the space of boundary conditions.

and a vector

$$Z(X_B) \in Z(B) = \otimes_{v \in B} Z(F_v)$$

to X_B.

Arithmetic Topology and TQFT?

A 4d arithmetic TQFT will naturally assign a vector space

$$Z(X)$$

to X.

A category

$$Z(F_v)$$

to F_v.

and an object

$$Z(X_B) \in Z(B) = \otimes_{v \in B} Z(F_v)$$

to X_B.
Compare work of Ben-Zvi, Sakellaridis, and Venkatesh in the function field case.

There, also get vector

$$Z(\mathscr{C}) \in Z(X)$$

associated to a category $\mathscr{C}$ of boundary conditions for the theory.

IV. Examples of arithmetic actions

Arithmetic Actions

For technical reasons, we will assume throughout that F is totally complex.

Would like to define

$$S : \mathscr{M}(X, R) = H^1(\pi_1(X), R) \longrightarrow K$$

as well as path integrals

$$\int_{\rho\in\mathscr{M}(X,R)} \exp(-S(\rho))d\rho$$

possibly also on more general fields and/or related moduli spaces.

Arithmetic Duality

Let μ_n be the n-th roots of 1. Then

$$H^3(X, \mu_n) = H^3(\mathrm{Spec}(\mathscr{O}_F), \mu_n) \simeq \frac{1}{n}\mathbb{Z}/\mathbb{Z}.$$

This follows from

$$1 \longrightarrow \mu_n \longrightarrow \mathbb{G}_m \xrightarrow{(\cdot)^n} \mathbb{G}_m \longrightarrow 1,$$

leading to

$$H^3(X, \mu_n) \simeq H^3(X, \mathbb{G}_m)[n].$$

Meanwhile

$$H^3(X, \mathbb{G}_m) \simeq \mathbb{Q}/\mathbb{Z}.$$

Arithmetic Duality

Local class field theory:

$$H^2(F_v, \mathbb{G}_m) \simeq \mathbb{Q}/\mathbb{Z}$$

Global class field theory:

$$0 \longrightarrow H^2(F, \mathbb{G}_m) \xrightarrow{\text{loc}} \oplus_v H^2(F_v, \mathbb{G}_m) \xrightarrow{\Sigma} \mathbb{Q}/\mathbb{Z} \longrightarrow 0.$$

$$0 \longrightarrow H^2(X_B, \mathbb{G}_m) \xrightarrow{\text{loc}_B} \oplus_{v \in B} H^2(F_v, \mathbb{G}_m) \xrightarrow{\Sigma} \mathbb{Q}/\mathbb{Z} \longrightarrow 0.$$

But

$$\oplus_{v \in B} H^2(F_v, \mathbb{G}_m) = H^2(\partial X_B, \mathbb{G}_m),$$

so that

$$coker(\text{loc}_B) \simeq H^3_c(X_B, \mathbb{G}_m) \simeq H^3(X, \mathbb{G}_m).$$

Finite Arithmetic Chern-Simons Functionals

Assume $\mu_n \subset F$. Then

$$H^3(X, \mathbb{Z}/n) \simeq H^3(X, \mu_n) \simeq \frac{1}{n}\mathbb{Z}/\mathbb{Z},$$

so we get a map

$$\mathrm{inv} : H^3(\pi_1(X), \mathbb{Z}/n) \longrightarrow H^3(X, \mu_n) \simeq \frac{1}{n}\mathbb{Z}/\mathbb{Z}.$$

Let R have trivial $\pi_1(X)$-action. On the moduli space

$$\mathscr{M}(X, R) = \mathrm{Hom}(\pi_1(X), R)//R,$$

of continuous representations of $\pi_1(X)$, a Chern-Simons functional is defined as follows.

Finite Arithmetic Chern-Simons Functionals

The functional will depend on the choice of a cohomology class (a level)

$$c \in H^3(R, \mathbb{Z}/n).$$

Then

$$CS_c : \mathscr{M}(X, R) \longrightarrow \frac{1}{n}\mathbb{Z}/\mathbb{Z}$$

is defined by

$$\rho \mapsto \rho^*(c) \in H^3(\pi_1(X), \mathbb{Z}/n) \mapsto \mathrm{inv}(\rho^*(c)).$$

Finite Arithmetic Chern-Simons Functionals

Example:

Let $R = \mathbb{Z}/n$. Then

$$\mathscr{M}_X = Hom(\pi_1(X), \mathbb{Z}/n) = H^1_{et}(X, \mathbb{Z}/n).$$

Take $c \in H^3(R, \mathbb{Z}/n)$ to be given as

$$a \cup \delta a,$$

where $a \in H^1(R, \mathbb{Z}/n) = \mathrm{Hom}(\mathbb{Z}/n, \mathbb{Z}/n)$ is the class coming from the identity map, while

$$\delta : H^1(R, \mathbb{Z}/n) \longrightarrow H^2(R, \mathbb{Z}/n)$$

is the Bockstein map coming from the extension

$$0 \longrightarrow \mathbb{Z}/n \longrightarrow \mathbb{Z}/n^2 \longrightarrow \mathbb{Z}/n \longrightarrow 0.$$

Then

$$CS_{a\cup\delta a}(\rho) = \mathrm{inv}(\rho^*(a) \cup \rho^*(\delta a)).$$

BF-theory

Have a function

$$H^1(X, V) \times H^1(X, D(V)) \xrightarrow{BF} \frac{1}{n}\mathbb{Z}/\mathbb{Z}$$

defined by

$$(a, b) \mapsto \mathrm{inv}(da \cup b)$$

For this, V is a finite n-torsion group scheme that admits a suitable Bockstein map

$$d : H^1(X, V) \longrightarrow H^2(X, V)$$

and $D(V)$ is the Cartier dual.

Variant:

$$H^1(X_B, V) \times H^1_c(X_B, D(V)) \xrightarrow{BF} \frac{1}{n}\mathbb{Z}/\mathbb{Z}$$

Remark on arithmetic differentials

The Bockstein map

$$d : H^1(X, \mathbb{Z}/n) \longrightarrow H^2(X, \mathbb{Z}/n)$$

is very much like a differential. In crystalline cohomology of varieties over perfect fields of positive characteristic, Bockstein maps on crystalline cohomology sheaves are used to construct the De Rham-Witt complex.

In general, whenever you have an extension

$$0 \longrightarrow V \longrightarrow E \longrightarrow V \longrightarrow 0,$$

there is a differential

$$H^1(X, V) \longrightarrow H^2(X, V)$$

that can be used to construct arithmetic functionals.

More general differentials arise from deformation theory.

V. Some simple arithmetic path integrals

Arithmetic Chern-Simons

[Joint work with H. Chung, D. Kim, G. Pappas, J. Park, H. Yoo]

Let $n = p$, a prime and assume the Bockstein map

$$d : H^1(X, \mathbb{Z}/p) \longrightarrow H^2(X, \mathbb{Z}/p)$$

is an isomorphism.

Then

$$\sum_{\rho \in H^1(X,\mathbb{Z}/p)} \exp[2\pi i CS(\rho)]$$

$$= \sqrt{|Cl_X[p]|} \left(\frac{\det(d)}{p} \right) i^{\left[\frac{(p-1)^2 \dim(Cl_X[p])}{4}\right]}.$$

Arithmetic BF-theory: [Joint work with Magnus Carlson]

$$BF : H^1(X, \mu_n) \times H^1(X, \mathbb{Z}/n) \longrightarrow \frac{1}{n}\mathbb{Z}/\mathbb{Z},$$

$$(a, b) \mapsto \mathrm{inv}(da \cup b).$$

Proposition

For $n >> 0$,

$$\sum_{(a,b)\in H^1(X,\mu_n)\times H^1(X,\mathbb{Z}/n)} \exp(2\pi i BF(a, b))$$

$$= |Cl_X[n]||\mathcal{O}_X^\times/(\mathcal{O}_X^\times)^n|.$$

Compare with

$$\frac{L^{(r)}(Triv, 0)}{r!} = -|Cl_X|\|\det(\mathcal{O}_F^\times)\|$$

Arithmetic BF-theory

Similarly, if E is an elliptic curve with Neron model $\mathscr{E}$, then we have

$$0 \longrightarrow \mathscr{E}[n] \longrightarrow \mathscr{E}[n^2] \longrightarrow \mathscr{E}[n] \longrightarrow 0$$

for n coprime to the conductor and the orders of component groups of $\mathscr{E}$.

This gives us a map

$$BF : H^1(X, \mathscr{E}[n]) \times H^1(X, \mathscr{E}[n]) \longrightarrow \frac{1}{n}\mathbb{Z}/\mathbb{Z},$$

as

$$(a, b) \longrightarrow \mathrm{inv}(da \cup b).$$

Arithmetic *BF*-theory

Proposition

Assume Sha(E) is finite. For $n >> 0$ as above,

$$\sum_{(a,b)\in H^1(X,\mathscr{E}[n])\times H^1(X,\mathscr{E}[n])} \exp(2\pi i BF(a,b))$$

$$= |Ш(A)[n]||E(F)/n|^2.$$

Compare

$$\frac{L^{(r)}(T_pE,0)}{r!} = (\prod_v c_v)|Ш_E|\|\det(E(F))\|^2$$

Questions

In the paper 'Relative Langlands Duality' of Ben-Zvi, Sakellaridis, and Venkatesh, there is a construction of L-functions in a space

$$\Gamma(\mathscr{M}(X, R), \mathscr{L})$$

of sections of a line bundle on $\mathscr{M}(X, R)$ via a conjectural four-dimensional arithmetic TQFT.

Can one construct arithmetic L-functions via ATQFT?

Is there a QFT explanation of the arithmetic Langlands correspondence?