Forget-me-not particle

An explicit time integrator method to advect inertial particle with memory

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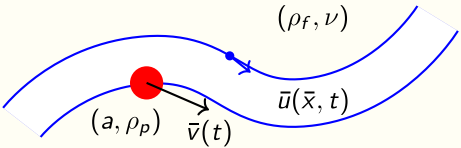
Inertial particle in viscous flow



The Maxey-Riley Equations (M&R, Gatignol,1983)

$$\dot{y} = v(t),$$

$$\dot{v} = \underbrace{-\alpha(v-u)}_{\text{STOKES' DRAG}} - \gamma \underbrace{\left(\frac{v_0 - u(y_0)}{\sqrt{\pi t}} + \int_0^t ds \frac{\dot{v} - \dot{u}}{\sqrt{\pi(t-s)}} \right)}_{\text{BASSET-BOUSSINESQ "HISTORY" FORCE}} + \underbrace{f(t, u(y(t)))}_{\text{NONLINEAR + OTHER FORCING}} .$$

$$R = \frac{2\rho_p/\rho_f + 1}{3}, \quad \alpha^{-1} = \frac{R}{3}\frac{a^2/\nu}{\tau_f}, \quad \gamma^{-1} = \frac{R}{3}\sqrt{\frac{a^2/\nu}{\tau_f}}, \quad f(t) = \frac{1}{R}\frac{Du}{Dt}_{y(t)} + \text{body force}$$

The Basset-Boussinesq History Force

$$\overbrace{\mathscr{H}(t)}^{\text{History force}} = \underbrace{\frac{v_0 - u(y_0)}{\sqrt{\pi t}}}_{\text{singular at t=0}} + \overbrace{\int_0^t d\tau}^{\text{non-local}} \underbrace{\frac{1}{\sqrt{\pi(t-\tau)}}}_{\substack{\text{Basset kernel:} \\ \text{time-dependent \&} \\ \text{slowly-decaying}}} \overbrace{\left(\dot{v}(\tau) - \dot{u}(\tau)\right)}^{\text{slip acceleration}}$$

It is a numerically expensive term to handle.

Building blocks for iterative numerics

The Idea: An **identical evolution rule** to advance into the future state given a current state

a model semilinear ODE

$$\dot{x}(t) = cx + N(x,t)$$

$$\downarrow$$

$$\underbrace{x(t+h)}_{\text{future state}} = \overbrace{e^{-ch}}^{\text{time-independent propagator}} \underbrace{x(t)}_{\text{current state}} + \int_0^h e^{-c(h-s)} N(t+s, x(t+s))ds$$

Cox & Matthews (2002), Hochbruck, et al (2010)

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- if $c = 0$, **nonlinear** integral can be dealt with a **standard time-integrator schemes** such as forward-Euler, Runge-Kutta, etc.
- if $N = 0$, we retrieve **exact** solution to the **linear problem**

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Our system, however, is: $\dot{x}(t) = (cx + H(t,x)) + N(x,t)$.

Search for the exact linear propagator

Idea is to **invert this entire linear operator** exactly,

$$q(t) := v - u,$$

$$\dot{q} + \alpha q + \gamma\Big(\frac{q_0}{\sqrt{\pi t}} + \int_0^t ds \frac{\dot{q}(s)}{\sqrt{\pi(t-s)}}\Big) = 0, \quad q(0) = q_0.$$

...then hopefully we find our linear propagator.

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Heat equation to the rescue!

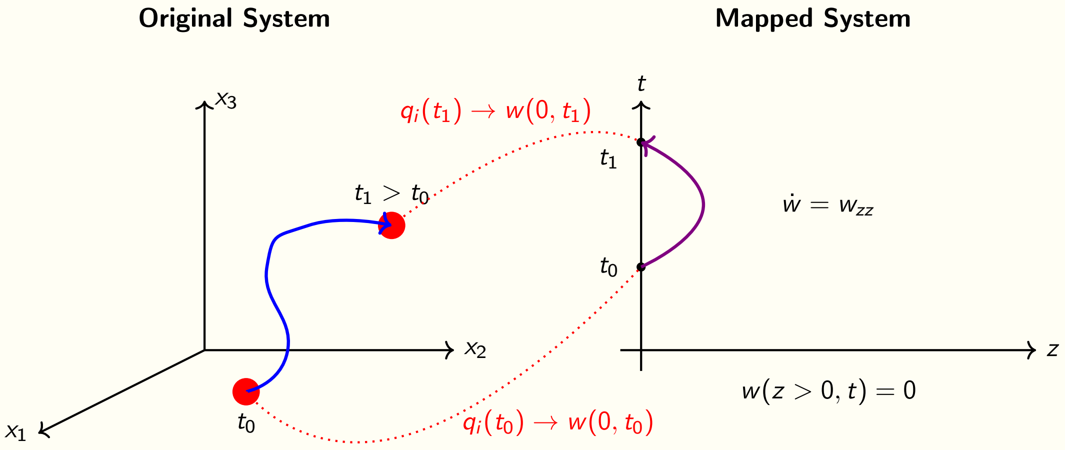
Consider a diffusing variable $w(z,t)$ in a $(1+1)$ domain. Then we have the following **Dirichlet-Neumann map -**

$$w_z(0,t) = -\frac{w(0,0)}{\sqrt{\pi t}} - \int_0^t ds \frac{\dot{w}(0,s)}{\sqrt{\pi(t-s)}}$$

Uncanny resemblance of the RHS to the history force.

A different viewpoint

SG Prasath, et al (2019)



$$\dot{q} + \alpha q + \gamma \underbrace{\left(\frac{q_0}{\sqrt{\pi t}} + \int_0^t ds \frac{\dot{q}(s)}{\sqrt{\pi(t-s)}} \right)}_{\text{nonlocal}} = f(q)$$

$$\dot{w}(0,t) + \alpha w(0,t) - \gamma \underbrace{w_z(0,t)}_{\text{local}} = f(w)$$

Rewritten system

$$q(t+h) = \chi(h;\alpha,\gamma)q(t) + \int_0^h \chi(h-s;\alpha,\gamma)f(s,q(s),u(y(s))) + H(t,q)$$

where

$$\chi(x;\alpha,\gamma) := -\frac{2}{\pi}\int_{-\infty}^{\infty} \frac{ike^{-k^2x}}{-k^2+ik\gamma+\alpha}$$

we have found our exact linear integrator!

Some properties of our integrator:

- $\chi(0) = 1, \chi(x \to \infty) = 0, \dot{\chi}(0) \to \infty.$
- $\chi(x_1)\chi(x_2) \neq \chi(x_1 + x_2)$

loss of semigroup behavior $\implies$ the extra term

We still don't have an identical rule for iteration.

Rewritten system

We have an evolution equation for $\mathscr{H}(t)$.

$$\chi(h;\alpha,\gamma)q(t) + H(t,q) = \# \int_{-\infty}^{\infty} k e^{-k^2 h} \mathscr{H}(t;k)dk$$

new state description: $(y(t), q(t)) \to (y(t), q(t), \mathscr{H}(t;k))$

$$y(t+h) = y(t) + \int_0^h (q(t+s) + u(y(t+s), t+s)),$$

$$q(t+h) = \chi(h)q(t) + \int_0^h \chi(h-s) f(q(t+s), t+s) + F(\mathscr{H}(t))$$

$$\mathscr{H}(t+h;k) = e^{-k^2 t} \mathscr{H}(t;k) + \int_0^h e^{-k^2(h-s)} \Big(q(t+s) + g(k) f(t+s, q(t+s)) \Big) \leftarrow \text{k\# equations}$$

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Incur additional task per iteration → fixed computational cost

Store one additional function $\mathscr{H}(t)$ at every step instead of entire history of states

Handling nonlinearity

Approximate the integral,

$$\int_0^h \chi(h-s) f(q(t+s), t+s) \approx h \sum_{m=1}^{\#stages} b_m(h) f(t+\alpha_m h)$$

For a choice of # of stages, the **error** is

$$\epsilon(t) = \left| \int_0^h \chi(h-s) f(q(t+s), t+s) - h \sum_{m=1}^{\#stages} b_m(h) f(t+\alpha_m h) \right| \sim \mathcal{O}(h^p)$$

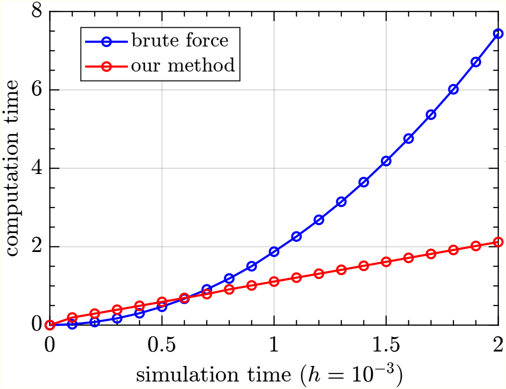
- **Taylor series expansion** of the nonlinear forcing about the state at t,
- **Extent of truncation** of the series $\implies$ **conditions on** $\{b_m\}$, and **order of accuracy,** p.

Computational performance

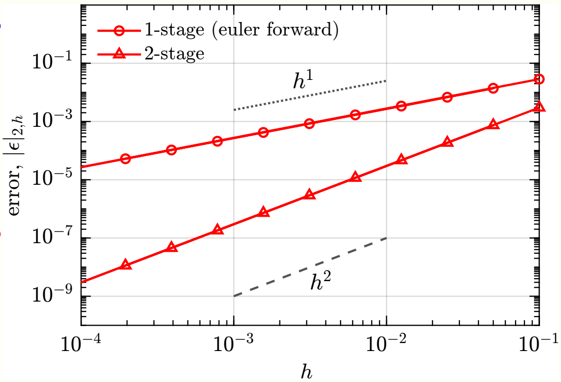
Model system

$$\dot{q} + \alpha q + \gamma\Big(\frac{q_0}{\sqrt{\pi t}} + \int_0^t \frac{\dot{q}(s)}{\sqrt{\pi(t-s)}}ds\Big) = \sin(\lambda t)$$

$$(\alpha = 1, \gamma = \sqrt{2}), \lambda = 5$$



(a) Computational time: **quadratic growth** (brute force) versus **linear growth** (our method)

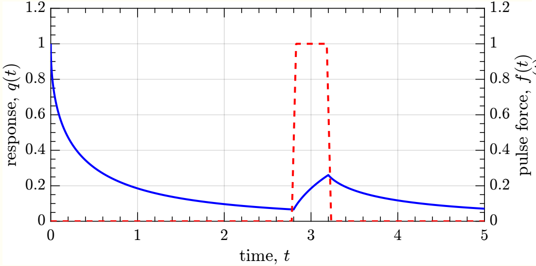


(b) **Error convergence rates** for different schemes

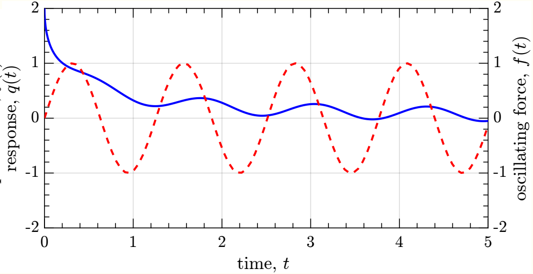
Summary

- An **explicit** iterative scheme of **tunable accuracy** with the **history term** preserving the **common syntax** of numerical methods.
- **Mitigated** the **increasing computational** and **memory storage costs**.
- With the computational issues out of the way, this will **facilitate long-time numerical experiments of particle with history effects**, possibly **help understand role of history force in turbulence**.

Some numerical experiments with linear forcing



(c) response of particle with memory to pulse force



(d) particle in an oscillating (in time) flow