3/5/23 ICTS Bangalore

Duality for C^*BG III: a structural approach

We recall for $R \to k$ (so for a local ring & its residue field)

<u>Def^n</u>: (a) R is <u>Gorenstein of shift a</u> if $\mathrm{Hom}_R(k,R) \simeq \Sigma^a k$

(b) R has <u>Gorenstein duality of shift a</u> if $\mathrm{Cell}_k R \simeq \Sigma^a R^\vee$.

If R is a k-algebra & (like C^*BG) connective or coconnective with finite dimension in each degree then $R^\vee \simeq \mathrm{Hom}_k(R,k)$

<u>Remark</u>: If R has GorD^a then it is Gor^a.

because
$$\mathrm{Hom}_R(k,R) \simeq \mathrm{Hom}_R(k, \mathrm{Cell}_k R) \simeq \mathrm{Hom}_R(k, \Sigma^a R^\vee) = \mathrm{Hom}_R(k, \Sigma^a \mathrm{Hom}_k(R,k)) \simeq \mathrm{Hom}_R(R \otimes_k k, \Sigma^a k) \simeq \Sigma^a k$$

In commutative algebra $\mathrm{Gor}^a \Rightarrow \mathrm{GorD}^a$, but not in general.

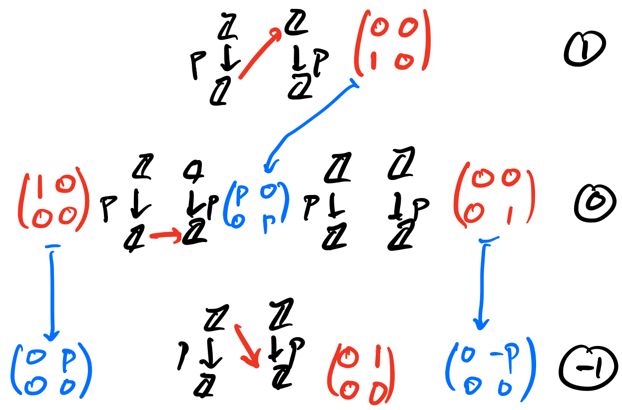
<u>Example Context 2</u> The ambient category is chain complexes of ab gps

R a CDGA eg $R = \mathbb{Z}$

k a field $k = \mathbb{Z}/p$

$\mathcal{E} = \mathrm{Hom}_R(k,k)$ $\mathcal{E} = \mathrm{Hom}(\mathbb{Z}/p, \mathbb{Z}/p) = M_2\mathbb{Z}$.

$\pi_*\mathcal{E} \Leftarrow \mathrm{Ext}^{**}_{\pi_*R}(k,k)$ $\pi_*\mathcal{E} = \mathrm{Ext}^*_{\mathbb{Z}}(\mathbb{Z}/p, \mathbb{Z}/p) = \Lambda_{\mathbb{Z}/p}(\overset{(-1)}{\tau})$



<u>Example Context 2</u>: The ambient category is spectra

R commutative ring spectrum

k an Eilenberg-MacLane spectrum of a field. (H omitted!)

$\mathcal{E} = Hom_R(k,k)$ (a noncommutative ring spectrum)

For a space X

$R = C^*X := map(X, k)$

(notation since $\pi_* C^*X = [X_+, Hk]$
$= H^*(X;k)$)

$\mathcal{E} = Hom_{C^*X}(k,k)$
$\simeq C_*(\Omega X)$ — Eilenberg-Moore (Dwyer, Shipley)

<u>Provided</u>: Either X 1-conn
Or X is 0-conn, char $k = p$
& X is p-complete
& $\pi_1 X$ is a finite p-group

<u>Note</u>: If $X = BG$ for a p-gp this is satisfied
& more generally if
$X = BG^{\wedge}_p$
since $\pi_1 BG^{\wedge}_p = G/O^pG$.

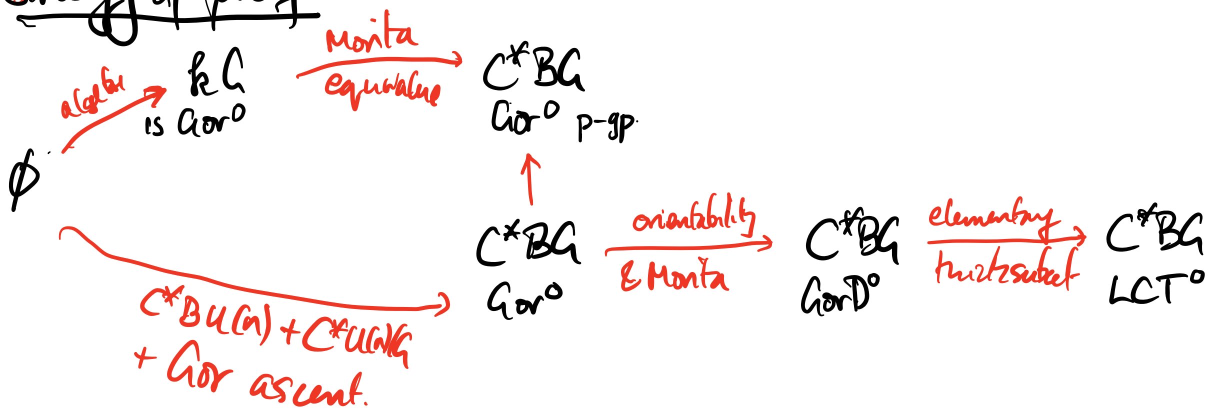
Theorem: C^*BG is Gorenstein of shift 0 [$\mathrm{Hom}_{C^*BG}(k, C^*BG) \simeq k$]

C^*BG has Gorenstein duality of shift 0 [$\mathrm{Cell}_k C^*BG \simeq C_*BG$]

H^*BG has a LCSS. [$\mathrm{Cell}_k C^*BG \simeq \Gamma_{\mathfrak{m}} C^*BG$]

$H^*_{\mathfrak{m}}(H^*BG) \Rightarrow H_*BG$ as above

__Strategy of proof__



Two of these methods involve Morita theory, so we do that next.

Theorem: If k is small then $k\text{-cell-}R\text{-mod} \simeq \text{mod-}\mathcal{E}$.

Proof:

$$M \overset{E}{\longmapsto} \operatorname{Hom}_R(k, M)$$

$$X \otimes_{\mathcal{E}} k \overset{T}{\longleftarrow\!\!|} X$$

counit $TE = \operatorname{Hom}_R(k,M) \otimes_{\mathcal{E}} k \longrightarrow M$

k-cellular ✓ Clearly $\simeq$ if $M = k$ & hence for M built from k

$$\left(\begin{array}{l} \operatorname{Hom}_R(k, \operatorname{Hom}_R(k,M) \otimes_{\mathcal{E}} k) \longrightarrow \operatorname{Hom}_R(k,M) \\ \quad \cong \\ \operatorname{Hom}_{\mathcal{E}}(\operatorname{Hom}_R(k,k), \operatorname{Hom}_R(k,M)) \end{array} \right)$$

$$X \longrightarrow \operatorname{Hom}_R(k, X \otimes_{\mathcal{E}} k) = ETX \qquad //$$

Example: $R = \mathbb{Z}$

$k = \mathbb{Z}/p$; $k \simeq (\mathbb{Z} \overset{p}{\leftarrow} \mathbb{Z})$ & so $\mathcal{E} = \operatorname{Hom}_{\mathbb{Z}}(k, k) = M_2(\mathbb{Z})$

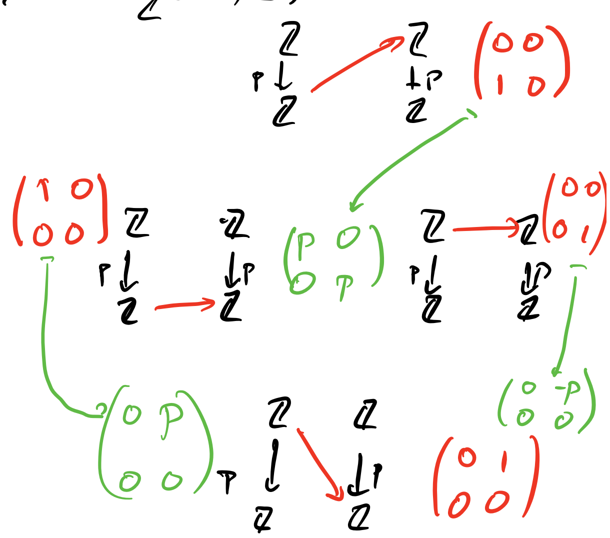
$p\text{-tors-}\mathbb{Z}\text{-mod} \simeq \text{mod-}\mathcal{E}$.

$\mathbb{Z}/p^\infty \longmapsto \operatorname{Hom}_{\mathbb{Z}}(\mathbb{Z}/p, \mathbb{Z}/p^\infty) = \mathbb{Z}/p$

$\operatorname{Hom}_{\mathbb{Z}}(\mathbb{Z}/p^\infty, \mathbb{Z}/p^\infty) \simeq \operatorname{Hom}_{\mathcal{E}}(\mathbb{Z}/p, \mathbb{Z}/p)$

$\cong$

$\hat{\mathbb{Z}}_p$



Theorem: The map $\varepsilon : \mathrm{Hom}_R(k, M) \otimes_{\mathcal{E}} k \longrightarrow M$ is k-cellular approximation.

Remark: This continues to hold if k is just proxy small (ie $R \models K \Rightarrow k$.)

$$\begin{array}{c} K \\ \uparrow \\ k \end{array}$$

Proof: $\mathrm{Hom}_R(K, \mathrm{Hom}_R(k, M) \otimes_{\mathcal{E}} k) \longrightarrow \mathrm{Hom}_R(K, M)$

$$\simeq$$

$$\mathrm{Hom}_{\mathcal{E}}(\mathrm{Hom}(k, K), \mathrm{Hom}(k, M))$$

[But now consider

$$\mathrm{Hom}_R(X, M) \to \mathrm{Hom}_{\mathcal{E}}(\mathrm{Hom}(k, X), \mathrm{Hom}(k, M))$$

Equiv for $X = k$, hence for $X = K$]

//

Remark: Note also there is a double centralser map $R \longrightarrow \mathrm{Hom}_{\mathcal{E}}(k, k)$

Provided k is proxy small, this is well behaved & has the character of a completion

Example 1: If $R \to k$ is a classical local ring then $R \longrightarrow \mathrm{Hom}_{\mathcal{E}}(k, k)$ is m-adic completion

Example 2: If $R = C^* X$ & X satisfies one of the two conditions then

$$C^*X \xrightarrow{\simeq} \mathrm{Hom}_{C_*\Omega X}(k,k)$$

by Rothenberg-Steenrod

Proof of C^*BG Gor^0 for G a p-gp

$$R^\vee \otimes_R k \simeq k \otimes_{\mathcal{E}} k \otimes_R k \simeq k \otimes \mathcal{E}^\vee$$

Now apply $\mathrm{Hom}_k(\ ,k)$

$$\mathrm{Hom}_R(k, R^{\vee\vee}) \simeq \mathrm{Hom}_k(R^\vee \otimes_R k, k) \simeq \mathrm{Hom}_k(k \otimes \mathcal{E}^\vee, k)$$

$$\mathrm{Hom}_R(k,R) \simeq \mathrm{Hom}_R(k,R^{\vee\vee}), \qquad \mathrm{Hom}_k(k \otimes \mathcal{E}^\vee, k) \simeq \mathrm{Hom}_R(k,\mathcal{E})$$

Since G is a p-gp $C_*(\Omega BG^\wedge_p) = C_*(\Omega BG) = C_*(G) = kG$ //

Proof that $\mathrm{Gor}^0 \Rightarrow \mathrm{GorD}^0$ for C^*BG

Supposing R is Gor^a we have

$$\mathrm{Hom}_R(k,R) \simeq \Sigma^a k$$

$$\mathrm{Hom}_R(k, \mathrm{Cell}_k R) \simeq \mathrm{Hom}_R(k,R), \qquad \Sigma^a k \simeq \mathrm{Hom}_R(k, \Sigma^a R^\vee)$$

Note that $R^\vee = C_*BG = \varinjlim C_*BG^{(n)}$

So $R^\vee$ is already cellular.

This is an equivalence of R-modules, but if it were an equivalence of right $\mathcal{E}$-modules we could apply $\otimes_{\mathcal{E}} k$ & have

$$\mathrm{Cell}_k R \simeq \mathrm{Hom}_R(k, \mathrm{Cell}_k R) \otimes_{\mathcal{E}} k \simeq \mathrm{Hom}_R(k, \Sigma^a R^\vee) \otimes_{\mathcal{E}} k \simeq \Sigma^a R^\vee$$

as required.

Finally we observe there is a unique $\mathcal{E}$ action on k since

$$\pi_* \mathcal{E} = H_* \Omega(BG_p^\wedge)$$

is connective with $\pi_0 \mathcal{E} = H_0 \Omega BG_p^\wedge = k[\pi_1 BG_p^\wedge] = k[G/O^pG]$ (finite p-gp)

Now k is of characteristic p so $\exists !$ action of $k[G/O^pG]$ //.

Remark: Strickland's direct approach

To show Gorenstein duality it suffices to show

$$\mathrm{Hom}_{C^*BG}(k, \hat{C}^*BG) \simeq 0$$

since then

$$\mathrm{Hom}_{C^*BG}(k, C_*BG^{ad}) \xrightarrow{\simeq} \mathrm{Hom}_{C^*BG}(k, C^*BG)$$

Σ if ad orientable

$$\mathrm{Hom}_{C^*BG}(k, \Sigma^{ad} C^*BG^\vee)$$

$$\| $$

$$\Sigma^{ad} k.$$

//.

[It seems that the proof of this uses stable Koszul-type constructions of C^*BG, $\hat{C}^*BG$ So not clear it is more elementary]

[Lecture 4 starts here]

Proof that C^*BG is Gorenstein. We choose n large enough that there is a faithful representation $G \hookrightarrow SU(n)$.

First note $U(n)/G$ is an orientable manifold of dim^n $n^2 = \dim U(n)$ so that $C^* U(n)/G$ is Gorenstein of shift $-n^2$.

Next note $C^*BU(n)$ is Gorenstein of shift $+2(1+2+\cdots+n)+n$
$= +n(n-1)+n$
$= +n^2$.

[Note: π_*R Gorenstein $\Rightarrow R$ is Gorenstein by virtue of the SS $\mathrm{Ext}^*_{\pi_*R}(k, \pi_*R) \Rightarrow \pi_*\mathrm{Hom}_R(k, R)$.]

Now we have a cofibre sequence of algebras

$$C^*BU(n) \longrightarrow C^*BG \longrightarrow C^*BG \otimes_{C^*BU(n)} k$$

$\boxed{+n^2}$ $\qquad \simeq$ Eilenberg–Moore

$$C^*U(n)/G$$

$\boxed{-n^2}$

It remains to prove that we can deduce that the Gorenstein condition ascends from the base to the fibre.

<u>Lemma</u> (Gorenstein Ascent): Provided Q is fin dim^l in any cofibre sequence $S \longrightarrow R \longrightarrow R \otimes_S k =: Q$

we have

$$\mathrm{Hom}_R(k, R) \simeq \mathrm{Hom}_Q(k, \mathrm{Hom}_S(k, S) \otimes_k Q)$$

Hence if S is Gor^s & Q is Gor^q

then R is Gor^{s+q}. //.

//

<u>Remark</u>: This proves C^*BG is Gor^d for $d = \dim(G)$ provided T_eG is orientable in that $\pi_0 G$ acts trivially on $H^d(S^{T_eG}; k)$.

__Algebraic cellularization__. We need to show $\mathrm{Cell}_k R \simeq \Gamma_{\mathfrak{m}} R$ for $R = C^* BG$.

First of all we recall the definition of $\Gamma_I R$ for $I = (x_1, \dots, x_n) \lhd R_*$.

As before we define
$$K_\infty(x) = \mathrm{fibre}(R \to R[\tfrac{1}{x}])$$
& $K_\infty(\underline{x}) = K_\infty(x_1, \dots, x_n) = K_\infty(x_1) \otimes_R \cdots \otimes_R K_\infty(x_n)$

Now take
$$\Gamma_I M = K_\infty(\underline{x}) \otimes_R M .$$

The construction instantly gives a SS

__Lemma__:
$$H_I^*(R_*, M_*) \Longrightarrow \pi_* \Gamma_I M$$

Hence $\pi_* \Gamma_I M$ is I-power torsion & only depends on $\sqrt{I}$ //

__Lemma__: $\Gamma_I M \longrightarrow M$ is $K(\underline{x})$ cellularization.

__Proof__: First note $K_\infty(I)$ is $K(\underline{x})$ cellular.

$$\begin{array}{ccccc}
R & \xrightarrow{x} & R & \quad & K(x) \\
\| & & \downarrow x & & \downarrow \\
R & \xrightarrow{x^2} & R & & K(x^2) \\
\| & & \downarrow x & & \downarrow \\
R & \xrightarrow{x^3} & R & & K(x^3) \\
\| & & \downarrow x & & \downarrow \\
 & & & & \\
R & \longrightarrow & R[\frac{1}{x}] & & K_\infty(x)
\end{array}$$

Hence $K_\infty(x) = \underrightarrow{\mathrm{holim}}_m\, K(x^m)$.

To see it's a $K(\underline{x})$ equivalence, note Cofibre $K_\infty(I) \to R$ has a finite filtration with subquotients $R[\frac{1}{x_{i_1} \cdots x_{i_r}}]$ & $[K(\underline{x}), R[\frac{1}{x_{i_1} \cdots}]] = 0$ //

Now take $I = \mathfrak{m} = H^{>0}(BG) = \pi_{<0} C^* BG$.

<u>Proposition:</u> $k \models K(\underline{x}) \vdash k$. (ie k is proxy small)

& hence $Cell_k(M) \simeq Cell_{K(\underline{x})}(M) \simeq \Gamma_I M$.

<u>Proof</u>: $H_* K(\underline{x}) \Leftarrow H_*(K(\underline{x}; R_*))$ is a finite dim^l vector space so

$$k \models K(\underline{x})$$

For $K(\underline{x}) \vdash k$

we note $k \otimes_R K_\infty(\underline{x}) \simeq k$ (since $k \otimes_R R[\frac{1}{x_i}] \simeq 0$).

Thus

$$K(\underline{x}) \vdash K_\infty(\underline{x}) = R \otimes_R K_\infty(\underline{x}) \vdash k \otimes_R K_\infty(\underline{x}) \simeq k \quad //$$

Extensions:

① For G compact Lie if $ad(G)$ is orientable over k C^*BG is Gor^d where $d = \dim(G)$

② For G an orientable virtual Poincaré duality group with geometric const^n of classifying space (eg S-arithmetic) C^*BG is Gor^e where $e = -\mathrm{vcd}(G)$.

③ Formal framework for the profinite case doesn't seem to be available.

NPS	$K'(n) = v_n^{-1}\hat{P}(n)$
$\langle BP \wedge K'(n) \rangle = \langle K(n) \rangle$	

Further examples

Gorenstein Spaces 1988.

Ⓐ If $k = \mathbb{Q}$, the notion of Gorenstein space is due to Félix/Halperin/Thomas & there are many examples, ← any finite Postnikov system. though they focused on the elliptic case. However the Gorenstein duality statement is still of interest

Example: $S^3 \times S^3 \longrightarrow X \longrightarrow \mathbb{C}P^\infty \times \mathbb{C}P^\infty$ classified by u^2, uv.

$H^*X = \mathbb{Q}[u,v,p]/(u^2, uv, up, p^2)$ 1 0 2 0 1 1 1 1 1 ...

Gor of shift -6 $+2$ $= \ell - 4$

One may check $H^0_m = \Sigma_2 \mathbb{Q}$, $H^1_m = \mathbb{Q}[v]^\vee \otimes (\Sigma^{-3}\mathbb{Q} \oplus \Sigma^2\mathbb{Q})$

$0 \to H^1_m H^*X \to \Sigma^{-4} H^*(A)^\vee \longrightarrow \Sigma^{-2}\mathbb{Q} \to 0.$

Ⓑ Of course if R_* is Gorenstein of shift R, so is R via the SS $\mathrm{Ext}_{R_*}(k, R_*) \Rightarrow \pi_* \mathrm{Hom}_R(k, R)$.

For example this applies to ku, $BP\langle n \rangle$, & many others.

Specifically $\mathbb{Z} \to \mathbb{F}_p$ is Gorenstein of shift -1

$ku \longrightarrow \mathbb{Z}$ -3

$ku \to \mathbb{F}_p$ is Gorenstein of shift -4

Gorenstein transfer: Given $S \longrightarrow R \longrightarrow k$ with $Hom_S(R,S) \simeq \Sigma^{-b} R$

We can transfer Gorenstein property between R & S.

$$Hom_R(k,R) \simeq \Sigma^{-b} Hom_R(k, Hom_S(R,S))$$
$$\simeq \Sigma^{-b} Hom_S(k,S)$$

Thus R $Gor^a \Leftrightarrow S$ Gor^{a-b}

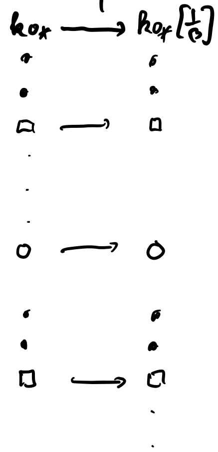
There is a well known cofibre sequence $\Sigma^1 ko \xrightarrow{\eta} ko \longrightarrow ku$ (sometimes known as connective Wood's Theorem since it states $ko_*(\Sigma^{-2}\mathbb{C}P^2) = ku_*$)

Hence we may take $S = ko$
$R = ku$

$$(\Sigma^{+1} ko, ko) \leftarrow (ko, ko) \leftarrow (ku, ko)$$
$$\Sigma^{-1} ko \qquad ko$$

& see $Hom_{ko}(ku, ko) \simeq \Sigma^{-2} ku$

& hence $ko \to \mathbb{F}_p$ is Gor of shift $-4-2=-6$



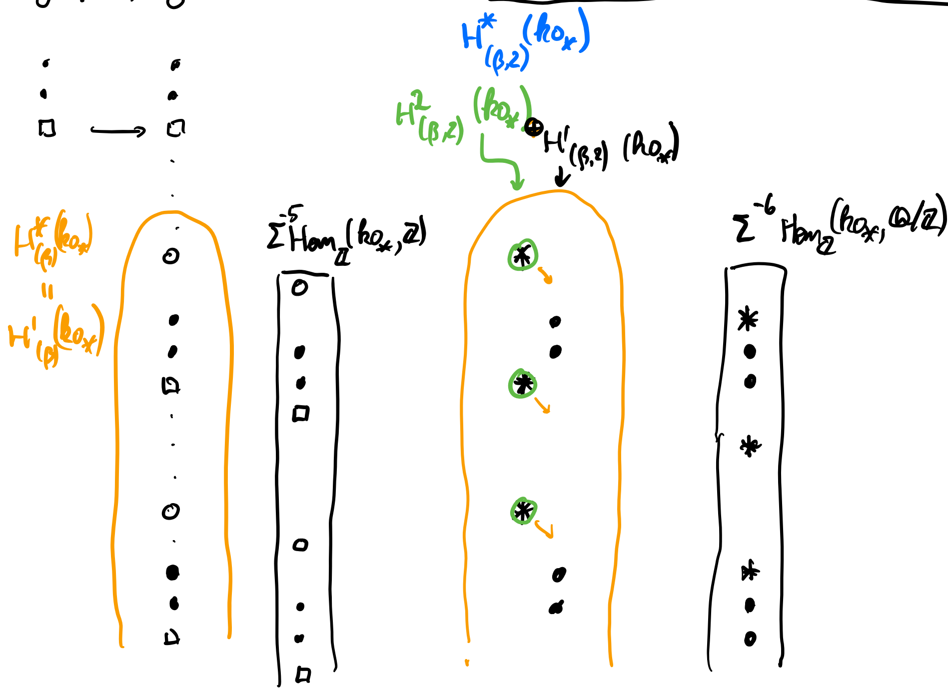
Brown-Comenetz & Anderson duality

If J is injective $\exists J^X$ (aka $I_J X$)

$[T, J^X]_* = Hom_{\mathbb{Z}}(\pi_* T \wedge X, J)$

Hence $\mathbb{Z}^X \to \mathbb{Q}^X \to \mathbb{Q}/\mathbb{Z}^X$

$0 \to Ext(\pi_{*-1}(X), \mathbb{Z}) \to \pi_* \mathbb{Z}^X \to Hom_{\mathbb{Z}}(\pi_*(X), \mathbb{Z}) \to 0$



Note: $\mathrm{Hom}_{ko}(\mathbb{Z}, ko) \simeq \Sigma^{-6}\mathbb{Z} \simeq \mathrm{Hom}_{ko}(\mathbb{Z}, \Sigma^{-5}\mathbb{Z}^{ko})$

$\mathrm{Hom}_{ko}(\mathbb{Z}, \mathrm{Cell}_{\mathbb{Z}} ko) \qquad \mathrm{Hom}_{ko}(\mathbb{Z}, \Sigma^{-5}\mathrm{cell}\,\mathbb{Z}^{ko})$

$$\Gamma_\beta ko \simeq \mathrm{Cell}_{\mathbb{Z}} ko \simeq \Sigma^{-5}\mathbb{Z}^{ko}$$

$\mathrm{Hom}_{ko}(\mathbb{F}_2, ko) \simeq \Sigma^{-6}\mathbb{F}_2 \simeq \mathrm{Hom}_{ko}(\mathbb{F}_2, \Sigma^{-5}\mathbb{Q}/\mathbb{Z}^{ko})$

$\mathrm{Hom}_{ko}(\mathbb{F}_2, \mathrm{Cell}_{\mathbb{F}_2} ko) \qquad \mathrm{Hom}_{ko}(\mathbb{F}_2, \Sigma^{-5}\mathrm{cell}_{\mathbb{F}_2}\mathbb{Q}/\mathbb{Z}^{ko})$

$$\Gamma_{\beta,2} ko \simeq \mathrm{Cell}_{\mathbb{F}_2} ko \simeq \Sigma^{-6}(\mathbb{Z}/2^\infty)^{ko}.$$

Note: $H^*_{\beta,2}(M) \Leftarrow H^*_2 H^*_\beta M$

Comment: The relationship between these & $H^*_{\mathfrak{m}}(H^{**}A(1)) \Rightarrow \Sigma^{(-6,0)} H^{**}(A(1))^\vee$ needs elucidation.

Example: $tmf \to \mathbb{Z}$ is Gor^{-22} & has GorD^{-22}

$$\Gamma_{(\beta,M)} tmf \simeq \mathrm{Cell}_{\mathbb{Z}} tmf \simeq \Sigma^{-22}\mathbb{Z}^{tmf}$$

$tmf \to \mathbb{F}_p$ is Gor^{-23} & has GorD^{-23}

$$\Gamma_{(2,\beta,M)} tmf \simeq \mathrm{Cell}_{\mathbb{F}_2} tmf \simeq \Sigma^{-23}(\mathbb{Z}/2^\infty)^{tmf}$$

$|\beta| = 8$

$|M| = 192$

Comment: The relationship between this & $H^*_{\mathfrak{m}}(H^{**}A(1)) \Rightarrow \Sigma^{(-23,0)} H^{**}(A(1))^\vee$ needs elucidation.

Example: If S is regular Gorenstein of shift a augmented over $k = \mathbb{F}_p$

then $THH(S; \mathbb{F}_p)$ is Gorenstein of shift $-a-3$

Proof: $S \to k \to Q$ shows Q is Gor^{-a}.

Now Dundas lemma gives $Q \to THH(S;k) \to THH(k)$

$-a \qquad\qquad -3$ (by Bökstedt)

The hypotheses show $Q = \varprojlim Q^{(n)}$ gives an approximation to $THH(S; k)$ through fibrations with ascent //.