Interacting QFT on Causal Sets

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Outline

1. Background
2. Free quantum Field theory on Causal sets
3. A diagrammatic expansion for in-in correlators

Background

A causal set is defined as a locally finite partially ordered set.
Take a pair $(\mathcal{C}, \preceq)$ where $\mathcal{C}$ is a set with a partial order relation $\preceq$ which satisfies:

- $\forall\ a \in \mathcal{C},\ \ a \preceq a$ Reflexivity
- $\forall\ a, b \in \mathcal{C},\ \ a \preceq b \preceq a \Rightarrow a = b$ Acyclicity
- $\forall\ a, b, c \in \mathcal{C},\ \ a \preceq b \preceq c \Rightarrow a \preceq c$ Transitivity
- $\forall\ a, c \in \mathcal{C}, |\ [a, c]\ | < \infty$, where the set $[a, c] := \{b \in \mathcal{C}|\ a \preceq b \preceq c\}$ is a causal interval and $|X|$ is the cardinality of a set X. Locally finiteness

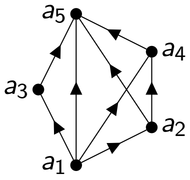
MOTTO:

"Order + Number = Geometry"

Representation

$$\mathcal{C} = \{a_1, a_2, a_3, a_4, a_5\}$$

$$\begin{gathered} a_1 \preceq a_1, a_1 \preceq a_2, a_1 \preceq a_3, a_1 \preceq a_4, a_1 \preceq a_5, \\ a_2 \preceq a_2, a_2 \preceq a_4, a_2 \preceq a_5, \\ a_3 \preceq a_3, a_3 \preceq a_5, \\ a_4 \preceq a_4, a_4 \preceq a_5, a_5 \preceq a_5, \end{gathered}$$



$$C = \begin{pmatrix} 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \quad L = \begin{pmatrix} 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Sprinkling: generating causal sets from continuum

Select points in (M, g) uniformly at random via a Poisson distribution and impose a partial ordering via the induced spacetime causality relation.

$$P(|\mathcal{C} \cap V| = n) = \frac{(\rho V)^n e^{-\rho V}}{n!}, \tag{1}$$

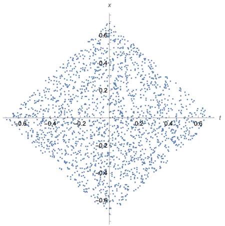


Figure: CS with 1000 elements approximated by a portion of $1+1$ Minkowski

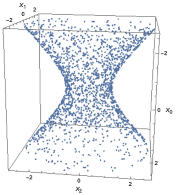


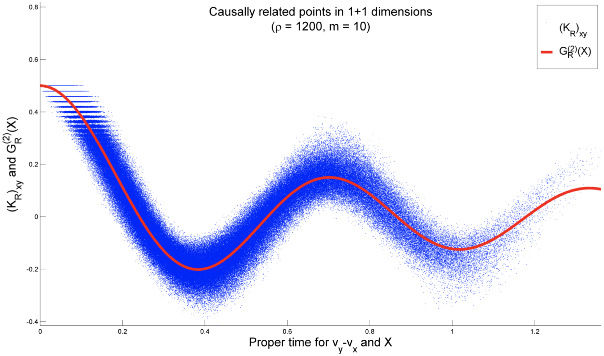
Figure: CS with 1000 elements approximated by dS_2

This process is Lorentz invariant! only uses the invariant volume measure.

Free QFT on Causal sets

- Discrete structure $\implies$ No tangent space $\implies$ No equation of motion.
- Start with the retarded propagators K^R_{xy}: hops and stops model at each element of the trajectory.

$$K_R^{(2D)} := \frac{1}{2}C(\mathbb{I} + \frac{1}{2}\frac{m^2}{\rho}C)^{-1} \tag{2}$$



Sorkin-Johnston vacuum

- Associate a field operator $\phi(x)$ to each $x \in \mathcal{C}$ and impose the Peierls bracket,

$$[\phi(x), \phi(y)] = i\Delta_{xy} = i(K^R_{xy} - K^R_{yx}). \tag{3}$$

Note: $[\phi(x), \phi(y)] = 0$ if $x \natural y$.

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- $i\Delta$ is **skew-symmetric** and **Hermitian** $\implies$ The eigenvectors of $i\Delta$ can be used to define a Gaussian vacuum state $|0\rangle$ by requiring that,

$$\langle 0|\phi(x)\phi(y)|0\rangle = Pos(i\Delta) \tag{4}$$

  This is the Sorkin-Johnston (SJ) vacuum : UNIQUE on curved backgrounds as well!

Free Decoherence functional

Any expectation values can be computed in the path integral approach by integrating over all *pairs* of field configurations, $\xi, \bar{\xi} \in \mathbb{R}^N$. Denote the space of fields as $\mathcal{F} \cong \mathbb{R}^N$.

Given a causet $(\mathcal{C}, \preceq)$ with $|\mathcal{C}| = N$, the *measure* of this integral is the decoherence functional

$$\begin{aligned} D_0(\xi, \bar{\xi}) &= \langle 0|\delta(\phi_1 - \bar{\xi}_1)\delta(\phi_2 - \bar{\xi}_2)...\delta(\phi_N - \bar{\xi}_N)\delta(\phi_N - \xi_N)...\delta(\phi_2 - \xi_2)\delta(\phi_1 - \xi_1)|0\rangle \\ &= \delta(L(\xi, \bar{\xi}))\, e^{i\Delta S(\xi, \bar{\xi})} \end{aligned}$$

for i = 1 . . . N is any *natural labelling* of all the elements $x \in \mathcal{C}$, i.e $x_j \prec x_k \Rightarrow j < k$ and for some linear function L and quadratic function $\Delta S(\xi, \bar{\xi})$.
Analogous to $\Delta S[\gamma, \gamma'] = S[\gamma] - S[\gamma']$ for continuum quantum theory.
The delta-functions in $D_0(\xi, \bar{\xi})$ act to causally/anti-causally order the corresponding functions of the field operators

Causal Ordering

Define the causal ordering operator C whose action on a product of two fields is,

$$C[\phi(x)\phi(y)] = \begin{cases} \phi(x)\phi(y) & \text{if } x \succ y \\ \phi(y)\phi(x) & \text{if } x \prec y, \end{cases} \tag{5}$$

- For a spacelike pair of points $x\natural y$: $C[\phi(x)\phi(y)] = \phi(x)\phi(y) = \phi(y)\phi(x)$.
- In a labelled causal set: ordering a product of operators by decreasing label from left to right $=$ causal ordering, *e.g.* $\phi(4)\phi(4)\phi(2)\phi(1)$

Causal ordering is the causal set analogue of the time ordering of the continuum.
Define the Feynman propagator as

$$\Delta^F_{xy} = \langle C[\phi(x)\phi(y)] \rangle. \tag{6}$$

Free “Feynman” 2-point function

The delta-functions in $D_0(\xi, \bar{\xi})$ act to causally/anti-causally order the corresponding functions of the field operators

For example, to get the value of the 2-point function, $W_{xy} = \langle 0|\phi_x \phi_y|0\rangle$, for two fixed causet elements $x, y \in \mathcal{C}$, we integrate $\bar{\xi}_x \xi_y$ against this measure, i.e.

$$\int_{\mathbb{R}^{2N}} d^N\bar{\xi} d^N\xi \, D_0(\xi, \bar{\xi}) \, \bar{\xi}_x \xi_y = W_{xy} \tag{7}$$

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Let $y \prec x$, then

$$\begin{aligned} \int d^N\xi \int d^N\bar{\xi} \; D_0(\xi,\bar{\xi})\xi^x\xi^y &= \langle \phi^x\phi^y \rangle \\ &= \langle C[\phi^x\phi^y] \rangle \\ &= \Delta^F_{xy} \,. \end{aligned} \tag{8}$$

The Heisenberg field in the continuum

In the continuum, the Heisenberg field $\phi^H(t, \mathbf{x})$ is related to the interaction picture field $\phi(t, \mathbf{x})$ via,

$$\phi^H(t, \mathbf{x}) = U^\dagger(t, t_0)\phi(t, \mathbf{x})U(t, t_0). \tag{9}$$

where,

$$U(t, t_0) = T\left[e^{-i\int_{t_0}^{t} H(t)dt}\right] \quad (t \geq t_0) \tag{10}$$

is the time-evolution operator and where H is the interacting Hamiltonian in the interaction picture.

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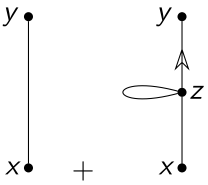
Recipe:

- Replace the time integral by a sum over causal set points
- Replace the time-ordering T with the causal ordering C. Under the action of C, all field commutators vanish and we can express the exponential of a sum as a product of exponentials.

Example: $\mathcal{H}(z) = g_z \phi_z^4$

Consider interaction point z in a total ordered: $x \prec z \prec y$.

$$\langle C \phi_x^H \phi_y^H \rangle = \langle e^{-ig_z \phi_z^4} \phi_y e^{ig_z \phi_z^4} \phi_x \rangle =$$



The field expansion terminates at a finite order in the interaction coupling. This order increases with the order of the interaction Hamiltonian and with the number of points to the past of x which are contained in the interaction region.

MANIFESTLY CAUSAL: Each internal vertex is connected to at least one external vertex by at least one directed path.

Interacting Decoherence functional

The interacting decoherence functional is,

$$D_g(\xi, \bar{\xi}) = D_0(\xi, \bar{\xi})\, e^{-i\mathcal{V}_{int}(\xi)+i\mathcal{V}_{int}(\bar{\xi})}. \tag{11}$$

where $\mathcal{V}_{int}(\xi) = \sum_{x=1}^{N} \mathcal{P}_x(\xi_x)$ with each local $\mathcal{P}_x$ is a real polynomial, that may vary from element to element.

A generating functional for in-in correlators

The in-in generating functional is given in terms of the Interacting DF:

$$\mathcal{Z}^{in-in}[J,\bar{J}] = \int d^N\xi\, d^N\bar{\xi}\, D_g(\xi,\bar{\xi})\, e^{-iJ.\xi}\, e^{i\bar{J}.\bar{\xi}}, \tag{12}$$

where J and $\bar{J}$ are two independent sources. For example, the in-in causally ordered 2-point correlator is given by

$$i\frac{\partial}{\partial J_x} i\frac{\partial}{\partial J_y} \mathcal{Z}^{in-in}[J,\bar{J}]\Big|_{J=0,\bar{J}=0} = \left\langle C\left[\phi^H_x \phi^H_y\right]\right\rangle , \tag{13}$$

with similar expressions with more derivatives for the causally ordered in-in n-point correlators. Derivatives with respect to $\bar{J}$ similarly result in anti-causally ordered products of field operators.

A generating functional for in-out correlators

The generating functional for the causally ordered in-out correlators is not given in terms of the interacting decoherence functional but closely

$$\mathcal{Z}^{in-out}[J] = \frac{\int d^N\xi d^N\bar{\xi}\, D_0(\xi,\bar{\xi})\, e^{-i\mathcal{V}_{int}(\xi)}\, e^{-iJ\cdot\xi}}{\int d^N\xi\, d^N\bar{\xi}\, D_0(\xi,\bar{\xi})\, e^{-i\mathcal{V}_{int}(\xi)}}\,, \tag{14}$$

Now we have,

$$i\frac{\partial}{\partial J_x} i\frac{\partial}{\partial J_y} \mathcal{Z}^{in-out}[J]\Big|_{J=0} = \frac{\langle \hat{S}\, C[\phi_x^H \phi_y^H]\rangle}{\langle \hat{S}\rangle}\,, \tag{15}$$

where the S-operator is given by,

$$\hat{S} = C[\prod_{z\in\mathcal{C}} e^{-i\mathcal{H}(z)}] \tag{16}$$

The expansion DOES NOT TERMINATE at a finite order even if the causal set itself is finite. The diagrams are identical to those of the continuum, with $\langle \hat{S}\rangle$ given by the sum of vacuum bubble diagrams.

The S-matrix on a causal set

Scattering amplitudes are given by the overlap of an in- and an out-state. Taking 2-to-2 scattering as an example, the associated amplitude is,

$$ {}_{out}\langle\lambda_3,\lambda_4|\lambda_1,\lambda_2\rangle_{in} = \langle\lambda_3,\lambda_4;N|\lambda_1,\lambda_2;1\rangle = \langle\lambda_3,\lambda_4|\hat{S}|\lambda_1,\lambda_2\rangle, \tag{17} $$

where on the RHS, $|\lambda,\lambda'\rangle$ are the particle states of the free theory and $\hat{S}$ reduces to,

$$ \hat{S} = C[\prod_z e^{-i\mathcal{H}(z)}], \tag{18} $$

where the product is over points z in the interaction region.

Outlook: The discrete cosmological collider

- Can we compute cosmological correlators on a causal set background? Yes! We can also define an S-matrix.
- A new tool for cosmological collider physics, can produce predictions to compare against cosmological data to test for spacetime discreteness
- Can also help with developing techniques for continuum cosmological spacetimes, for instance defining a unique vacuum state.
- Can offer a novel regularization of the continuum, since there are no UV divergences on a causal set.

Summary

- Causal Set Theory is an approach to quantum gravity in which spacetime is fundamentally discrete.
- It's a tool for new discoveries of non- local and Lorentz-invariant physics.
- New developments are enabling us to make concrete predictions, including for cosmological collider physics.

THANKS FOR LISTENING!