Erratum to the paper 'The Atiyah Singer Index Theorem' by M S Raghunathan in 'Contemporary Mathematics'

The proof of the theorem given in 7.1 of the above paper is incorrect. The assertion "When M is odd dimensional the asymptotic expansion of the heat kernel is in odd powers of $t^{1/2}$ so that for any vector bundle E on M (with a unitary connection), T_E is zero."does not make sense as the twisted signature operators are defined only om manifolds of even dimension. However the analytic index of *any* elliptic operator on an odd dimensional manifold is 0. For this one has to invoke the following fact:

Theorem 1. *Let M be an oriented smooth Riemannian manifold of dimension m and V be a complex vector bundle on M with a hermitian inner product along the fibres. Let L be a smooth non-negative self adjoint elliptic differential operator of order d from V to itself. Let Λ' be the set of eigen values of L and $\Lambda = \{0\} \cup \Lambda' = \{\lambda_q |\ q \in \mathbb{N}, \lambda_0 = 0, \lambda_q < \lambda_{q+1} \forall q \in \mathbb{N}\}$. Let $\mathbb{H} = \Gamma(M, V)_2$ be the Hilbert space of all square integrable sections of V and for $q \in \mathbb{N}$, let $\mathbb{H}_q = \{v \in \mathbb{H}|\ L(v) = \lambda_q \cdot v\}$. Then $\mathbb{H}$ is an orthogonal direct sum of the finite dimensional subspaces $\{\mathbb{H}_q |\ q \in \mathbb{N}\}$. For $t > 0$, the operator $e^{-t \cdot L}$ (which restricts to multiplication by $e^{-t \cdot \lambda_q}$ on $\mathbb{H}_q$) is compact and has a smooth 'Heat' kernel $K_L(t, x, y)$. a smooth section of the bundle $V \otimes V^* (\simeq Hom(V, V))$ on $M \times M$ depending smoothly on $t > 0$ as well. If for $q \in N$, $\{\varphi_{qi}|\ 1 \le i \le n_q$ is a basis of $\mathbb{H}_q$,*

$$K_L(t, x, y) = \sum_{q \in \mathbb{N}} e^{-t \cdot \lambda_q} \cdot (\sum_{1 \le i \le n_q} (\varphi_{qi}(x) \otimes \bar{\varphi}_{qi}(y).$$

$K_L(t, x, x)$ has an asymptotic expansion (in powers of t as $t \to 0$:

$$K_L(t, x, x) = \sum_{n=0}^{\infty} t^{(n-m)/d} \cdot k_n(x)$$

*as $t \to 0^+$. If m is **odd** $k_n(x) = 0$ for all n even.*

Now let E, F be vector bundles on the Riemannian manifold M equipped with hermitian metrics and D be an elliptic operator from E to F. Let D^* be the adjoint of D. Then the above theorem is applicable to D^*D and DD^*. Moreover D maps any eigen space of D^*D corresponding to any non-zero eigen value λ in $\Gamma(M, E)$ isomorphically onto the eigen space of DD^* in $\Gamma(M, F)$ corresponding to the same eigen value. It follows that $\int_M \text{Trace}(K_{D*D}(t, x, x) - K_{DD^*}(t, x, x)) =$ Trace($e^{-t \cdot D^*D}$)-Trace($e^{-t \cdot D^*D}$) $= a(D) = -a(D^*)$ is independent of t.

It is not difficult to see now from the above theorem applied to both D^*D and DD^* that $a(D) = 0$ when dim.M is odd.

That $t(D) = 0$ is proved (correctly) in 7.1 of the paper. We refer to Lemma 1.7.4 in the book Invariance Theory, The Heat Equation, and The Atiyah-Singer Index Theorem, Publish or Perish Inc., USA, 1984. by Peter B Gilkey. It is available for free download on the internet.