Quantum trajectories and Page-curve entanglement

Manas Kulkarni

International Centre for Theoretical Sciences
ICTS
Tata Institute of Fundamental Research
tifr

Contents

Part-A

Collective theory for Page Curve–like Dynamics of a Freely Expanding Fermionic Gas
Saha, **MK**, Dhar (PRL 2024)

Madhumita Saha (ICTS)

Abhishek Dhar (ICTS)

Part-B

Quantum trajectories, effective interaction and page curve
Ganguly, Gopalakrishnan, Naik, Agarwalla, **MK**
(arXiv:2501.12110)

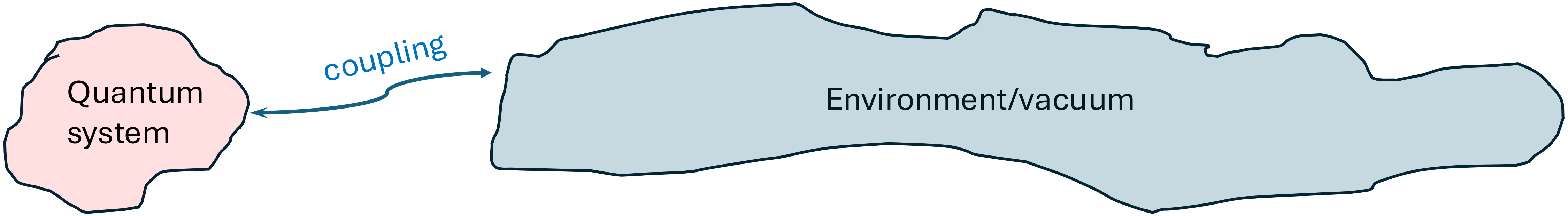
Katha Ganguly
(IISER Pune)

Preethi Gopalakrishnan
(Luxembourg)

Atharva Naik
(ICTS)

Bijay Agarwalla
(IISER Pune)

Central Platform



- Number of particles that leave the system as a function of time ? This is same as number of particles that enter the vacuum.
- What about density profiles, currents, number fluctuations ?
- How long does it take to empty a system ?
- What about entanglement entropy between system and its complement (environment) ?
- Is there a field theory description that captures essential features ?
- Is there some connection between entanglement entropy and thermal entropy ?
- How imperfections, effective interactions, weak measurements gets encoded in quantities mentioned above ?

Page (1993, 2013)

- Possible relevance to black holes

- Entanglement between black holes and the radiation starting from the unentangled initial state of just the back hole

- As the black hole radiates, the effective Hilbert space dimension of the radiation increases and there will be a corresponding increase in the entanglement entropy.

- However, this increase has to stop at some time when the black hole and radiation have the same Hilbert space dimensions. Beyond this time (referred to as the “Page time”), the entropy has to decrease.

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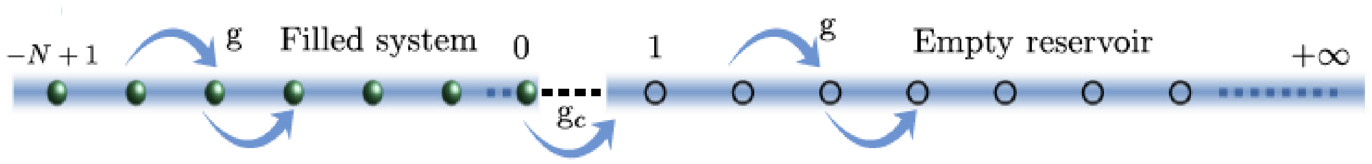

$$\hat{H} = \sum_{i,j=-N+1}^{\infty} h_{ij}\hat{c}_i^\dagger \hat{c}_j$$

$$h_{i,j} = -g(\delta_{i,j+1} + \delta_{i+1,j}) \ \forall \ i, \ j \neq 1, \ 0.$$

Calabrese, Cardy (2005)
Bertini, Fagotti, Piroli, Calabrese (2018)
Alba, Bertini, Fagotti, Piroli, Ruggiero (2021)
and many more..

Very recent: Kehrein (2024), Glatthard (2024,2025)

- The fact that g_c is not equal to g is why we call it “defect”. We consider three types of defects: conformal, hopping, onsite.

Exact Numerics

Length of reservoir

Total Hamiltonian $\hat{H} = \sum_{i,j=-N+1}^{N_b} h_{ij}\hat{c}_i^\dagger \hat{c}_j$

Correlation matrix

$C_{ij} = \langle c_i^\dagger c_j \rangle$

Dynamics

$C(t) = e^{iht} C(0) e^{-iht}$.

Quantities of interest that can be extracted from correlation matrix

Density $\rho(i) = \langle \hat{c}_i^\dagger \hat{c}_i \rangle$

Current $I = 2g_c \mathrm{Im}[\langle \hat{c}_0^\dagger \hat{c}_1 \rangle]$

Eigenvalues of part of correlation matrix

Von-Neumann Entanglement Entropy $S = \mathrm{tr}_s[\rho_s \log \rho_s] = -\sum_{l=1}^{N} [(1-m_l)\log[1-m_l] + m_l \log m_l]$

Particle number fluctuations $\kappa_2 = \langle \hat{\mathcal{N}}^2 \rangle - \langle \hat{\mathcal{N}} \rangle^2 = \sum_{\ell=1}^{N} m_\ell (1-m_\ell)$

Generalized Hydrodynamic Description

Reviews: Doyon (2020)
Essler (2022)

- The evolution of integrable systems observed on large time and length scales is described by generalized hydrodynamics

- The idea is that the system is a gas of quasiparticles that carry fixed momentum labels k and has a phase-space density $n_t(x, k)$.

Euler equation: $$(\partial_t + \sin[k]\partial_x)n_t(x,k) = 0.$$

in our case

Coarse grained wigner function

- This equation needs to be solved with appropriate boundary conditions

- From $n_t(x, k)$, quantities of interest can be extracted such as density, average current, "hydrodynamic" entropy

- We can essentially get analytical solutions

Solutions to Generalized Hydrodynamic Description

Recall: $(\partial_t + \sin[k]\partial_x)n_t(x,k) = 0.$

Solution on infinite line: $n_t(x,k) = n_0(x - t\sin(k), k)$ where $n_0(x,k) = \theta(-x) - \theta(-x-N)$
(boost the function)

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For our case, it is crucial to consider boundary conditions which is going to be instrumental in capturing Page-Curve

Transmission ← → Reflection

$$n_t(x>0,k) = n_t(x>0,k>0) = \sum_{s=0}^{\infty} T_k R_k^s \big(\theta(-x - 2sN + t\sin[k]) - \theta(-x - 2sN - 2N + t\sin[k])\big).$$

$$n_t(x<0,k>0) = \sum_{s=0}^{\infty} R_k^s \big(\theta(-x - 2sN + t\sin[k]) - \theta(-x - 2sN - 2N + t\sin[k])\big)$$

$$n_t(x<0,k<0) = \sum_{s=0}^{\infty} R_k^{s+1} \big(\theta(x - 2sN - t\sin[k]) - \theta(x - 2sN - N - t\sin[k])\big) + R_k^s \big(\theta(-x + 2sN + t\sin[k]) - \theta(-x + 2sN - N + t\sin[k])\big)$$

defect g_c is encoded in reflection and transmission coefficients

Quantities of interest

Density profile from hydrodynamics: $$\rho(x) = \int_{-\pi}^{\pi} \frac{dk}{2\pi} n_t(x,k)$$

See also: Pandey, Bhat, Dhar, Goldstein, Huse, M. K, Kundu, Lebowitz (2023)

Hydrodynamic/Thermodyanmic (Yang-Yang) entropy :
(not Von-Neumann entanglement entropy)

$$s_{\rm hydro}(x) = -\int_{-\pi}^{\pi} \frac{dk}{2\pi} \left[n_t(x,k) \log(n_t(x,k)) + (1-n_t(x,k)) \log(1-n_t(x,k)) \right]$$

$$S^{(S)}_{\rm hydro} = \int_{-N}^{0} dx\, s_{\rm hydro}(x), \quad S^{(R)}_{\rm hydro} = \int_{0}^{\infty} dx\, s_{\rm hydro}(x)$$

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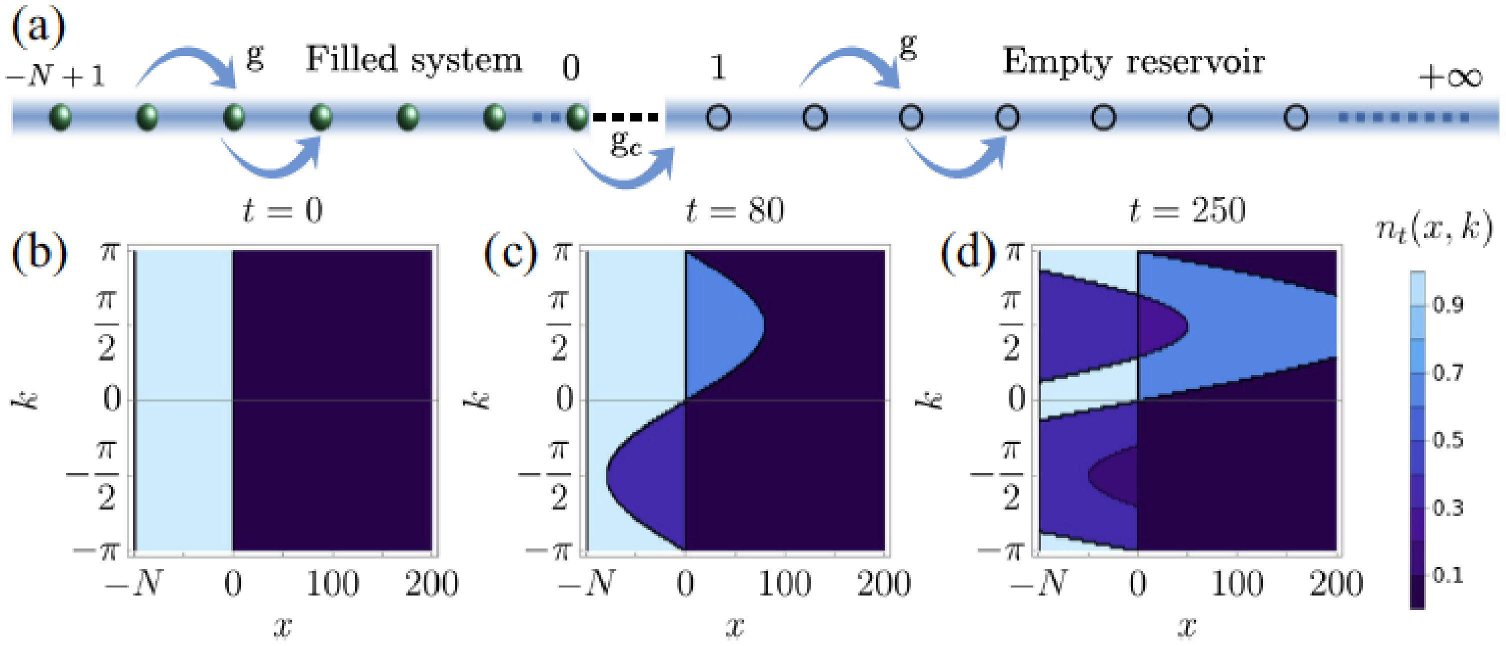
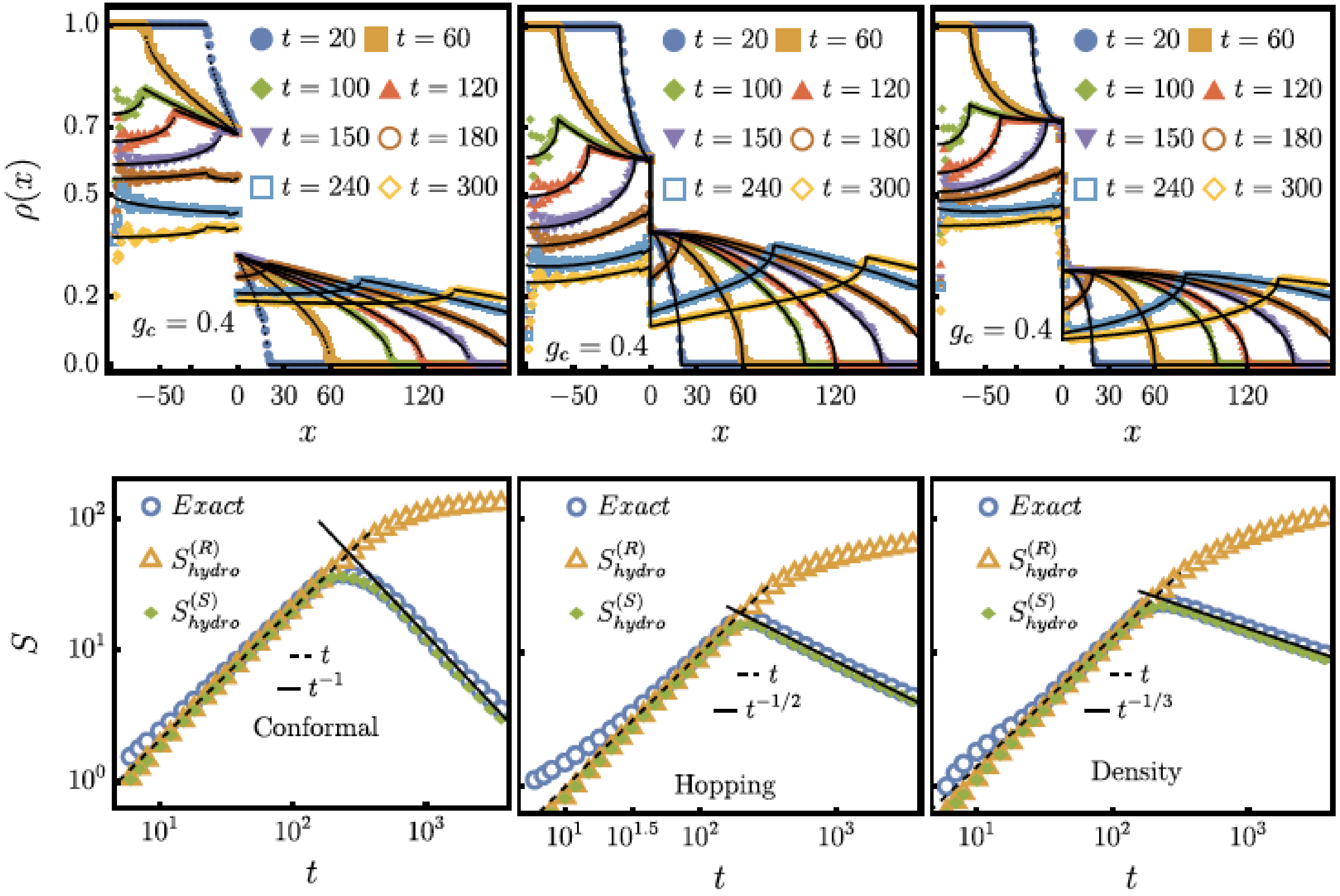
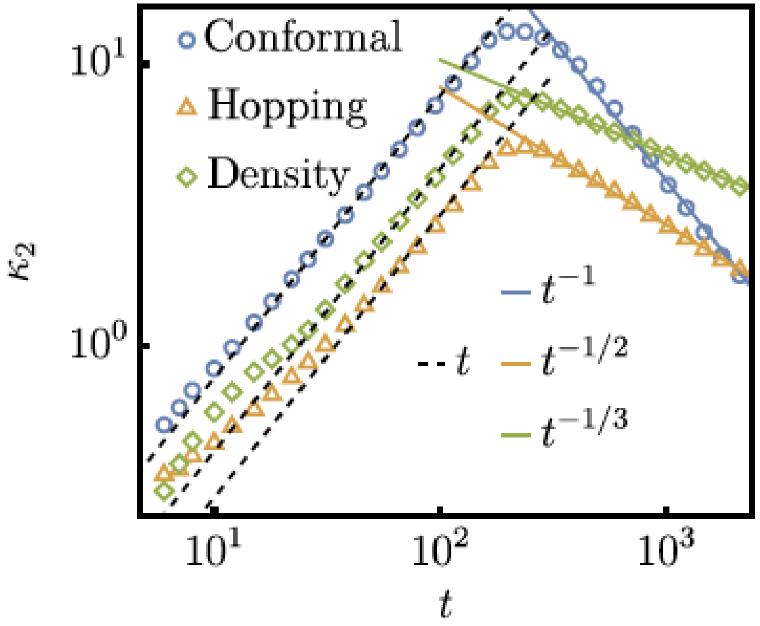


Phase-space dynamics

See also: M.K., Mandal, Morita (2018)

Density evolution and Page-Curve Entanglement





Recall:

$$S^{(S)}_{\text{hydro}} = \int_{-N}^{0} dx\, s_{\text{hydro}}(x), \qquad S^{(R)}_{\text{hydro}} = \int_{0}^{\infty} dx\, s_{\text{hydro}}(x)$$

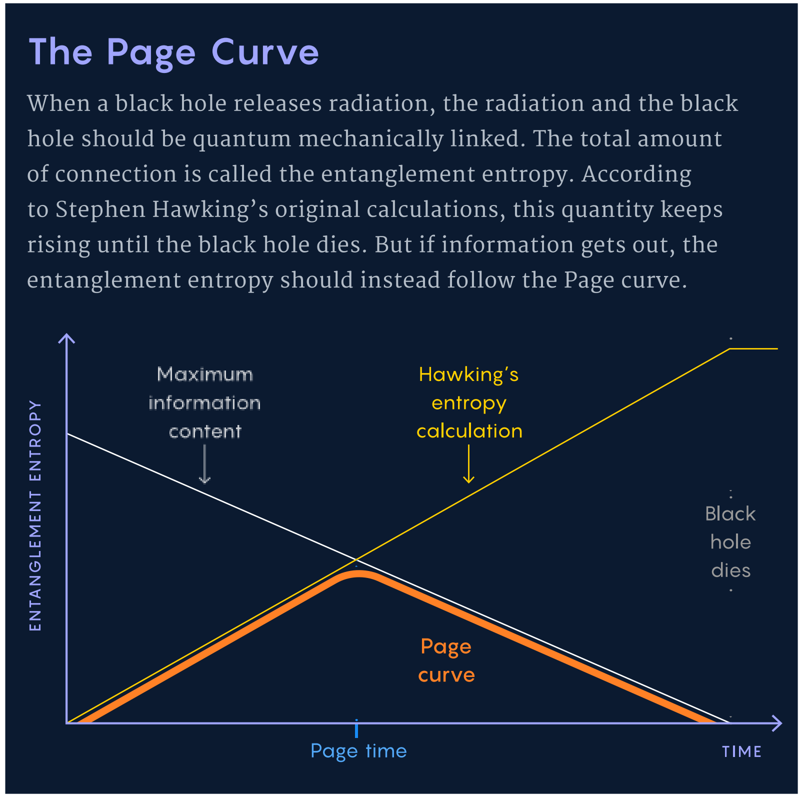
Exact: $S = -\sum_{l=1}^{N} [(1 - m_l)\log[1 - m_l] + m_l \log m_l]$

$$\kappa_2 = \langle \hat{\mathcal{N}}^2 \rangle - \langle \hat{\mathcal{N}} \rangle^2$$

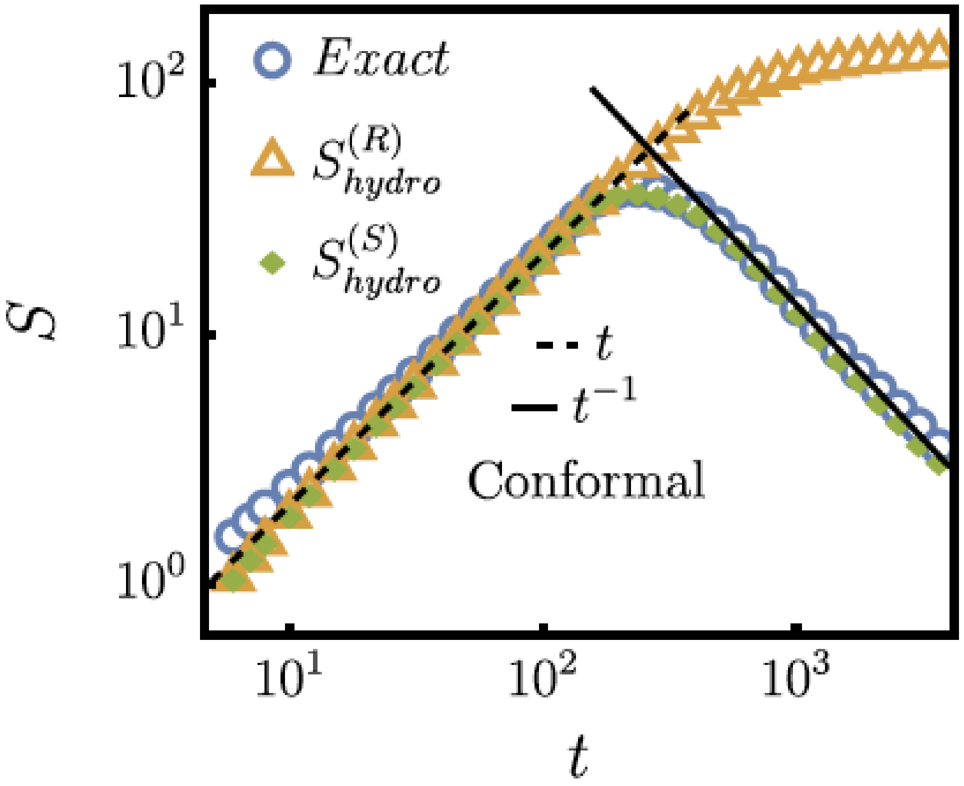
Can analytically extract early time (setting s=0) and also late time (using Poisson summations)

Conclusions to Part A

Saha, **MK**, Dhar (PRL 2024)



Samuel Velasco, Quanta Magazine



Review on entanglement in SYK and its generalizations: Zhang (2022)

- Numerically and analytically amenable platform to capture essential features of a Page curve
- Semiclassics enables us to understand analytically the page curve, both early and late times
- Yang-Yang/thermodynamics entropy of the system remarkably agrees well with the von-Neumann entanglement entropy
- On the other hand, reservoir entropy only agrees at early times and keeps on increasing.
  (Black hole analogy: Radiation entropy computed from semi-classical theories keeps increasing ?)

What about systems with effective interactions or imperfections ?

Ganguly, Gopalakrishnan, Naik, Agarwalla, **MK** (arXiv:2501.12110)

- How is the impact of effective interactions, or imperfections encoded in usual quantities of interest ?
- Are there some universal properties ?

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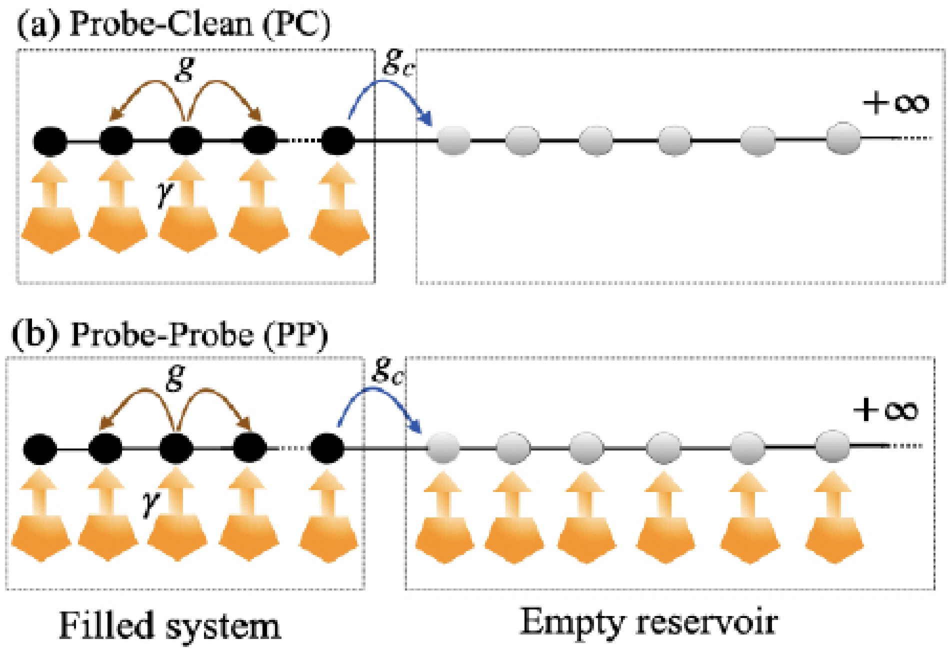

$$H = \sum_{i,j=1}^{L} h_{ij}\hat{c}_i^\dagger \hat{c}_j$$

$$h_{i,j} = -g\,(\delta_{i,j+1} + \delta_{i+1,j}), \;\; \forall\, i,j \neq L_S+1, L_S$$

Number of "lattice sites"

Number of "probes"

$$\frac{d\rho}{dt} = -i[\hat{H},\rho] + \gamma \sum_{i=1}^{L_P} \left(\hat{n}_i \rho\, \hat{n}_i - \frac{1}{2}\{\hat{n}_i,\rho\}\right)$$

$$L = L_S + L_R$$

Dephasing strength

Important: such "probes"

(i) Often mimic interactions by inducing scattering effects [for e.g., Ferreira et al, PRL 2024 and many more]

(ii) Often mimic environmental imperfections [H.J. Carmichael, Statistical Methods in Quantum Optics and many more]

- Lindblad Master equation well suited to study both the dynamics and steady states of the expectation values of observables that depend linearly on density matrix $\langle \hat{\mathcal{O}} \rangle_t = \mathrm{Tr}[\rho(t)\, \hat{\mathcal{O}}]$

- Quantities that are inherently nonlinear in ρ(t) fall outside the realm of Lindblad equation, and careful unraveling of the underlying density matrix evolution is warranted.

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Two kinds of unraveling (quantum trajectories)

Cao, Tilloy, De Luca (2019)
Alberton, Buchhold, Diehl (2021)
Dolgirev, Marino, Sels, Demler (2020)
and many more works.

Stochastic Unitary Unraveling (SUU)
Onsite fluctuating noise added to the Hamiltonian thereby making the evolution stochastic while maintaining its unitarity

Quantum State Diffusion (QSD)
Unraveling mimicking weak measurement of the local particle number at each site

Questions investigated – Temporal growth and decay of entanglement
Scaling of Page value
Scaling of Page time
Number of fermions radiated into the reservoir
Relation between entanglement production rate and current

Before proceeding further, let us discuss the unraveling procedure

Stochastic Unitary Unraveling (SUU)

Let us recall the dephasing Lindblad equation $$\frac{d\rho}{dt} = -i[\hat{H},\rho] + \gamma \sum_{i=1}^{L_P} \left(\hat{n}_i \rho \hat{n}_i - \frac{1}{2}\{\hat{n}_i, \rho\} \right)$$

- This describes the time evolution of the mixed state ρ(t), which is an ensemble average of pure states
- These pure states evolve stochastically in time.
- Under SUU protocol, the lattice system is subjected to onsite fluctuating Gaussian noise $\xi_i(t)$ at each lattice site i.
- It mimics onsite imperfections

$$\hat{H}_T(t) = \hat{H} + \sum_{i=1}^{L_P} \xi_i(t)\, \hat{n}_i$$

$$\overline{\xi_i(t)\, \xi_j(t')} = \gamma\, \delta_{ij}\, \delta(t-t')$$

→

$$|\psi_\xi(t+dt)\rangle \approx e^{-i\hat{H}dt - i\sum_{j=1}^{L_P} d\xi_j^t\, \hat{n}_j} |\psi_\xi(t)\rangle$$

$$\overline{d\xi_i^t} = 0 \qquad d\xi_i^t\, d\xi_j^t = \gamma\, dt\, \delta_{ij}$$

↓

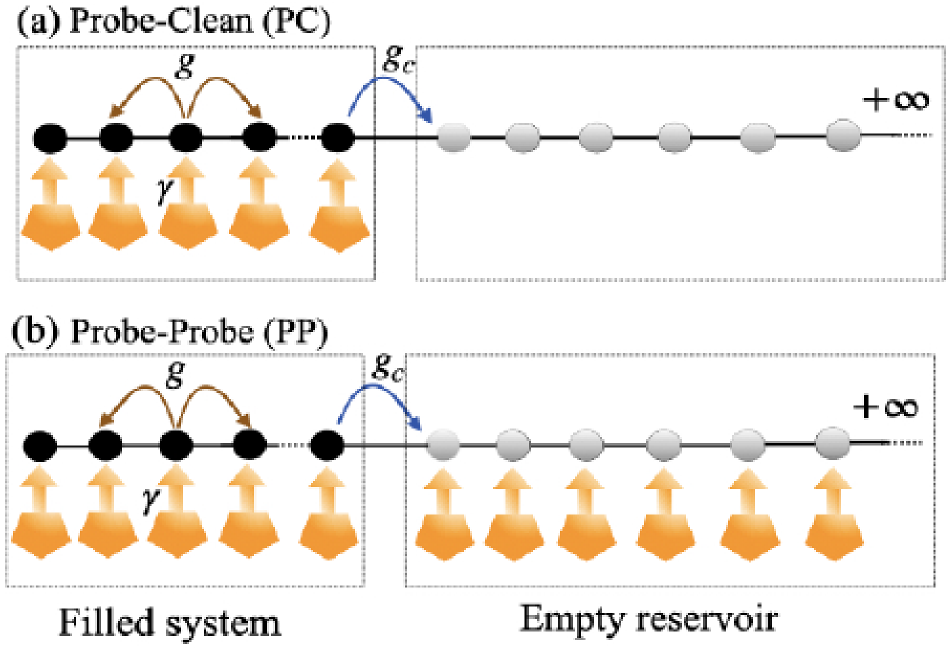
$$|\psi_\xi(t+dt)\rangle = \left[\hat{\mathbb{I}} - i\hat{H}dt - i\sum_{j=1}^{L_P} d\xi_j^t\, \hat{n}_j - \frac{\gamma}{2}\sum_{j=1}^{L_P} \hat{n}_j\, dt\right] |\psi_\xi(t)\rangle$$

Stochastic Unitary Unraveling (SUU)

$$|\psi_\xi(t)\rangle = \prod_{l=1}^{N} \Big(\sum_{k=1}^{L} U_{kl}(t)\, \hat{c}_k^\dagger \Big) |0\rangle$$

L x L_S Matrix

$$U(t+dt) = \begin{cases} \mathrm{diag}(e^{-id\xi_1^t}\ e^{-id\xi_2^t}\ e^{-id\xi_3^t}\ \ldots e^{-id\xi_{L_S}^t}\ 1 \ldots 1\ 1)\, e^{-ihdt}\, U(t), & \text{(PC case)} \\ \mathrm{diag}(e^{-id\xi_1^t}\ e^{-id\xi_2^t}\ e^{-id\xi_3^t}\ \ldots \ldots\ e^{-id\xi_L^t})\, e^{-ihdt}\, U(t). & \text{(PP case)} \end{cases}$$



We can finally extract the correlation matrix for a given trajectory

$$C_{ij}^{\xi}(t) = [U(t)U^\dagger(t)]_{ji}$$

- From this matrix, one can compute various quantities including entanglement entropy
- At the very end, we perform average over quantum trajectories.

Quantum State Diffusion (QSD)

- Another widely employed stochastic protocol that also mimics the same Lindblad dynamics upon averaging, is the Quantum State Diffusion (QSD)
- Stochastic dynamics under this protocol describes the evolution of a quantum system subjected to continuous weak measurements of local particle number n_i
- These measurements can be performed by coupling the system to a detector and making projective measurement of the detector with measurement operator M_μ

$$\hat{M}_\mu(dt) = \prod_{j=1}^{L_P} \left[\frac{2\gamma dt}{\pi}\right]^{1/4} e^{-\gamma dt(\mu_j - \hat{n}_j)^2}$$

Measurement outcomes of the projective measurement on the detector

where $\mu = \{\mu_j\}$

$$|\psi(dt, \{\mu_i\})\rangle = \frac{1}{\mathcal{N}_c}\hat{M}_\mu(dt)|\psi(0)\rangle$$

μ_j is a Gaussian stochastic variable which can be written as

$$\mu_j = \langle \hat{n}_j \rangle_0 + \frac{d\xi_j^0}{2\gamma dt}$$

Quantum State Diffusion (QSD)

After some calculations, one can show

$$|\psi_\xi(t+dt)\rangle = \left[\hat{\mathbb{I}} - i\hat{H}\,dt + \sum_{j=1}^{L_P} d\xi_j^t\,(\hat{n}_j - \langle \hat{n}_j\rangle_t) - \frac{\gamma}{2}\sum_{j=1}^{L_P}(\hat{n}_j - \langle \hat{n}_j\rangle_t)^2\,dt\right]|\psi_\xi(t)\rangle$$

We then get

$$|\psi_\xi(t)\rangle = \prod_{k=1}^{N}\Big(\sum_{j=1}^{L} U_{jk}(t)\,c_j^\dagger\Big)\,|0\rangle$$

$$U(t+dt) = \begin{cases} \mathrm{diag}(e^{d\xi_1^t + \frac{\gamma}{2}(2\langle n_1\rangle_t - 1)\,dt}, \dots e^{d\xi_{L_S}^t + \frac{\gamma}{2}(2\langle n_{L_S}\rangle_t - 1)\,dt},\ 1, \dots 1,\ 1)\,e^{-ihdt}\,U(t), & \text{(PC case)} \\ \mathrm{diag}(e^{d\xi_1^t + \frac{\gamma}{2}(2\langle n_1\rangle_t - 1)\,dt}, \dots\dots\ e^{d\xi_L^t + \frac{\gamma}{2}(2\langle n_L\rangle_t - 1)\,dt})\,e^{-ihdt}\,U(t), & \text{(PP case)} \end{cases}$$

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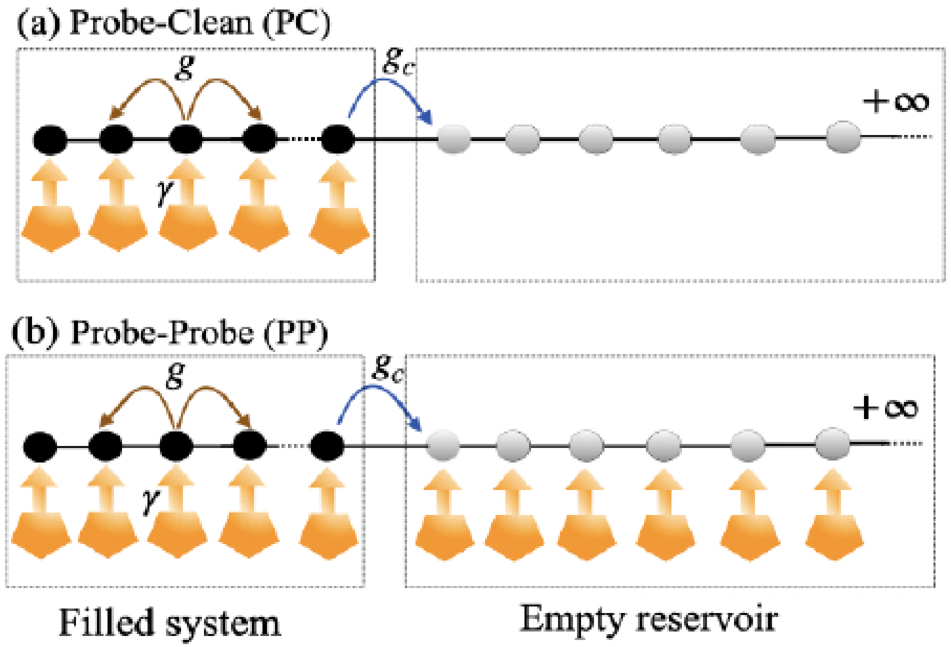
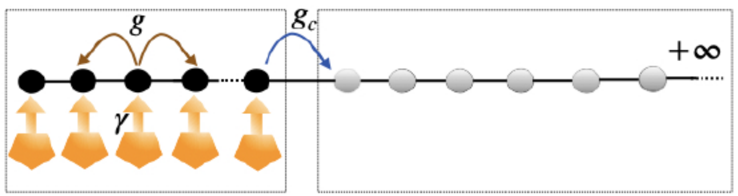
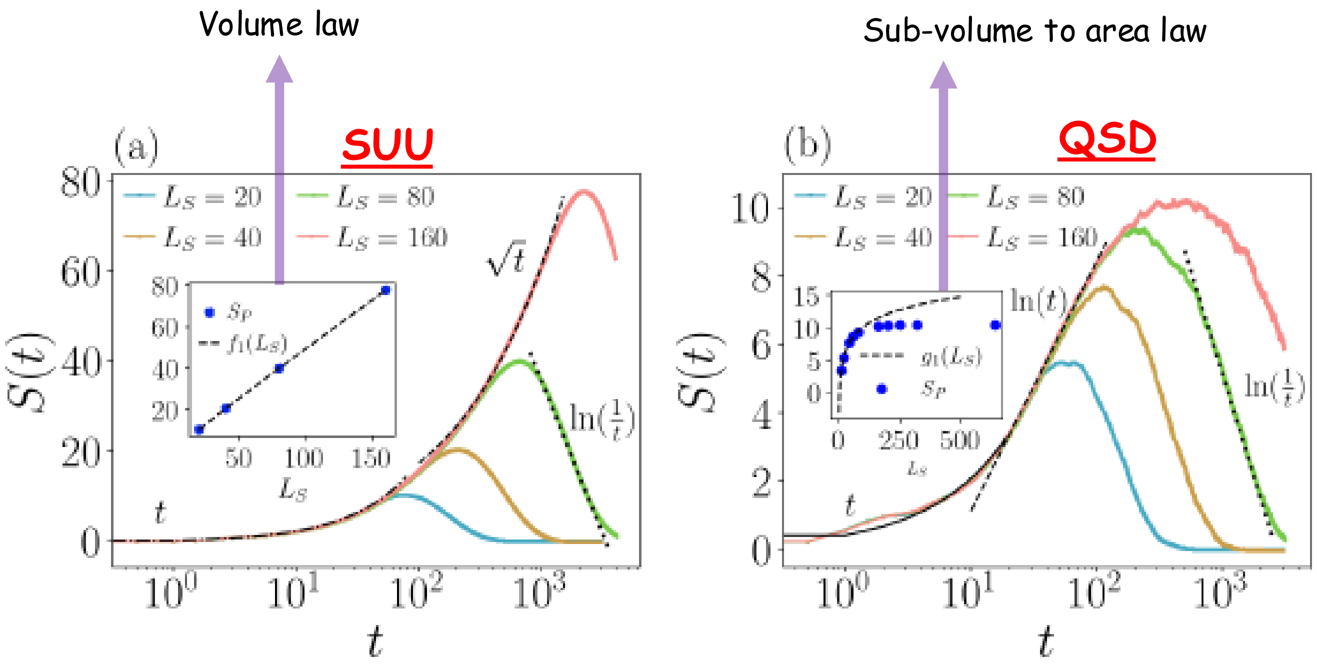


- Presence of $\langle n_i\rangle_t$ makes the dynamics nonlinear – effect of measurement back action

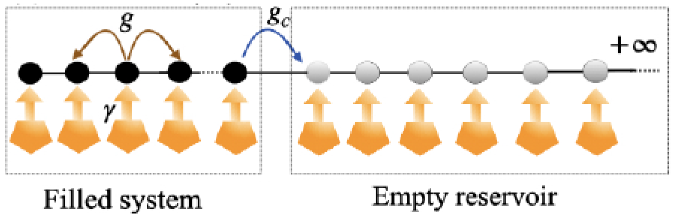
- Recall that in SUU, there is no such measurement back-action.

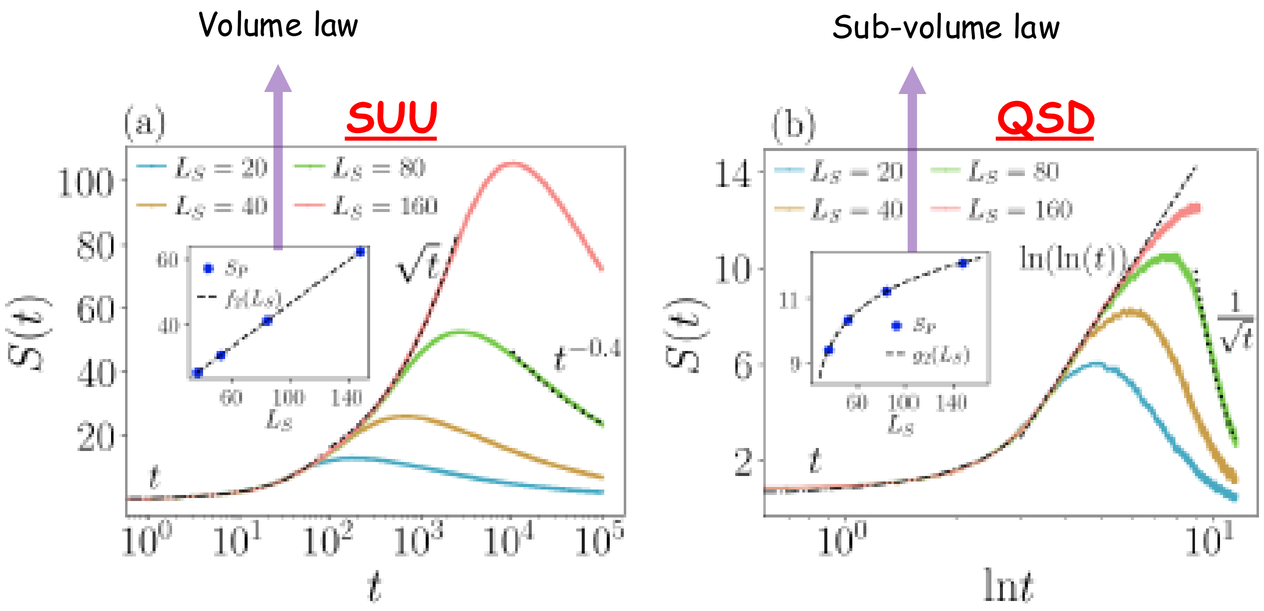
Probe-clean setup





Probe-probe setup



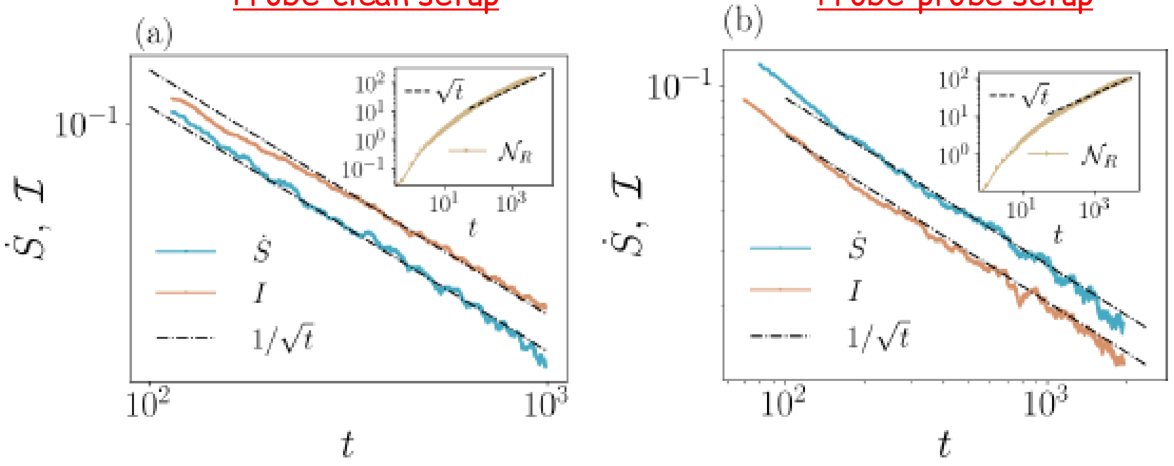


Ganguly, Gopalakrishnan, Naik, Agarwalla, **MK**
(arXiv:2501.12110)

Interesting connection between entropy production rate and current

Ganguly, Gopalakrishnan, Naik, Agarwalla, **MK** (arXiv:2501.12110)

Probe-clean setup   Probe-probe setup



In the SUU protocol, we find that the following seems to hold

$$\frac{dS(t)}{dt} \propto \mathcal{I}(t), \quad t < t_P, \quad \text{for both PC and PP case}$$

Relation seizes to exist in monitoring protocols such as QSD, which is somewhat expected since the EE is sensitive to the underlying measurement scheme while the right-hand side is completely independent of the unraveling procedure.

Conclusions to Part B

Ganguly, Gopalakrishnan, Naik, Agarwalla, **MK** (arXiv:2501.12110)

- Examined the entanglement entropy (EE) Page curve in an effectively interacting system following quantum trajectory approach.

- Depending on the unravelling protocol and the setup, we observe distinct time dynamics for the growth and the decay of EE.

- SUU protocol always leads to diffusive growth

- QSD protocol due to local monitoring always leads to much slower growth of EE. The maximum attainable value of EE, i.e., the Page value, shows interesting system-size scaling (volume law, sub-volume law, area law)

- Find strong numerical evidence supporting the relation between the entropy production rate and particle current up to Page time.

Thank You