The granular geometry of large N matrix models and 2D string theory

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"Quantum Information, Quantum Field Theory and Gravity"

ICTS, Bangalore, 28 August, 2024

with
Ajay Mohan https://arxiv.org/abs/2406.07629
Adwait Gaikwad, Ajay Mohan (in progress)
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Earlier: with
Avinash Dhar, Anirvan Sengupta, Nemani Suryanarayana, Spenta Wadia.

1. Introduction

Matrix field theories have been argued to describe emergent geometries in the large N limit (e.g. AdS/CFT, c=1 matrix model etc).

In this talk, we will point out that one may encounter certain inconsistencies in insisting on a continuum emergent geometry from matrix models.

We will show this by focusing on matrix models (space time-independent matrices), as well as matrix quantum mechanics (which includes the c=1 model).

In the case of large N matrix QM, we will find that the inconsistencies are resolved by a lattice geometry ("granular geometry"). We will briefly hint at a similar resolution for time-independent matrix models.

Here, by "geometry" we mean a bosonic field theory representation of the matrix theory.
(Note that in 2D string theory, the only dynamical field is a scalar field).

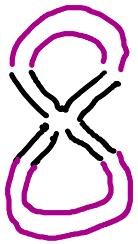
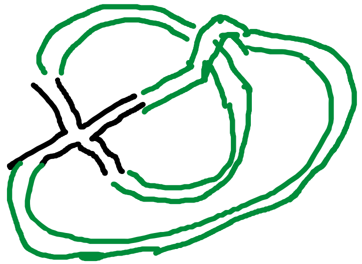
2. Matrix models

Let us first consider matrix models (space-time independent) to see the problem.

Consider computing $I_p = \frac{1}{Z}\int \prod_{ij} dM_{ij}\, e^{-NS}\, \mathcal{O}_p$ $\qquad \mathcal{O}_p = \frac{\mathrm{Tr}\, M^p}{N}\,,\quad S = \mathrm{Tr}\underbrace{\left(\frac{1}{2}M^2 + g\,\frac{M^4}{4}\right)}_{V(M)}$ $\qquad M$ = hermitian

Take p=4, g=0

Feynman digrams

$I_4 =$ $2\times$  $+$ 1  $= 2 + \frac{1}{N^2}$

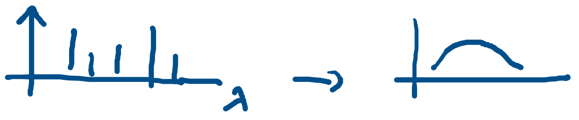
Is there a continuum description (cf. bulk dual of CFT) of the above computation?

$M_{ij} \to (\lambda_i, \Omega_{ij})$, $\prod_{ij} dM_{ij} = \prod_i d\lambda_i \, \Delta^2(\lambda)$ $\Delta(\lambda) = \prod_{i<j}(\lambda_i - \lambda_j)$ = van der Monde determinant

$$I_p = \frac{1}{Z}\int \prod_i d\lambda_i \exp\left[-N \sum_i \underbrace{\left(\frac{\lambda_i^2}{2} + g\frac{\lambda_i^4}{4}\right)}_{V(\lambda_i)} + \sum_{i\neq j} \ln|\lambda_i-\lambda_j| \}\right]\left(\sum_{i=1}^{N} \frac{\lambda_i^p}{N}\right)$$

Attempt at a continuum description

$$\rho(\lambda) = \frac{1}{N}\sum_{i=1}^{N}\delta(\lambda-\lambda_i) =$$



$$\prod_{i=1}^{N} d\lambda_i \longrightarrow \prod_{\lambda\in R} d\rho(\lambda) \qquad \text{....... (1)}$$

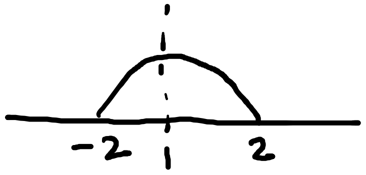
This is clearly not justified at finite N, however large. Nevertheless, let us soldier on.....

$$I_p = \frac{1}{Z}\int \prod_\lambda d\rho(\lambda)\, J[\rho]\, \exp\left[-N^2\left\{\int d\lambda\, \rho(\lambda)V(\lambda) + \int_{\lambda\neq\lambda'} d\lambda\, d\lambda'\, \rho(\lambda)\rho(\lambda')\ln|\lambda-\lambda'|\right\}\right] \int \rho(\lambda)\lambda^p$$

Gives a saddle point solution at large N

Large N: saddle point

$g=0$ $\qquad \rho_0(\lambda) = \frac{1}{2\pi}\sqrt{4-\lambda^2}\;\theta(2-|\lambda|)$



Wigner semi-circle law

$p=4$ $\qquad I_4 = \int d\lambda\, \rho_0(\lambda)\, \lambda^4 = 2$

Recall the exact result: $I_4 = 2 + \frac{1}{N^2}$

Question: Can we obtain the $\frac{1}{N^2}$ correction from the $\rho(\lambda)$ path integral?

Answer: NO!

$\langle\rho(\lambda)\rangle = \rho_0(\lambda) + \frac{1}{N}\langle\,\delta\rho(\lambda)\rangle = \rho_0(\lambda)$, $\quad\langle\,\delta\rho(\lambda)\rangle$ **=0, since the action is strictly quadratic.**

Double scaling limit, $N \to \infty, g \to g_c$ also does not help. Possible Jacobian can help? No.

Problem really lies with the change of variable (1), from N variables λ_i to an uncountable infinity of variables $\rho(\lambda)$!

One way to see this is that a generic $\rho(\lambda)$ violates trace identities (Cayley-Hamilton) for any N, however large.

At any N, however large, $\mathrm{tr}\, M^{N+p}$, $p \geq 1$ is writeable in terms of $\mathrm{tr}\, M^{N-q}$, $q \geq 0$

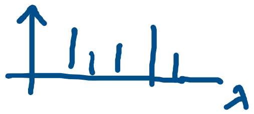
e.g. $N = 2$ $\quad \lambda_1^3 + \lambda_2^3 - \frac{3}{2}(\lambda_1 + \lambda_2)(\lambda_1^2 + \lambda_2^2) + \frac{1}{2}(\lambda_1 + \lambda_2) = 0$

$$\mathrm{Tr}\, M^3 - \frac{3}{2}\,\mathrm{Tr}\, M\, \mathrm{Tr}\, M^2 + \frac{1}{2}(\mathrm{Tr}\, M)^3 = 0$$

$$\Rightarrow \int d\lambda\, \rho(\lambda)\lambda^3 - \frac{3}{2}\int \rho(\lambda)\lambda \int \rho(\lambda')\lambda'^2 + \frac{1}{2}\left(\int d\lambda\, \rho(\lambda)\lambda\right)^3 = 0$$

No smooth $\rho(\lambda)$ would ever satisfy such trace identities!

Actual $\rho(\lambda) = \frac{1}{N}\sum_{i=1}^{N}\delta(\lambda - \lambda_i)$ is a very kinky function



Indeed, from definition, $\rho(\lambda)$ is a distribution rather than a function. Hence a functional integral over $\rho(\lambda)$ is not valid.

3. Matrix Quantum Mechanics

Consider computing the following vev:

$$I_p = \frac{1}{Z}\int \mathcal{D}M(t)\, e^{i\frac{N}{g}\int_0^T \mathrm{Tr}\left(\frac{\dot{M}^2}{2} - V(M(t))\right)dt}\ \frac{1}{N}\mathrm{Tr}\, M^p(t)$$

$\Delta(\lambda) = \prod_{i<j}(\lambda_i - \lambda_j)$ = van der Monde determinant

$$\longrightarrow \frac{1}{Z}\int \prod_t \prod_{i=1}^{N} d\lambda_i(t)\ \Delta(\lambda(0))\,\Delta(\lambda(T))\ \exp\left[\frac{iN}{g}\int_0^T \sum_{i=1}^{N}\left(\frac{\dot{\lambda}_i^2}{2} - V(\lambda_i(t))\right)dt\right]$$

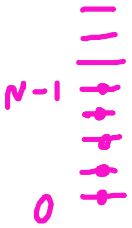
$\rightarrow$ eigenvalues become fermions

$\Rightarrow$ N free fermions on the real line, with a single-particle hamiltonian

$$\hat{h} = \frac{\hat{p}^2}{2} + V(\hat{\lambda})$$

$$\hat{h}\,\chi_n(\lambda) = \epsilon_n\,\chi_n(\lambda) \qquad n = 0, 1, 2, \cdots \infty$$

For $V = \frac{1}{2}\,\omega^2\,\lambda^2$, $\quad \chi_n(\lambda) = H_n(\sqrt{\omega}\,\lambda)\exp\left(-\frac{\omega\,\lambda^2}{2}\right)$, $\quad \epsilon_n = \left(n + \frac{1}{2}\right)\omega$



$$I_4 = \frac{1}{N}\sum_{n=0}^{N-1}\int d\lambda\ \lambda^4\, |\chi_n(\lambda)|^2 = \frac{1}{2\omega^2}\left(1 + \frac{1}{2N^2}\right)$$

Is there a "continuum" description which reproduces this?

Let us try the eigenvalue density description again.

Since the eigenvalues have become fermions, the eigenvalue density has the following alternative description: $\rho(\lambda) = \psi^{+}(\lambda)\psi(\lambda)$

where $\psi(\lambda) = \sum_n \chi_n\,(\lambda)\,\psi_n$, $\psi^{+}(\lambda) = \sum_n \chi_n^{*}\,(\lambda)\,\psi_n^{+}$

Das-Jevicki 1990, Sengupta-Wadia 1990 wrote the following density representation (collective variable) of the matrix path integral

$$I_p = \frac{1}{Z}\int \mathcal{D}\zeta(\lambda)\ \frac{1}{N}\int_\lambda \rho(\lambda)\lambda^p\ e^{iS_c}, \quad S_c = N^2\int dt\, d\lambda \left\{ \frac{\dot{\zeta}^2}{2\,\partial_\lambda \zeta} - \frac{\pi^2}{6}(\partial_\lambda \zeta)^3 - V(\lambda)\,\partial_\lambda \zeta \right\} \qquad \partial_\lambda \zeta \equiv \frac{\rho(\lambda)}{N}$$

Saddle point solution: $\partial_\lambda \left(\frac{\pi^2}{2}(\partial_\lambda \zeta)^2 + V(\lambda) \right) = 0$

$$\rho_0(\lambda) = \frac{1}{\pi}\sqrt{2(\epsilon_F - V(\lambda))} \quad \text{where } \epsilon_F \text{ is determined by } \int \rho_0(\lambda)\,d\lambda = N$$

For harmonic oscillator potential $V = \frac{1}{2}\,\omega^2\,\lambda^2$ we find that $\epsilon_F = N\,\omega$

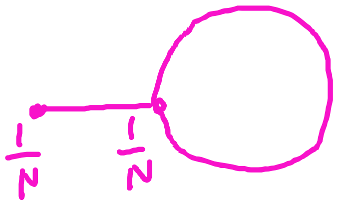
$$\therefore\ I_4 = \frac{1}{N}\int_\lambda \rho_0(\lambda)\lambda^4 = \frac{1}{2\omega^2}$$

which reproduces the leading term of the exact result. How about the sub-leading O(1/N²) correction?

The sub-leading term is given by looking at fluctuations around the saddle point:

$$\rho(\lambda) = \rho_0(\lambda) + \frac{1}{N}\delta\rho(\lambda)$$

$$\langle \rho(\lambda) \rangle \supset \frac{1}{N}\langle \delta\rho(\lambda) \rangle = \frac{1}{N}\left\langle \delta\rho(\lambda)\, e^{i\frac{1}{N}\int (\delta\rho)^3} \right\rangle_0 =$$



The loop integral involves a coincident propagator, and is divergent; this gives

$$I_4 = \frac{1}{2\omega^2}\left(1 + \frac{1}{N^2}\cdot\infty\right)$$

Recall that the exact result is

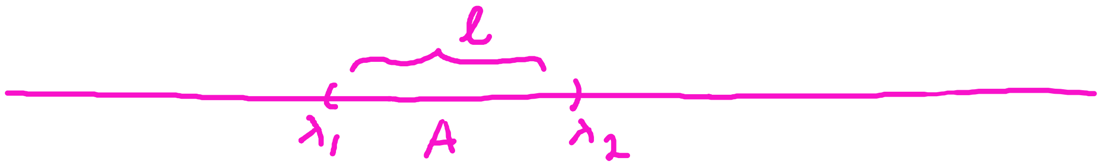
$$I_4 = \frac{1}{2\omega^2}\left(1 + \frac{1}{N^2}\cdot\frac{1}{2}\right)$$

Once again, we do not get the correct result since the change of variable from $\{\lambda_i(t), i = 1,2,\ldots,N\}$ to $\{\rho(\lambda,t), \lambda \in R\}$ is not exact except in the $N \to \infty$ limit.

We will resolve the problem below by finding an **Exact transformation of variables from the eigenvalues to a lattice boson field** where the lattice contains precisely N points.

3.1 Entanglement Entropy

Before describing the new boson, let us mention another important calculation, that of entanglement entropy (EE), which also shows a problem with the eigenvalue density representation:



We ask the question: what is the ground state EE of an interval $A = (\lambda_1, \lambda_2)$ in the eigenvalue direction?

We can answer the question in various ways:

EE of an N-particle quantum mechanics: target space EE. **(Das, Kaushal, Liu, Mandal, Trivedi)**

This is equivalent to computing the usual QFT EE in the fermion theory.

This is given by the following formula:

$$S_A = \int_A dx \sum_{n=0}^{N-1} \chi_n^*(x)\chi_n(x) - \int_A dx \int_A dx' |\sum_{n=0}^{N-1} \chi_n^*(x)\chi_n(x')|^2.$$

Entanglement Entropy (fermionic formulation)

$$S_A = \int_A dx \sum_{n=0}^{N-1} \chi_n^*(x)\chi_n(x) - \int_A dx \int_A dx' |\sum_{n=0}^{N-1} \chi_n^*(x)\chi_n(x')|^2. \quad \text{....... (1a)}$$

Note that: this is obviously finite!
(In the 2D string theory context computed and emphasized by **Das 1995;** later refined by **Hartnoll-Mazenc 2015**)

In case of free fermions (box potential), the above evaluates to the formula well-known from the statistical mechanics literature :

$$S_A = \frac{1}{3}\log(\frac{\lambda_2 - \lambda_1}{\epsilon}), \quad \epsilon = \frac{L}{N} \quad \text{.....(2)}$$

For fermions in an external potential $V(\lambda)$ [**Das-Hampton-Liu 2022, see also Sensarma's talk**]

$$S_A = \frac{1}{3}\log(p_F(\lambda_0)\frac{\lambda_2 - \lambda_1}{\hbar}) \qquad P_F(\lambda_0) = \sqrt{2(E - V(\lambda_0)} \quad \text{.....(2a)}$$

Here, we have assumed that $V(\lambda)$ varies slowly inside the interval A. More sophisticated formulae exist.
[LeCroix-Majumdar-Schehr]

Can we get these formulae from the density theory (collective variable theory)?

Entanglement Entropy (density boson formulation)

For a bosonic field theory, the EE of a region A is given by the formula

$$S_A = Tr\,[\left(C+\tfrac{1}{2}\right)\log\left(C+\tfrac{1}{2}\right) - \left(C-\tfrac{1}{2}\right)\log(C-\tfrac{1}{2})] \quad \text{....(3)}$$

where $C^2 = X.P,\ \mathrm{X} = \langle\, \phi_x\, \phi_y \rangle, \mathrm{P} = \langle\, \Pi_x\, \Pi_y \rangle, \quad x, y \in A$

If we consider the collective variable theory for $V(\lambda) = 0$, and restrict to quadratic fluctuations, we get a massless relativistic scalar field, for which eq. (3) gives the well-known result (**Holzhey-Wilczek, Calabrese-Cardy**)

$$S_A = \frac{1}{3}\log(\frac{\lambda_2 - \lambda_1}{\epsilon}), \quad \epsilon = \text{short-distance ultraviolet cut-off} \quad \text{.....(3a)}$$

Note that this result is divergent. It agrees with eq.(2) **IF** we choose $\epsilon = \frac{L}{N}$!

We will find that eq.(2), as well as eq. (2a) can indeed be reproduced in terms of a lattice boson, with the above lattice spacing.

How about the full collective theory? In a new variant **[Das-Jevicki 2022],** it is shown that the Fermionic EE (1a), in case $V(\lambda) = 0$, has a bosonic interpretation, leading to the finite result (2), although it is not clear what the bosonic EE (3) yields in that case, or what happens if $V(\lambda) \neq 0$

(see Sumit's talk)

Summary so far….

1. A continuum description of matrix models in terms of an eigenvalue density $\boldsymbol{\rho(\lambda)}$ is inconsistent with trace identities, which prevents an exact change of variables from the matrix degrees of freedom to $\boldsymbol{\rho(\lambda)}$.

2. The $\boldsymbol{\rho(\lambda)}$ description fails to reproduce $\frac{1}{N}$ expansion of moments like Tr (M^p), $\langle \mathbf{Tr}\ (\boldsymbol{M^p}) \rangle \neq \int \boldsymbol{d\lambda}\ \langle \boldsymbol{\rho(\lambda)} \rangle\ \boldsymbol{\lambda^p}$, beyond the leading $(N = \infty)$ term. (In the c=1 model, it also fails to reproduce the perturbative expansion of the 2D string S-matrix).

3. In case of matrix quantum mechanics, the eigenvalue density description does not reproduce the leading large N expression for the entanglement entropy $\boldsymbol{S_A = \frac{1}{3}\log\big((x_2 - x_1)\underbrace{p_F(x)/\hbar}_{N/L}\big)}$ which is nonperturbative in N.

Coming attractions….

4. We will discuss an exact bosonization of matrix quantum mechanics (equivalently, an exact bosonization of 1D non-relativistic fermi gas), valid for any N, which incorporates the trace identities.

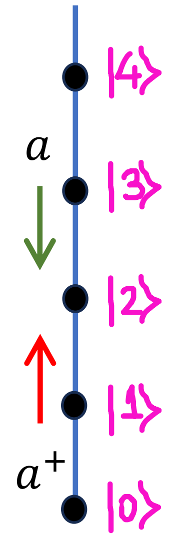
5. **The boson field lives on a lattice of N points (granular geometry)**, which are in one-to-one correspondence with the N eigenvalues.

6. This new bosonization solves the abovementioned problems, including that of the perturbative 2D string S-matrix.

4. Exact bosonization of Matrix QM (non-relativistic fermion)

We would like to find a bosonic description of matrix QM which resolves the problems with the eigenvalue density description. The main point that we would like to ensure is that the degrees of freedom on the bosonic side match with those of the matrix QM.

The way we achieve this is by finding an exact finite N bosonization of non-relativistic fermions (equiv. to MQM) **[Dhar-GM-Suryanarayana].**



N = 1

1-particle Hilbert space $\mathcal{H}$ = Span$\{|n\rangle, n = 0,1,\dots,\infty\}$ where $|n\rangle = \psi_n^+|0\rangle$

For a single particle, there is no distinction between fermions and bosons. We can trivially define oscillators a, a^+ such that

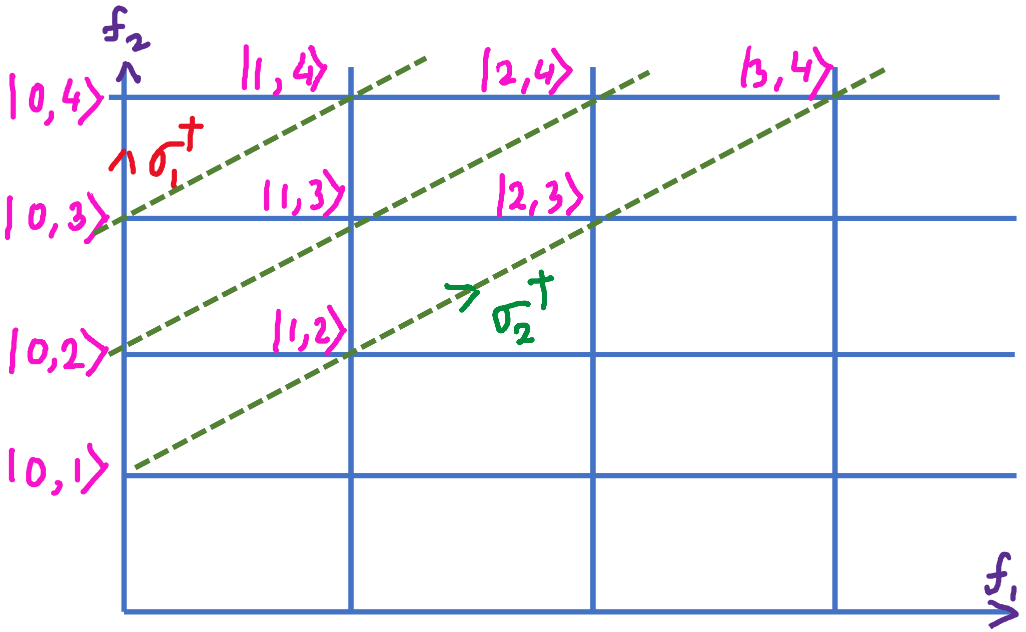
$[a, a^+] = 1$ and

$$|n\rangle = \frac{(a^+)^n}{\sqrt{n!}}|0\rangle,\ a^+|n\rangle = \sqrt{n+1}\,|n+1\rangle,\ a|n\rangle = \sqrt{n}\,|n-1\rangle, n = 0,1,2,\dots,\infty$$

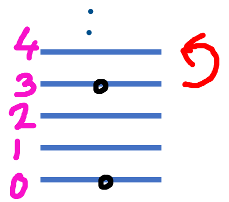
$$a^+ = \sum_{n=0,1,\dots,\infty} \sqrt{n+1}\,\psi_{n+1}^+\,\psi_n\,\delta(\psi_n^+\psi_n) \qquad a = \text{adjoint of } a^+$$

N = 2

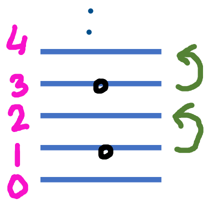
Hilbert space= $\mathcal{H} \wedge \mathcal{H}$ = Span{$|f_1, f_2\rangle$, $0 \le f_1 < f_2 < \infty$}



$$\sigma_1^\dagger |f_1, f_2\rangle = |f_1, f_2+1\rangle$$



$$\sigma_2^\dagger |f_1, f_2\rangle = |f_1+1, f_2+1\rangle$$



σ_1, σ_2 adjoint $\qquad \sigma_i |0,1\rangle = 0$

$$\sigma_i = \frac{1}{\sqrt{a_i^\dagger a_i}} a_i \qquad \sigma_j^\dagger = a_j^\dagger \frac{1}{\sqrt{a_j^\dagger a_j}}$$

$$a_1^\dagger |f_1, f_2\rangle = \sqrt{f_2 - f_1}\, |f_1, f_2+1\rangle$$

$$a_2^\dagger |f_1, f_2\rangle = \sqrt{f_1 + 1}\, |f_1+1, f_2+1\rangle$$

$$[a_i, a_j^\dagger] = \delta_{ij}$$

Bosonic Hilbert space = Span{ $|r_1, r_2\rangle = \prod_{i=1,2} \frac{(a_1^\dagger)^{r_1}(a_2^\dagger)^{r_2}}{\sqrt{r_1! r_2!}} |0\rangle$, $\quad r_1 \ge 0, r_2 \ge 0$ }

These definitions imply $|r_1, r_2\rangle = |f_1, f_2\rangle$, where $\quad r_2 = f_1, \quad r_1 = f_2 - f_1 - 1$

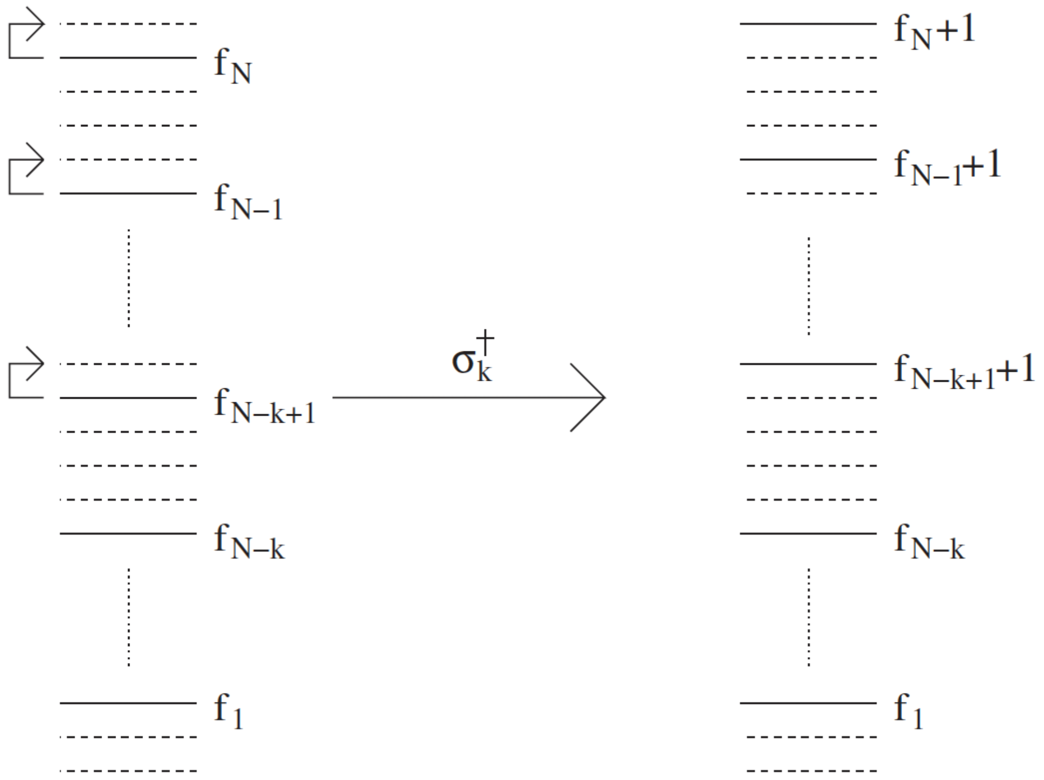


Figure 2: The action of $\sigma_k^\dagger$ on the state $|\vec{f}\rangle$ pushes the top k fermions up by one level, starting from top down. Clearly, if we act on the filled Fermi sea with $\vec{f} = 0, 1, 2, ..., N-1$, one gets the state $\vec{f} = 0, 1, 2, .., N-k-1, N-k+1, N-k+3, ..., N$, i.e. the operator creates a hole at depth k (see also Figure 1; also compare with the action of the Schur of antisymmetric n-tensor as detailed in Appendix D). The action of the adjoint σ_k involves pushing the top k fermions down, starting from down up, with the understanding that it target level is occupied, it annhilates the state. The picture is taken from [34].

$$\sigma_k \sigma_k^\dagger = 1,\ \sigma_k^\dagger \sigma_k = \theta(r_k - 1),\ [\sigma_k, \sigma_l^\dagger] = 0 \text{ if } k \neq l$$

$$\sigma_k = \frac{1}{\sqrt{a_k^\dagger a_k + 1}} a_k,\ \sigma_k^\dagger = a_k^\dagger \frac{1}{\sqrt{a_k^\dagger a_k + 1}}$$

$$[a_n, a_m^\dagger] = \delta_{nm},\ n, m = 1, 2, ..., N$$

Exact Heisenberg algebra

Finite number of bosonic oscillators (=> Finite UV cut-off)

$$|\vec{f}\rangle = |\vec{r}\rangle \equiv \prod_{n=1}^{N} \frac{(a_n^\dagger)^{r_n}}{\sqrt{r_n!}} |0\rangle_B$$

$|0, 1, 2, \cdots, N-1\rangle \equiv |F_0\rangle = |0\rangle_B \quad (r_i = 0, i = 1, \cdots, N)$

$$r_{N-k} = f_{k+1} - f_k - 1,\ k = 1, ..., N-1,\ r_N = f_1$$ .... (4)

Demystifying the correspondence

We wish to understand how, e.g. $\sigma_2^+ \, |0,1\rangle = |1,2\rangle$, $\sigma_1^+ \, |0,1\rangle = |0,2\rangle$. Note $|0,1\rangle = |F_0\rangle$= filled Fermi sea

Suppose that the 1-particle fermion wavefunctions are harmonic oscillator eigenfunctions: then

$$\langle z_1, z_2 | 0,1 \rangle = \begin{vmatrix} 1 & 1 \\ z_1 & z_2 \end{vmatrix} e^{-(|z_1|^2 + |z_2|^2)/2}$$

← Slater determinant of the Fermi ground state $|F_0\rangle$

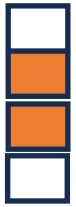
σ_2^+ is an operator which creates a hole at depth **2**. The Berezin representation of such an operator is a Schur polynomial corresponding to the representation ⊟ (YT of height **2**)

$$\chi_{⊟}(z_1, z_2) = z_1 z_2 = \tfrac{1}{2}(z_1 + z_2)^2 - \tfrac{1}{2}(z_1^2 + z_2^2)$$

Thus, $\langle z_1, z_2 | \sigma_2^+ | 0,1 \rangle = \chi_{⊟}(z_1, z_2) \langle z_1, z_2 | 0,1 \rangle = \underbrace{z_1 z_2 (z_2 - z_1)}_{\begin{vmatrix} z_1 & z_2 \\ z_1^2 & z_2^2 \end{vmatrix}} e^{-(|z_1|^2 + |z_2|^2)/2} = \langle z_1, z_2 | 1, 2 \rangle$

$|0,1\rangle$

$|1,2\rangle$



Hole at depth 2

Similarly $\sigma_1^+ \longrightarrow \chi_{□}(z_1, z_2) = z_1 + z_2$ $\qquad (z_1 + z_2)(z_2 - z_1) = \begin{vmatrix} 1 & 1 \\ z_1^2 & z_2^2 \end{vmatrix}$

Hole at depth 1

History: Holes = giant gravitons.

Bosonized Hamiltonian:

$r_{N-k} = f_{k+1} - f_k - 1,\ k = 1, ..., N-1,\ r_N = f_1 \Rightarrow \quad f_1 = r_N, f_2 = r_N + r_{N-1} + 1, \dots, \boldsymbol{f_n = r_N + r_{N-1} + \ \dots + r_{N-n+1} + n - 1}$

Hence, energy of the fermionic state $|f_1, f_2, \dots, f_N\rangle$ is

$= \sum_{n=1}^{N} \epsilon(f_n) = \sum_{n=1}^{N} \epsilon(r_N + r_{N-1} + \ \dots + r_{N-n+1} + n - 1)$ ....(5)

What bosonic Hamiltonian, acting on $|f_1, f_2, \dots, f_N\rangle = |r_1, r_2, \dots, r_N\rangle$, produces the result (5)?

$H_B = \sum_{n=1}^{N} \epsilon(\hat{n}_n), \quad \hat{n}_n = \sum_{i=n}^{N} \hat{a}_i^{+} \hat{a}_i + N - n$ .....(5a)

Thus, e.g., for fermions in a box, where $\epsilon(f_n) = \alpha\, f_n^2,\ \alpha = \frac{\hbar^2\pi^2}{L^2}$, we get

$H_B = \alpha \sum_{n=1}^{N} (\sum_{i=n}^{N} a_i^{+} a_i + N - n)2 = E_g + H_1 + H_2, \qquad E_g = \alpha \frac{N(N-1)(2N-1)}{6},$

$H_1 = 2\alpha\, N \sum_{k=1}^{N} k\, a_k^{+} a_k, \quad \mathrm{H}_2 = \frac{\alpha}{2} \sum_{k=1}^{N} \left(-k^2 a_k^{+} a_k + \left(\sum_{i=k}^{N} (a_i^{+} a_i) \right)^2 \right)$ ....(6)

The Bose-Fermi mapping is EXACT !

In the fixed fermion number sector, physical excitations are fermion bilinears $\psi_m^+\psi_n$. ALL such bilinears can be written exactly in terms of the N pairs of bosonic oscillators a_m^+, a_n and vice versa:

$$\psi_n^\dagger\,\psi_n = \sum_{k=1}^{N}\delta\Big(\sum_{i=k}^{N} a_i^\dagger a_i - n + N - k\Big), \qquad \cdots (6a)$$

$$\psi_{n+1}^\dagger\,\psi_n = \sigma_1^\dagger\,\delta\Big(\sum_{i=1}^{N} a_i^\dagger a_i - n + N - 1\Big) + \sum_{k=1}^{N-1}\sigma_k\,\sigma_{k+1}^\dagger\,\theta_+(a_k^\dagger a_k - 1)\,\delta\Big(\sum_{i=k+1}^{N} a_i^\dagger a_i - n + N - k - 1\Big), \quad \cdots$$

$$a_k^\dagger \equiv \sum_{m_k>m_{k-1}>\cdots>m_0}\sqrt{m_1-m_0}\,(\psi_{m_0}^\dagger\psi_{m_0})(\psi_{m_1+1}^\dagger\psi_{m_1})\cdots(\psi_{m_k+1}^\dagger\psi_{m_k}) \times\delta\Big(\sum_{m=m_0+1}^{m_1-1}\psi_m^\dagger\psi_m\Big)\,\delta\Big(\sum_{m=m_1+1}^{m_2-1}\psi_m^\dagger\psi_m\Big)\cdots\delta\Big(\sum_{m=m_{k-1}+1}^{m_k-1}\psi_m^\dagger\psi_m\Big) \times\delta\Big(\sum_{m=m_k+1}^{\infty}\psi_m^\dagger\psi_m\Big), \qquad k=1,2,\cdots,(N-1)$$

$$a_N^\dagger \equiv \sum_{m_N>m_{N-1}>\cdots>m_1}\sqrt{m_1+1}\,(\psi_{m_1+1}^\dagger\psi_{m_1})\cdots(\psi_{m_N+1}^\dagger\psi_{m_N}) \times\delta\Big(\sum_{m=m_1+1}^{m_2-1}\psi_m^\dagger\psi_m\Big)\cdots\delta\Big(\sum_{m=m_{N-1}+1}^{m_N-1}\psi_m^\dagger\psi_m\Big) \times\delta\Big(\sum_{m=m_N+1}^{\infty}\psi_m^\dagger\psi_m\Big).$$

Applications of the exact bosonization

GM-Mohan 2024
+ work in progress

1. Calculation of moments I_p

2. Entanglement entropy

Finite number of fermions N

1. $$I_p = \frac{1}{N}\langle F_0 | \int d\lambda\, \lambda^p\, \psi^\dagger(\lambda)\psi(\lambda)\, | F_0\rangle = \frac{1}{N}\sum_{n=0}^{\infty}\langle F_0|\psi_n^\dagger \psi_n|F_0\rangle \underbrace{\int d\lambda\, \lambda^p\, |\chi_n(\lambda)|^2}_{f_{p,n}} = \frac{1}{N}\sum_{n=0}^{N-1} f_{p,n}$$

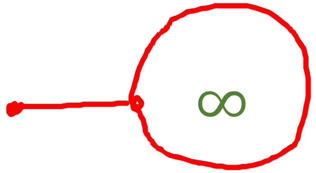
E.g., for fermions in harmonic oscillator potential $I_4 = \frac{1}{2\omega^2}\left(1 + \frac{1}{N^2}\cdot\frac{1}{2}\right)$

In terms of the bosons, the exact transcription on the previous page gives (using (6a))

$$I_p = \frac{1}{N}\sum_{n=0}^{\infty}\langle 0|\sum_{k=1}^{N}\delta\Big(\sum_{i=k}^{N} a_i^\dagger a_i - n + N - k\Big)|0\rangle f_{p,n} = \frac{1}{N}\sum_{n=0}^{\infty}\sum_{k=1}^{N}\delta(N-n-k)\, f_{p,n}$$

$$= \frac{1}{N}\sum_{k=1}^{N} f_{p,N-k} = \frac{1}{N}\sum_{m=0}^{N-1} f_{p,m}$$

Finite number of bosonic modes N (uv-finite)



In particular, $$I_4 = \frac{1}{2\omega^2}\left(1 + \frac{1}{N^2}\cdot\frac{1}{2}\right)$$

Collective field theory has infinite no. of modes

Thus, our bosonic theory reproduces the fermionic moment exactly (not a surprise).

For entanglement entropy, the question is slightly more involved.

The issue is as follows.

Consider a spatial region A in the fermionic theory.

We have so far found an oscillator bosonization of the fermion theory.

Of course, the Fermionic EE for the region A will have a (complicated) translation in terms of the bosonic Hilbert space.

The important question is, does the fermionic EE correspond to the EE of a certain “region” of a real space bosonic formulation of the above exact bosonization?

This is what we do next: to invent a real space version of the bosonic theory.

We will find that indeed the fermionic EE in a region A becomes the bosonic EE in a corresponding spatial region of the bosonic theory.

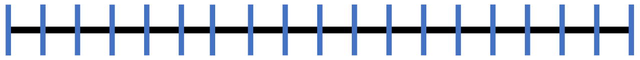
Introducing the real space: lattice boson

GM-Mohan 2024

Inventing local fields from the oscillators by introducing "space": first consider fermions in a box.

The linear dispersion relation in $H_1 = 2\alpha\ N\ \sum_{k=1}^{N} k\ a_k^+\ a_k$ suggests a bosonic field with EOM $\ddot{\Phi}(x,t) = \partial_x^2 \Phi(x,t)$ ...(6)

Since there are a finite number N of oscillators, we cannot have continuous space. We must have discrete space, with N lattice points, with the following normal mode expansion



$x_0 = 0$ $\quad$ $x_j = j\ \epsilon$ $\quad$ $x_{N+1} = (N+1)\epsilon = L$

$$\phi_j = \sum_{n=1}^{N} \frac{1}{\sqrt{2\omega_n}} \sqrt{\frac{2}{L}} \sin\left(\frac{n\pi j}{N+1}\right) [a_n + a_n^\dagger]$$

$$\pi_j = \sum_{n=1}^{N} i\epsilon \sqrt{\frac{\omega_n}{2}} \sqrt{\frac{2}{L}} \sin\left(\frac{n\pi j}{N+1}\right) [-a_n + a_n^\dagger]$$

Note that $[\phi_j, \pi_l] = i\ \delta_{jl}$ → **EXACT HEISENBERG ALGEBRA**

The Heisenberg algebra does not require specifying ω_n, but if we consider the lattice version of the EOM (6) , namely

$$\ddot{\phi}_j = \frac{\phi_{j+1} + \phi_{j-1} - 2\phi_j}{\epsilon^2}$$ we get $$\omega_n = \frac{2}{\epsilon} \sin\left(\frac{n\pi}{2(N+1)}\right)$$

Low energies: massless scalar

Recall that $|F_0\rangle \equiv |0,1,2,\ldots N-1\rangle = |0\rangle_B$. Low energy excitations above the Fermi sea correspond to creating holes at low depths, therefore the only non-zero occupation numbers $a_k^+ a_k$ are with small $k \ll N$.
For free fermions in a box, recall eq. (6)

$$H_1 = 2\alpha N \sum_{k=1}^{N} k\, a_k^+ a_k \,, \quad H_2 = \frac{\alpha}{2} \sum_{k=1}^{N} \left(-k^2 a_k^+ a_k + \left(\sum_{i=k}^{N} (a_i^+ a_i) \right)^2 \right)$$

Under the above conditions

$H_2 \ll H_1$ Also, $H_1 \approx \widetilde{H_1} = 2\alpha N \frac{L}{\pi} \sum_{k=1}^{N} \omega_k a_k^+ a_k \,, \quad since \quad \frac{L}{\pi}\omega_k = \frac{L}{\pi}\frac{2}{\epsilon}\sin\left(k \frac{\pi}{2(N+1)}\right) \approx k$

Now, $\widetilde{H_1}$ is the lattice Hamiltonian of a massless scalar in 1+1 dimensions, $\widetilde{H_1} \approx \sum_{j=1,2,\ldots,N} \pi(x_j)^2 + (\Delta_j \phi(x_j))^2$

Note that the ground state of $\widetilde{H_1}$, namely $|0\rangle_B$, is identical to that of the full bosonic Hamiltonian H. (both are given by the state $|0\rangle_B$ which is annihilated by the a_k). Hence EE for the ground state of H should be that of the relativistic scalar:

$S_A = \frac{1}{3}\log\frac{l}{\epsilon}$, where ϵ is the lattice cut-off. [Holzhey-Wilczek, Calabrese-Cardy, Peschel, …]

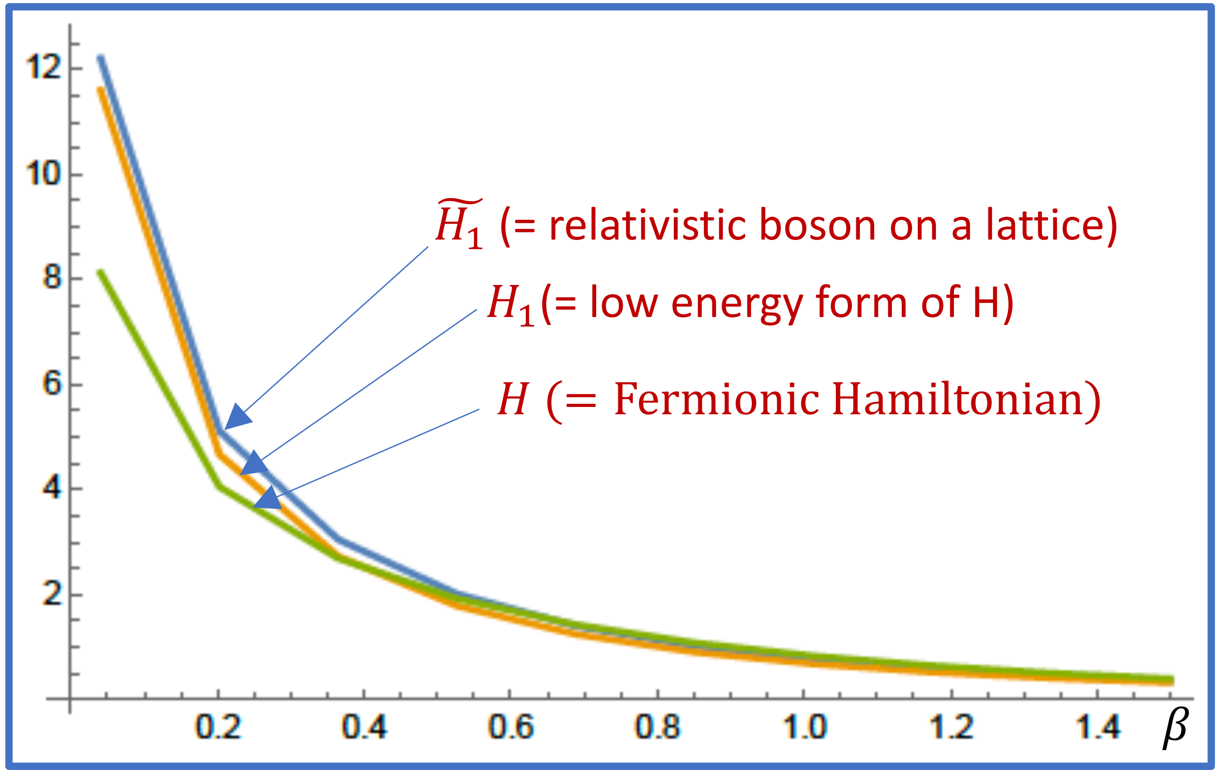
We verify this explicitly below. But before that …..

Low temperature partition function

$$H_1 = 2\alpha\, N \sum_{k=1}^{N} k\, a_k^+ a_k\,, \quad H_2 = \frac{\alpha}{2}\sum_{k=1}^{N}\left(-k^2\, a_k^+ a_k + \left(\sum_{i=k}^{N}(a_i^+ a_i)\right)^2\right), \quad H = H_1 + H_2$$

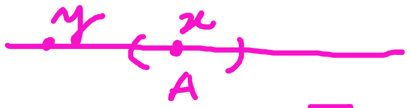
$$\widetilde{H_1} = 2\alpha\, N \frac{L}{\pi}\sum_{k=1}^{N}\omega_k\, a_k^+ a_k\,, \qquad \frac{L}{\pi}\,\omega_k = \frac{L}{\pi}\frac{2}{\epsilon}\sin\left(k\frac{\pi}{2(N+1)}\right) \approx k$$

$\widetilde{H_1}$ is the lattice Hamiltonian of a massless scalar in 1+1 dimensions, $\widetilde{H_1} \approx \sum_{j=1,2,\ldots,N} \pi(x_j)^2 + (\Delta_j \phi(x_j))^2$



Partition functions for various Hamiltonians as a function of inverse temperature β

Entanglement entropy: lattice boson



$$\Psi_0[\Phi(\cdot)] \sim e^{-\sum_{x,x'} K_{xx'}\Phi_x\Phi_{x'}}$$

$$\rho_A[\Phi(x), \tilde{\Phi}(x)] \qquad x \in A, \; y \in \bar{A}$$

$$= \int d\Phi_y \, \Psi_0^*[\Phi(x), \Phi(y)] \, \Psi[\tilde{\Phi}(x), \Phi(y)]$$

$$S_A = -\mathrm{Tr}_A \, \rho_A \log \rho_A$$

We can now compute the $\langle\phi\phi\rangle$ and $\langle\pi\pi\rangle$ correlators in the ground state

$$\langle\phi(j)\phi(k)\rangle = \sum_{m=1}^{N} \frac{1}{2\omega_m} f(m,j)f(m,k)$$

$$\langle\pi(j)\pi(k)\rangle = \epsilon^2 \sum_{m=1}^{N} \frac{\omega_n}{2} f(m,j)f(m,k),$$

with $f(m,j) = \sqrt{2/L}\sin(\pi mj/(N+1))$, and use Casini-Huerta method to compute EE in Mathematica to see that it gives the result:

$$S_A = \frac{1}{3}\log\left(\frac{Nl}{L}\right)$$

where l is the size of the interval A.

This PRECISELY reproduces the fermionic result! We thus have a bosonic description reproducing the (finite) EE.

The matching of the real spaces (box $\rightarrow$ box) is a surprise, since the bosonic field $\phi(x_j)$ is rather non-local in terms of the fermion field $\psi(x)$ or the fermion bilinear $\psi^+(x)\,\psi(y)$. The reason it works is that at low energies, the relation becomes approximately local.

Bosonic EE for fermions in a non-trivial external potential

Note that (e.g. from equation (5a)), even in the presence of a non-trivial external fermion potential, the state $|0\,\rangle_B$ remains the exact ground state of the bosonic Hamiltonian, even though it can be complicated and interacting.

The simple way to see it is that the fermionic Hamiltonian is minimized by the Fermi ground state $|F_0\rangle$, hence the corresponding state $|0\,\rangle_B$ must minimize the bosonic Hamiltonian which is an exact translation of the Fermionic Hamiltonian.

THIS SEEMS TO LEAD TO A PUZZLE!!

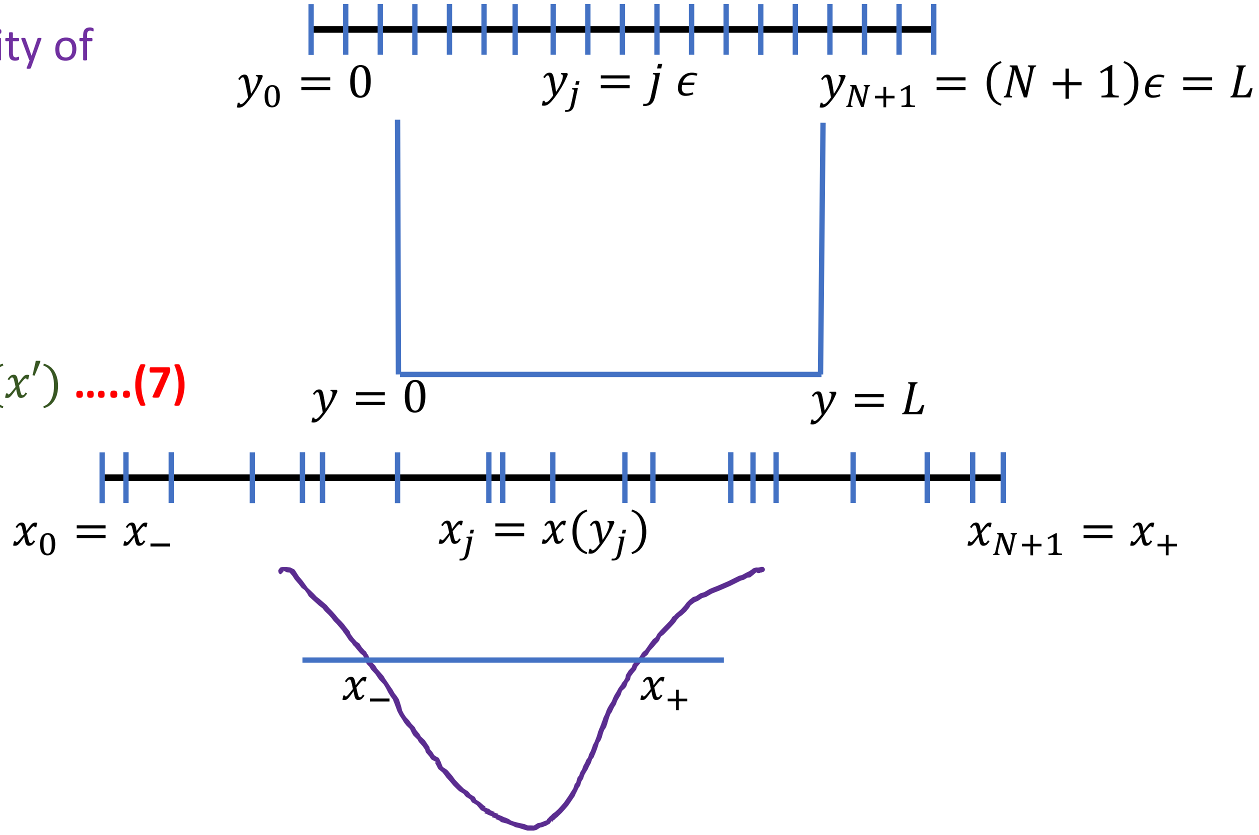
$\Psi_0[\Phi(x)] \sim e^{-\sum_{x,x'} K(x,x')\,\Phi(x)\Phi(x')}$ is the same as for a massless scalar. Hence, wouldn't the EE remain the same? But with a potential, the expression for the EE is different, and it depends on the potential

$$S(\lambda_1,\lambda_2) = \frac{1}{3}\log\left(\frac{(\lambda_2 - \lambda_1)\,p_F(\lambda_0)}{\hbar}\right),\quad p_F(\lambda) = \sqrt{2(\epsilon_F - V(\lambda))},$$

If the ground state is the same, how does the dependence on the potential $V(\lambda)$ come about**???**

Solution: let the potential determine the geometry of the lattice

The existence of a non-zero potential introduces inhomogeneity of space.



This corresponds to a reparameterization of the lattice

$$\sqrt{2E_N}\, y = 2\pi \frac{N}{L}\, y = \int_{x_-}^{x} dx \sqrt{2\left(\tilde{E}_N - V(x')\right)} = \int_{x_-}^{x} dx'\; p_F(x') \quad \text{.....(7)}$$

$$\phi(y_j) = \sum_{k=1}^{N} \frac{1}{\sqrt{2\,\omega_k}} \left(a_k\, e^{2\pi i\, j\frac{k}{N}} + a_k^+\, e^{-2\pi i\, j\frac{k}{N}}\right)$$

The conjugate field is defined by

$$\pi(y_j) = \frac{1}{i} \sum_{k=1}^{N} \frac{\sqrt{\omega_k}}{\sqrt{2}} \left(a_k\, e^{2\pi i\, j\frac{k}{N}} - a_k^+\, e^{-2\pi i\, j\frac{k}{N}}\right)$$

$\tilde{\phi}(x_j) = \phi(y_j)$, $\quad \tilde{\pi}(x_j) = \pi(y_j)$ $\quad$ so that $[\phi(x_j), \pi(y_l)] = i\,\delta_{jl}$ → **EXACT HEISENBERG ALGEBRA**

The EE is clearly $\boldsymbol{S[y_1, y_2] = \frac{1}{3}\log\left(\frac{(y_2 - y_1)N}{L}\right) = \frac{1}{3}\,log\big((x_2 - x_1)\,p_F(x_0)\big)}$ **reproducing the fermion EE of [Das et al]!**

(we have assumed $p_F(x)$ to be a slowly varying function of x in the interval $(x_1,\ x_2)$)

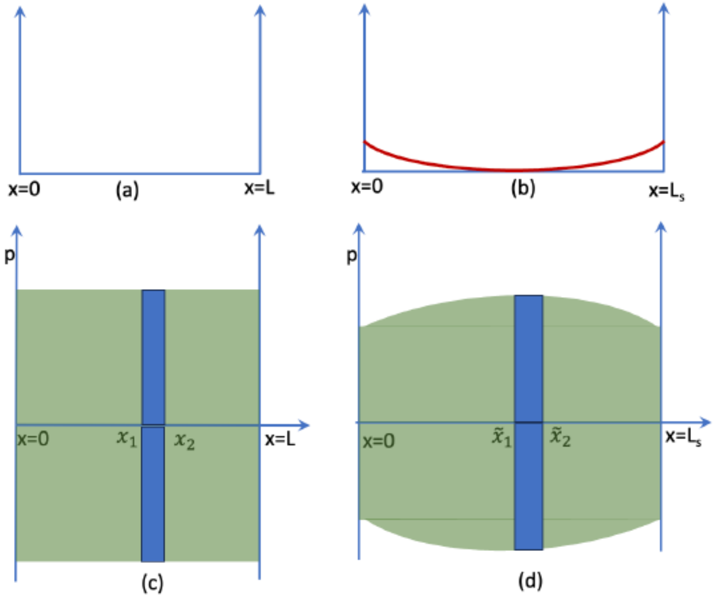
Justification for the coordinate transformation:

The ground state is semi-classically described by a fluid droplet occupying a certain region R, bounded by the Fermi surface.

One can make a canonical transformation of the single-fermion phase space; the fluid droplet changes its shape (without changing the area). The boundary of the new droplet can be interpreted as the Fermi surface of a new Hamiltonian.

One can ask: what is the canonical transformation needed to go from the free particle (in a box) Hamiltonian $h = \frac{p^2}{2}$ to $h' = \frac{p^2}{2} + V(\lambda)$?

Such a transformation can be found. When restricted on the Fermi surface, the canonical transformation projects onto a transformation of the x-coordinate. The coordinate transformation precisely turns out to be **(7).**



Why does fermionic real space map to the bosonic real space (albeit latticized)?

Locality

The EE of the bosonic theory for an interval A corresponds to an operator algebra of the bosonic fields $\phi(x), \pi(x)$, where $x \in A$.

From the expression of the bosonic oscillators in terms of the fermion bilinears, these local bosonic oscillators appear to be extremely non-local in terms of the fermion bilinears $\psi^{+}(x)\psi(y)$, involving operators which are outside of A

Why does the bosonic EE match with the fermionic EE then?

It can be shown that on low energy bosonic states (fermionic states near Fermi surface), the relation is approximately local.

5. C=1 matrix model

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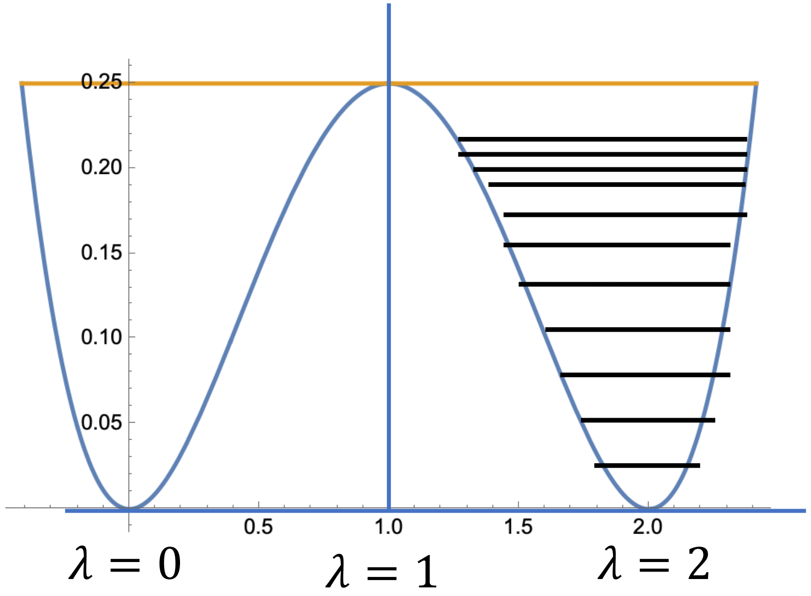
+ in progress

$$S = \beta N \int dt \{ \frac{1}{2} \mathrm{Tr}(\dot{M})^2 - \mathrm{Tr} V(M) \} \qquad V(\lambda) = \frac{1}{4}\lambda^2(\lambda-2)^2,$$



Entanglement entropy $S_A = \frac{1}{3}\log[2\frac{(\lambda_2 - \lambda_1)p_F(\lambda)}{\hbar}]$

where $p_F(\lambda) = \sqrt{2(\epsilon_F - V(\lambda))}$

As we saw above, this result can be obtained from the fermionic formulation as well as from the exact lattice boson described above.

Dual to 2D string:

$$\Psi^+(\lambda,t)\Psi(\lambda,t) = \rho(\lambda,t)$$

Density fluctuations near the Fermi surface map to close string tachyon

Double scaled c=1 matrix model

$$S=\beta N\int dt\{\frac{1}{2}\mathrm{Tr}(\dot{M})^2-\mathrm{Tr}V(M)\}\qquad V(\lambda)=\frac{1}{4}\lambda^2(\lambda-2)^2,$$

$$H\to\beta NH'',\ H''=\sum_{i=1}^{N}h(\frac{\partial}{\partial\lambda_i},\lambda_i),\qquad h(\frac{\partial}{\partial\lambda},\lambda)=-\frac{1}{(\beta N)^2}\frac{\partial^2}{\partial\lambda^2}+V(\lambda)$$

"Single scaling" $\quad$ '$\hbar$' $=\frac{1}{\beta N}\quad\to 0$

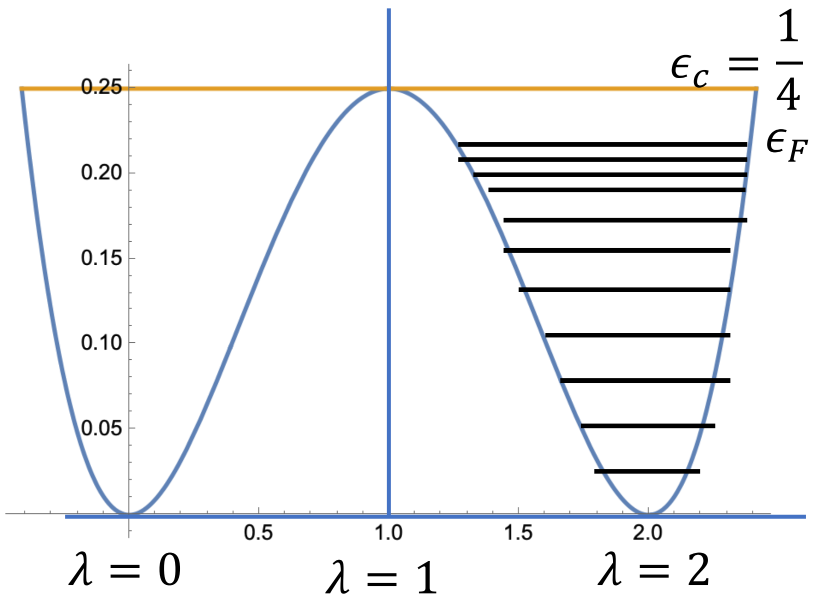
"Double scaling" $\quad N\to\infty,\beta\to\beta_c,\ \mu=-\beta N(\epsilon_F-\epsilon_c)$

→ Taking fermi level near the top

Also, zoom near the maximum $\lambda=1$ $\ \tilde{\lambda}=\sqrt{\beta N}(\lambda-1)$

$$p_F(\lambda)=\sqrt{2(\epsilon_F-V(\lambda))}=\sqrt{2(-\frac{\mu}{\beta N}-\frac{\tilde{\lambda}^2}{2\beta N}+\frac{\tilde{\lambda}^4}{4\,(\beta N)^2})}\sim 1/\sqrt{N}$$

$$\lambda_2-\lambda_1=(\tilde{\lambda}_2-\tilde{\lambda}_1)/\sqrt{\beta N}\sim 1/\sqrt{N}\qquad\frac{1}{\hbar}\sim N$$



$$V(\lambda)=\frac{1}{4}-\frac{(\lambda-1)^2}{2}+\frac{(\lambda-1)^4}{4}=\frac{1}{4}-\frac{\tilde{\lambda}^2}{2\beta N}+\frac{\tilde{\lambda}^4}{4\,(\beta N)^2}$$

$$\epsilon_F-\epsilon_c=\epsilon_F-\frac{1}{4}=-\frac{\mu}{\beta N}$$

Entanglement entropy S_A

$$= \frac{1}{3}\log\left(2\frac{(\tilde{\lambda}_2-\tilde{\lambda}_1)}{\sqrt{\beta N}}\beta N\sqrt{2\left(-\frac{\mu}{\beta N}+\frac{\tilde{\lambda}^2}{2\beta N}-\frac{\tilde{\lambda}^4}{4\beta^2 N^2}\right)}\right) \qquad \tilde{\lambda} = \sqrt{2\mu}\cosh\left(\frac{x}{l_s}\right)$$

$$= \frac{1}{3}\log\left(2(\tilde{\lambda}_2-\tilde{\lambda}_1)\sqrt{2\left(-\mu+\frac{\tilde{\lambda}^2}{2}\right)}\right) = \frac{1}{3}\left(\log\left(\frac{x_2-x_1}{l_s g_s}\sinh^2(x_0/l_s)\right)\right) \quad \text{… (8)}$$

This formula has a surprise, as the role of the short distance cut-off seems to be played by

$$\epsilon = l_s g_s, \qquad 12)$$

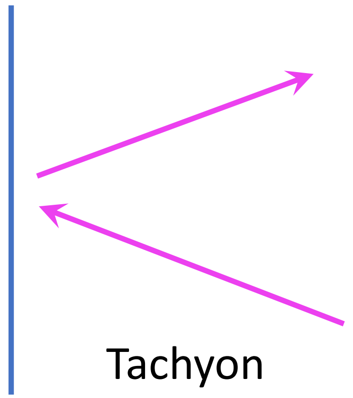
rather than just the string length l_s, which is what one *a priori* expects in a string theory.

The EE in eq. **(8)** was originally derived by **Das (1995)** and **Hartnoll-Mazenc (2015)** using the fermion theory. As we see, we have derived this here using the exact lattice boson theory.

(Note that (**see Sumit's talk**) in collective field theory, the EE is expected to be divergent in any order of perturbation theory).

Below, we will discuss the relation of this bosonic field with D0 branes. But, before that, we discuss how our lattice boson correctly captures loop corrections to 2D string S-matrix elements.

S- matrix of 2D string theory



Tachyon scattering off the wall

$\langle 0|\rho(1)\rho(2)|0\rangle$

c=1 Matrix model (= Fermion field theory) gives

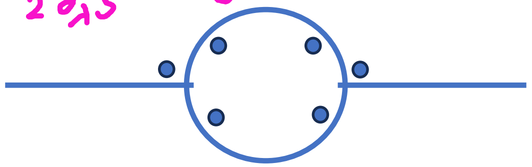
$$S(1 \to 1) = S_0 + g_s^2\, S_1, \qquad S_0 = \omega, \qquad S_1 = \frac{1}{24}\left(i\,\omega^2 + 2\,i\,\omega^4 - \omega^5\right) + O(g_s^4)$$

Sengupta-Wadia 1990, GM-Sengupta-Wadia 1991, Moore-Plessar-Ramgoolam 1991

S_1 corresponds to a 1-loop contribution in the string theory. This has been exactly reproduced by a torus world sheet calculation by **Yin et al 2018**.

In collective field theory, given by the cubic action, $S_c = N^2 \int dt\, d\lambda \left\{ \frac{\dot{\zeta}^2}{2\,\partial_\lambda \zeta} - \frac{\pi^2}{6}(\partial_\lambda \zeta)^3 - V(\lambda)\,\partial_\lambda \zeta \right\}$ $\quad \partial_\lambda \zeta \equiv \frac{\rho(\lambda)}{N}$

the above one-loop term corresponds to the following diagram



In **(GM-Sengupta-Wadia 1991)**, it was shown to be quadratically divergent. The divergence could be removed by a counter-term, but the remaining finite expression differed from S_1 above.

Result: In our new bosonic formulation, the above $S(1 \to 1)$ is exactly reproduced!

New c=1

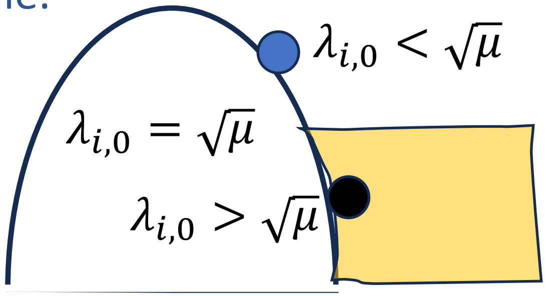
(**McGreevy-Verlinde, Klebanov-Maldacena-Seiberg, Berenstein, Gaiotto-Itzhaki-Rastelli**)

$M_{ij}(t) \equiv T_{ij}(t) =$ open string tachyon field on N D0 branes = dual to 2D closed string in a linear dilaton background

$V(M) = -\frac{1}{2\alpha'} M^2$ is reinterpreted as $V(T) = -\frac{1}{2\alpha'} T^2$ which is part of the spacetime tachyon potential, with mass$^2 = -\frac{1}{\alpha'}$

Eigenvalues of the matrix M, namely $\lambda_i(t)$, represent open string tachyon field on the i-th D0 brane.

Classical solution (with $l_s = \sqrt{\alpha'}$): $\lambda_i(t) = \sqrt{\mu}\sin(\pi\tilde{\lambda}_0)\cosh\left(\frac{t}{l_s}\right)$ **…(9)**



This is the rolling tachyon solution on an unstable D brane **(Sen)**

In this rolling tachyon background, string world-sheet action becomes (in $l_s = 1$ units)

$$S_{ws} = \int d^2\xi\, \partial X^\mu \partial X_\mu + \tilde{\lambda}_0 \int d\sigma \cosh(t(\sigma))$$

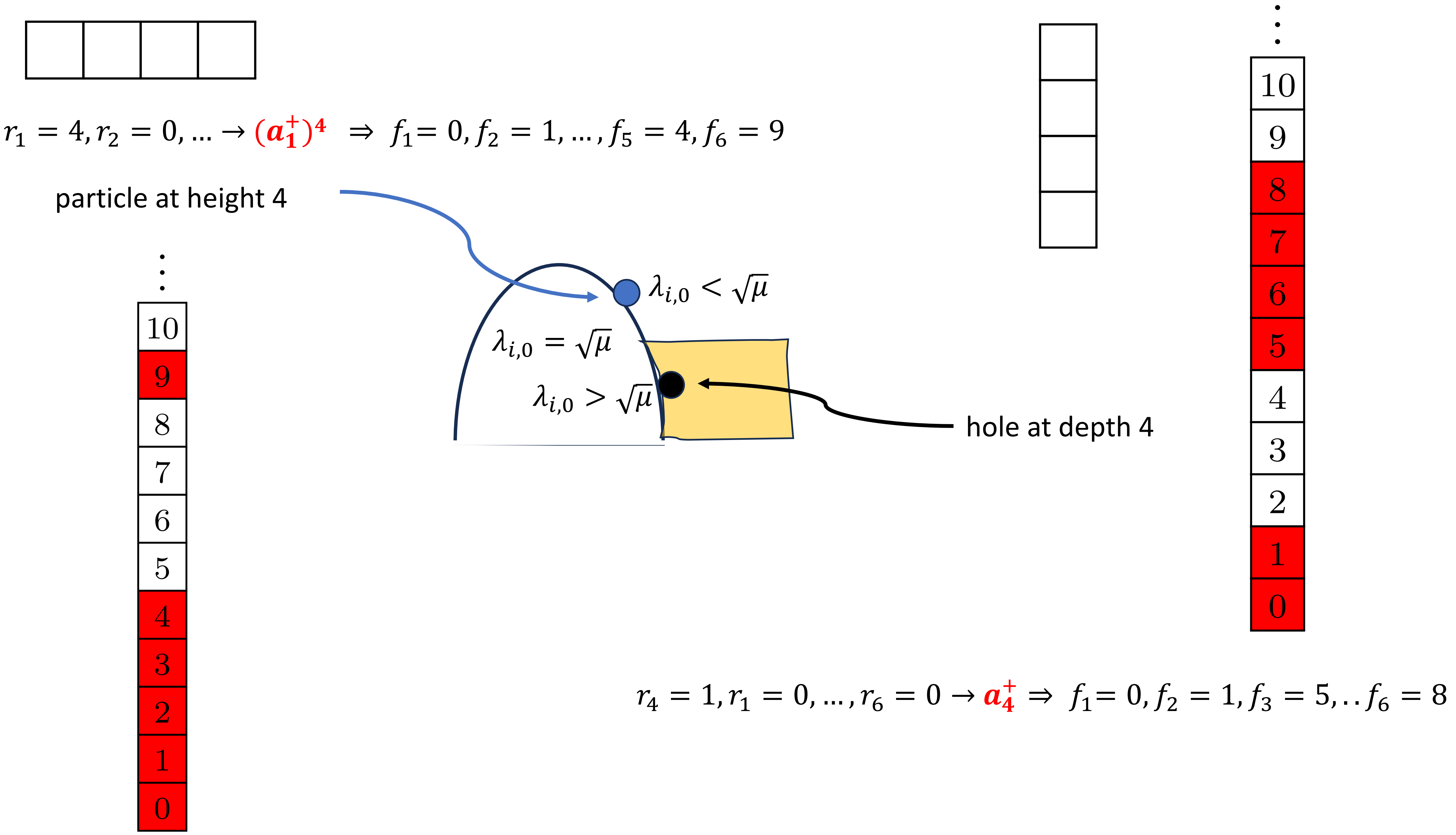
$V(\lambda, t)$= vertex operator related to the ground ring element $O_{1,2} + O_{2,1}$

Such a boundary deformation to a CFT corresponds to a boundary state $|Bd\rangle$

Closed string field $\int d\lambda\, \Psi^+(\lambda)\, \lambda\, \Psi(\lambda)$, generated by the D0 brane, is $\langle Bd|V(\lambda,t)|Bd\rangle = \sqrt{\mu}\sin(\pi\tilde{\lambda}_0)\cosh(t)$ **(same as (9))**

Eigenvalue above the fermi surface (particle): $\tilde{\lambda}_0$ = real, hence at $t = 0, \lambda_i \leq \sqrt{\mu}$ (see the blue dot)

Eigenvalue below the fermi surface (hole): $\tilde{\lambda}_0 = \frac{1}{2} + i\,\tilde{\tilde{\tilde{\lambda}}}_0$, $\tilde{\tilde{\tilde{\lambda}}}_0$ = real, hence at $t = 0, \lambda_i \geq \sqrt{\mu}$ (see the black dot)



$r_1 = 4, r_2 = 0, \ldots \rightarrow (a_1^{+})^4 \Rightarrow f_1 = 0, f_2 = 1, \ldots, f_5 = 4, f_6 = 9$
particle at height 4
$\lambda_{i,0} < \sqrt{\mu}$
$\lambda_{i,0} = \sqrt{\mu}$
$\lambda_{i,0} > \sqrt{\mu}$
hole at depth 4
10
9
8
7
6
5
4
3
2
1
0
10
9
8
7
6
5
4
3
2
1
0
$r_4 = 1, r_1 = 0, \ldots, r_6 = 0 \rightarrow a_4^{+} \Rightarrow f_1 = 0, f_2 = 1, f_3 = 5, \ldots f_6 = 8$

Lattice boson for unitary matrix model

GM-Ramgoolam-Suroshe, in progress

Unitary matrix model:

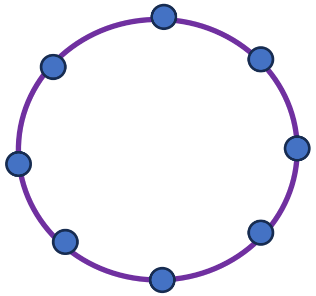
$$\int DU\, O(U) = \int \prod_{i=1,\dots,N} d^2 u_i \exp[-\sum_{k=1,\dots,N} \frac{\overline{u_k}\, u_k}{k}]\; O(u_i, \overline{u_i})$$

This can be rigorously shown for operators which are not "too large", i.e. when $\boldsymbol{O = \prod_{i=1,\dots,N} u_i^{P_i}\, \overline{u_i}^{Q_i}}$ **with** $\boldsymbol{\sum_j j\, P_j \le N,\ \sum_j j\, Q_j \le N,}$

The above equation can be rewritten in terms of a lattice field:

$$\chi(\theta_j) = \sum_{n=1,\dots,N} \frac{u_n}{\sqrt{n}} \exp(i\, n\, \theta_j) = \sum_{n=1,\dots,N} \frac{u_n}{\sqrt{n}} \exp\left(i\, 2\pi\, n \frac{j}{N}\right), \qquad \theta_j = 2\pi \frac{j}{N}$$

$$\int DU\, O(U) = \int \prod_{j=1,\dots,N} \mathrm{d}\chi(\theta_j)\, \mathrm{d}\overline{\chi(\theta_j)}\; \exp[-\sum_{j=1,\dots,N} \chi(\theta_j)\overline{\chi(\theta_j)}]\quad O(\chi(\theta_j), \overline{\chi(\theta_j)})$$



CONCLUSIONS

We pointed out subtleties with a continuum description of large N limit of matrix models: both for the time-independent matrix model (c=0), as well as for matrix QM (in particular c=1)

For matrix QM, a continuum description in terms of the eigenvalue density (the collective variable) gives divergent answers for (a) moments and (b) entanglement entropy, whereas they are finite in the matrix QM.

We found an exact lattice bosonization of matrix QM. It resolves the two problems mentioned above.

It appears that the exact boson field $\phi(x)$ is a second quantized field describing multiple D0 branes! It is perhaps natural that such a field theory comes with a cut-off $g_s l_s$.

What is the lesson for higher dimensions? E.g. for N=4 SYM ←→ AdS gravity, … ?

Thank you!