Games

All games considered are played by two players. Players make moves one after another. We call a player which makes the first move the *first player*. We say that a player has a *winning strategy* if the player can win the game regardless of the moves made by the other player.

1 Winning and losing positions

Problem 1. There is a pile of n stones. Player's move is to take 1 or 2 stones from the pile. The player who can not make a move loses the game. Who has a winning strategy?

Note that the answer could depend on n. Also see the end of the document for a hint

Problem 2. The same as Problem 1, but now player's move is to take a) $1, 2$ or 3 stones b) $1, 2, 3$ or 4 stones c) any number from 1 to k stones.

We can solve Problem 2a, using the notion of *winning position* and *losing position*. For instance, if in the beginning of your turn there are $1, 2$ or 3 stones – you are clearly in winning position, because you can win by a single move. If there are 4 stones, then you are in the losing position – indeed, after your move there will be $1, 2$ or 3 stones on the table left, so your opponent will be in the winning position. Following the same line of thinking we can determine is the position winning or losing for 5 stones, after that for 6 stones, etc.

Try to give (a bit recursive) definitions of winning and losing position in terms of positions where you can get in one move.

Problem 3. There is a pile of 300 stones. Player's move is to take at most half of the stones from the pile. The player who can not make a move loses the game. Who has a winning strategy?

Problem 4. The chess queen is located in the lower left corner of a 10×12 chessboard. In one move player can move the queen any number of cells to the right, up, or diagonally "right-up" (queen cannot be moved left or down). The player who can not make a move loses the game. Who has a winning strategy?

2 Unknown topic

If the problems in this section seem hard, see the name of the section in the end of the document.

Problem 5. There are several piles of stones. Player's move is to take any nonzero number of stones from any one pile. The player who can not make a move loses the game. Who has a winning strategy?

(a) two piles: 20 and 20 stones

(b) two piles: 30 and 20 stones

(c) three piles, 20 stones each

(d) four piles, 20 stones each

Problem 6. There is a round table. Player's move is to place a coin on a table in such a way that the coin does not overlap with other coins. All coins are of the same size. The player who can not make a move loses the game. Who has a winning strategy?

3 Theory

Problem 7. Suppose we have a game which is guaranteed to end after 100 moves. There are no ties. Prove that one of the players has a winning strategy.

As usual we assume that game does not depend on chance and there is no hidden information.

4 Optional problems

Problem 8. There is an infinite sheet of paper with a square grid – nodes connected by vertical and horizontal edges, similar to the grids found in notebooks. First player's move is to paint any unpainted edge in red, second player's move is to paint any unpainted edge in blue. If a closed loop of red edges forms at any moment in the game, the first player wins. Can the second player prevent this?

Problem 9. There is an infinite sheet of paper with a square grid. First player's move is to place crosses "X" in any two empty cells. Second player's move is to place one zero "O" in any one empty cell. The first player wins once there are 100 crosses in a row or column (without any gaps). Can the second player prevent this?

Problem 10. Numbers $1, 2, 3, \ldots, 2024$ are written on the board, separated by commas. The player's move is to replace any comma by $+$ or by $\times$. Once all commas are replaced, we are left with one big number on the board. The first player wins if this number is odd, otherwise the second player wins. Who has a winning strategy?

5 Hints

Hint for Problem 1 Solve the problem for $n = 1, 2, 3, 4, 5, 6, 7$ to see the pattern and the winning strategy.

Hint for unknown topic The topic of the section is "Symmetry".