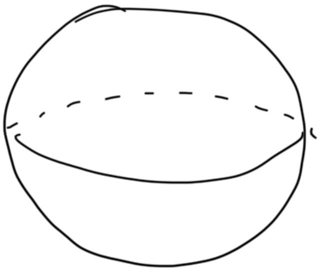
Geometry of the global nilpotent cone

Day 1: Intro to Higgs bundles

① 1.0 Examples

Line bundles

$\boxed{g=0 \quad \mathbb{P}^1}$

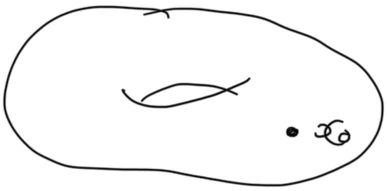


Birkhoff – Grothendie

$\mathcal{O}(d)$ $\qquad$ $\begin{array}{c}\mathrm{Pic}(\mathbb{P}^1)\\ \parallel \\ \mathbb{Z}\end{array}$

$$\begin{array}{ccc} E & \longrightarrow & \mathbb{P}^1 \quad \text{holo} \\ \cong & & \\ \bigoplus_{i=1}^{n} \mathcal{O}(d_i) & & \end{array}$$

$\boxed{g=1}$



$X = \mathbb{C}/\Lambda$

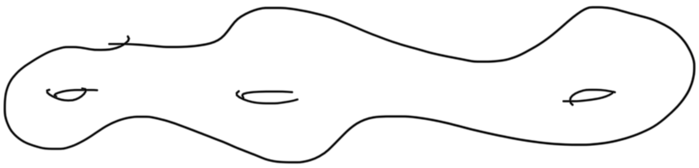
$\Lambda = \pi_1(X, x_0)$

$$\begin{tikzcd} aj_{x_0}: X \arrow[r] & \mathrm{Jac}^0(X) \cong H^{0,1}(X) / H_1(X,\mathbb{Z}) \\ x \arrow[r, mapsto] & \mathcal{O}(x_0 - x) \end{tikzcd}$$

$$\begin{tikzcd} H_1(X,\mathbb{Z}) \arrow[r, hook] & H^{0,1}(X) = \left(H^{1,0}(X)\right)^* \\ \gamma \arrow[r, mapsto] & \int_\gamma \cdot \end{tikzcd}$$

$$\begin{tikzcd} X \arrow[r, "aj_{x_0}"] & \mathrm{Jac}^0(X) \\ x \arrow[r, mapsto] & \int_{x_0}^{x} \cdot \end{tikzcd}$$

$$\boxed{g \geq 2}$$



$$\mathrm{Jac}^0(X) \cong H^{0,1}(X) / H_1(X,\mathbb{Z})$$

Topologically trivial bundle $\mathbb{L}$

Holo. structures on $\mathbb{L}$.

$\Updownarrow$ Newlander-Nirenberg

$$\bar{\partial}_{\mathbb{L}} : \Omega^0(\mathbb{L}) \longrightarrow \Omega^{0,1}(\mathbb{L})$$

- Leibniz rule
- $\bar{\partial}_{\mathbb{L}}^2 = 0$ (trivial for X kähler)

Classify up to iso, i.e. autos $\mathcal{G}$

$$\mathbb{L} \xrightarrow{f} \mathbb{L} \qquad f \in C^\infty(X, \mathbb{C}^\times$$

$$f \circ \bar\partial_1 = \bar\partial_2 \circ f \iff fs \text{ holo grace}$$

for $\bar\partial_1$ $\quad \bar\partial_2(f \cdot s) = 0 = f \circ \bar\partial_1 s$

$$\overset{\|}{f \cdot \bar\partial_2 s + \bar\partial f \cdot s} = 0 \iff \Big(\bar\partial_2 + \underbrace{f^{-1}\bar\partial f}_{\bar\partial \exp f}\Big) s =$$

So

$$\mathcal{G}_0^{\mathbb{C}} = \exp(\underset{\tilde f}{C^\infty(X, \mathbb{C})}) \circlearrowright \Omega^{0,1}(X)$$

complex structure

$$\tilde f \cdot \alpha = \alpha + \bar\partial \tilde f$$

$$\overset{\|}{\log(\exp \tilde f)}$$

So $$\Omega^{0,1}(X) \Big/ \mathcal{G}_0^{\mathbb{C}} = H^{0,1}(X)$$

$$\mathcal{G}_0^{\mathbb{C}} \hookrightarrow \mathcal{G}^{\mathbb{C}} \twoheadrightarrow \pi_0 \mathcal{G} = H_1(X, \mathbb{Z})$$

$$f \in \mathcal{G}^{\mathbb{C}} = C^\infty(X, \mathbb{C}^\times) \rightsquigarrow f_\# : \pi_1(X) \to \mathbb{Z}$$

① 1.1 Higgs bundles in rk 1

Def: a rk 1 Higgs bundle on X is a point of $T^*\mathrm{Jac}^0(X)$

$=: M_X(1,0)$ moduli space of rk 1 deg 0 Higgs bundles

$$T^*\mathrm{Jac}^0(X) \cong \mathrm{Jac}^0(X) \times H^0(K)$$

$$\left(T_{\bar\partial}\mathrm{Jac}^0(X)\right)^* = \left(H^{0,1}(X)\right)^* = H^{1,0}(X)$$

Complex structure I

- As a bundle over $\mathrm{Jac}^0(X)$

$M_X(1,0)$ has a $\mathbb{C}$ str.

- As a cotangent ⇝ holo. symplectic

w/ form

canonical 1-form.

$$d\theta^{c} = \Omega_I\left((\dot{\bar\partial}_1, \dot\varphi_1), (\dot{\bar\partial}_2, \dot\varphi_2)\right) =$$

$$\langle \dot{\bar\partial}_1, \dot\varphi_2 \rangle_{\text{serre}} - \langle \dot{\bar\partial}_2, \dot\varphi \rangle$$

$$\Omega_I = \omega_J + i\,\omega_K$$

Complex structure J

$T^*\mathrm{Jac}^0(X)$ is also a moduli space of flat connections.

Hodge theory

$$\begin{array}{ccccc} H^1(X,\mathbb{C}) & = & H^{1,0}(X) & \oplus & H^{0,1}(X) \\ \cong & & \cong & & \cong \\ \mathcal{H}_{\Delta_d} & & \mathcal{H}_{\Delta_{\bar\partial}} & & \mathcal{H}_{\Delta_\partial} \end{array}$$

harmonic (metric of X)

$(\bar\partial_L, \varphi) \in M(1,0)$

up to gauge $\bar\partial_L = \bar\partial + \underbrace{\alpha^{1,0}}_{\text{harmonic}}$

Rk $\partial + \bar\partial_L + (\alpha^{1,0})^*$ is flat.

$$M(1,0) \longrightarrow \{\text{Flat connection}\}$$

$$(\bar\partial_L, \varphi) \underset{\substack{\uparrow \\ \text{choose a} \\ \text{metric on } L}}{\longmapsto} \underbrace{\nabla_h}_{\text{chern}} + \underbrace{\varphi + \varphi^{*_h}}_{\varphi} = \nabla$$

$\exists!$ h s.t. $\boxed{\nabla^2 = 0}$

for ∇

The sol is harmonic

$$T^{*}\mathrm{Jac}^{0}(X) \underset{\text{diffeo}}{\cong} \mathrm{Hom}(\pi_1, \mathbb{C}^{\times}) \big/ \mathbb{C}^{\times} \cong (\mathbb{C}^{\times}$$

Kähler form conj

$$\begin{tikzcd}
H^1(X,\mathbb{C}) \times H^1(X,\mathbb{C}) \arrow[r, "\cup"] & \mathbb{C} \\
\mathrm{Hom}(\pi_1,\mathbb{C}) \quad \mathrm{Hom}(\pi_1,\mathbb{C}) \arrow[u, phantom, "\shortparallel"] & \\
(u \qquad v) \arrow[r, mapsto] & \sum_{i=1}^{g} u(A_i)v(B_i - u(B_i)v(B
\end{tikzcd}$$

In fact $T^{*}\mathrm{Jac}^{0}(X)$ is hyperkähler

1.2 Higher rank

$$N_X(n,d) = \{ E \ \overset{\text{(semi) stable}}{\vee} \ \mathrm{rk}\ n \ \deg d \ \text{holo v.b.} \} \big/ \text{S-equiv}$$

Def: E is (semi)-stable if

$\forall_{0 \neq} F \subsetneq E$ subbundle

$$\mu(F) := \frac{\deg F}{\mathrm{rk} F} < (\leq) \ \mu(E)$$

Examples

$g=0 \qquad N_X(n,d) = pt.$

$g=1 \qquad N_X(n,d) = \mathrm{Sym}^{(n,d)} X$

$M_X(n,d) = \{ \underbrace{(E,\varphi)}_{\text{(Semi)-stable}}$ → rk n deg d holo.v. ; → Higgs field $\in H^0(\mathrm{End}(E)\otimes K)$

S-e

(same defi. but $F \subsetneq E \quad \varphi(F) \not\subseteq F\otimes K$)

<u>Examples</u>

$g=0 \qquad M_X(n,d) = \emptyset$

$g=1 \quad M_X(n,d) = \mathrm{Sym}^{(n,d)}(T^*X)$

- $T^*N_X^s(n,d) \subset M_X(n,d)$ dense and open
- Ω_I extends to $M_X(n,d)^s \supsetneq T^*N_X^s$

<u>example</u> $\quad K^{1/2} \oplus K^{-1/2} \in M_X(n,d)^s \setminus T^*N$

$\underset{1}{\curvearrowright}$

This illustrates that

$U_n = \{(E,\varphi) \in M_X \mid E \notin N\}$

$\neq \emptyset$

$\tau \varphi$

- Non abelian Hodge theorem

$$M_X(n,d) \underset{C^\infty}{\cong} \mathrm{Hom}^*(\pi_1(X), GL(n,\mathbb{C})) \big/ GL_n(\mathbb{C}$$

$$\circlearrowleft$$

$J \hookrightarrow$ from GL_r

1.3 Mirror symmetry for Hitchin systems

$$\begin{tikzcd} X \arrow[dr] & & \check{X} \arrow[dl] \\ & B & \end{tikzcd}$$

CY varieties are mirrors if $\exists\, h, \check{h}$ s.t.

$$h^{-1}(b) \cong \left(\check{h}^{-1}(b)\right)^\vee$$

special Lagrangian tori

$$\begin{tikzcd} T^* Jac^0(X) = Jac^0(X) \times H^0(K) \arrow[d, "p_2"] \\ H^0(K) \end{tikzcd}$$

$p_2^{-1}(0) = Jac^0(X)$ which self dual

$$\Omega_I(Jac^0(X), Jac^0(X)) \equiv 0$$

⇒ special Lagr. in J & K

$M_X(0,1)$ is its own mirror pair.

Similarly in Higher rank

$$\begin{tikzcd} M_X \arrow[r, "h"] & \bigoplus_{i=1}^{n} H^0(X, K^i) =: B \\ (E,\varphi) \arrow[r, mapsto] & \det(\lambda \mathrm{Id} - \varphi) \end{tikzcd}$$

Spectral curve

$$X_b \xrightarrow[\text{ramified}]{n:1} X$$

$$x$$

$$\left\{ \begin{array}{l} \text{eigenvalues of} \\ \varphi(x) \quad \det(\lambda \mathrm{Id} - \varphi(x)) = b(x) \end{array} \right\}$$

X_b generically smooth

Thm : $h^{-1}(b) = \mathrm{Jac}^0(X_b)$ gric $b \in$

$M_X(n,0)$ ~~=~~is its own mirror