Stochastic thermodynamics of quantum jumps: entropy production, martingales and inefficient detection

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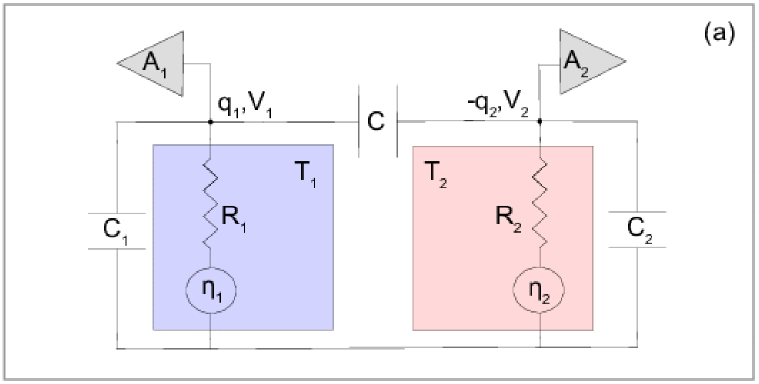
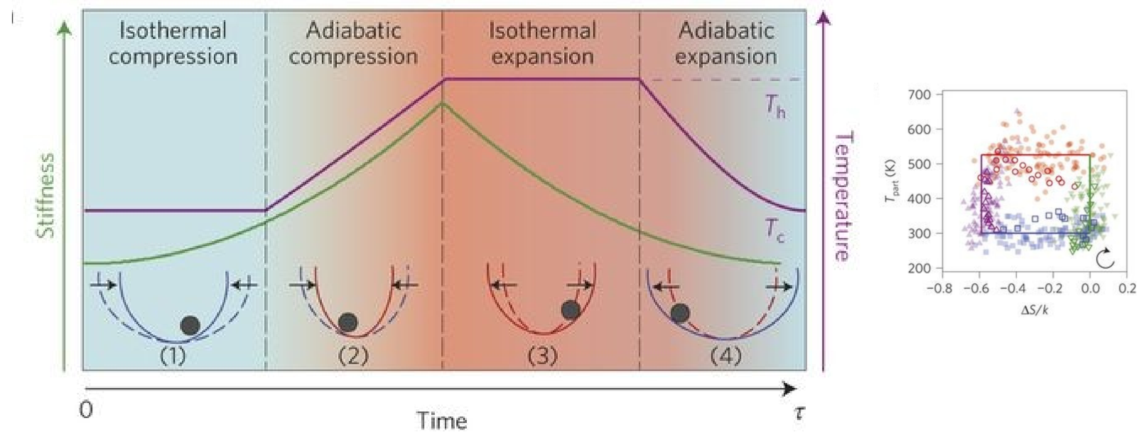
UNIT OF EXCELLENCE MARÍA DE MAEZTU

ICTS Bangalore 22 Jan 2025

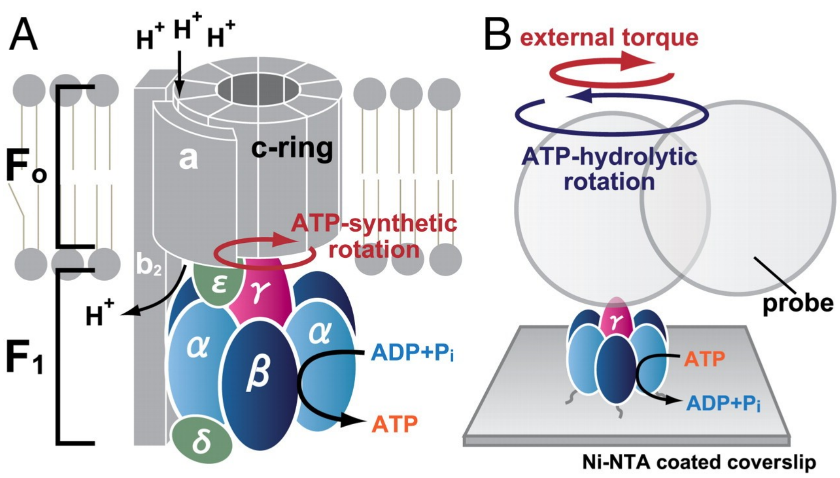
Electric circuits S. Ciliberto *et al.* PRL (2013)



Colloidal Carnot engine I.A. Martínez, et al. Nat. Phys (2016)

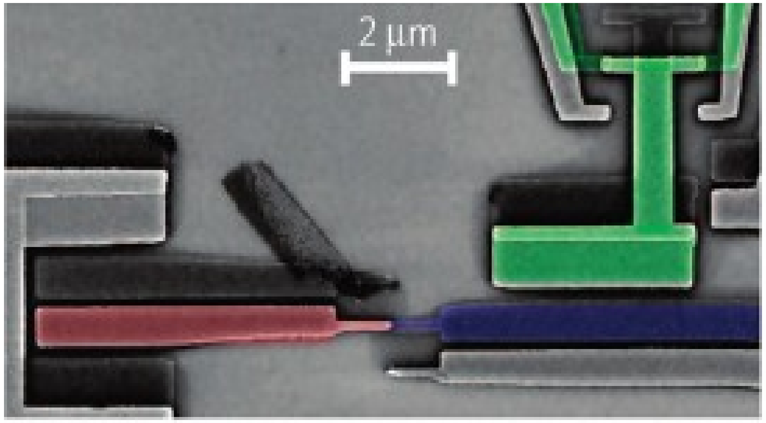


Molecular motors Toyabe *et al.* PNAS (2011)



Nanoelectronic devices

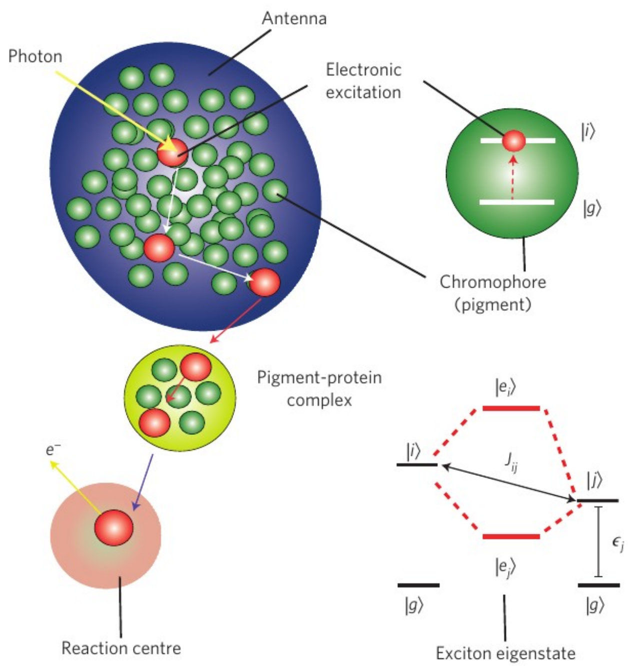
J.V. Koski *et. al. Nat. Phys. (2013)*



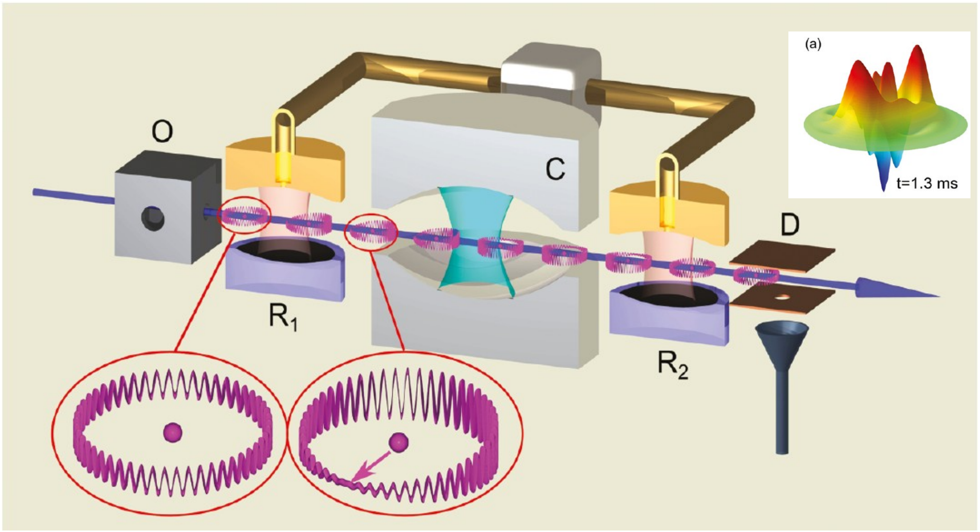
Review stochastic thermo experiments: S. Ciliberto, PRX (2017)

Light-harvesting complexes:

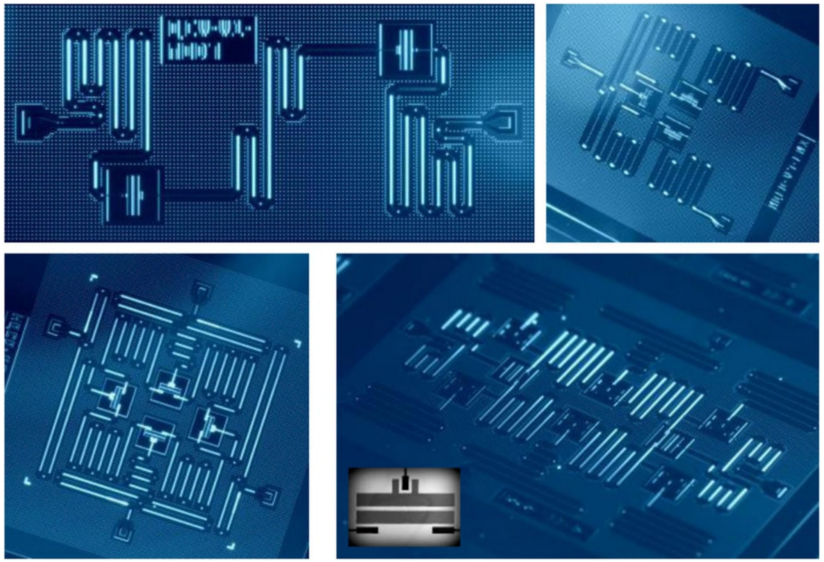
A. W. Chin *et al.* Nat. Phys. (2013)



Cavity and circuit QED experiments:

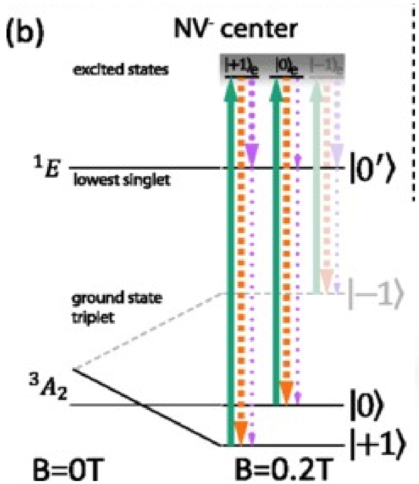


S. Haroche, Nobel lecture (2012)

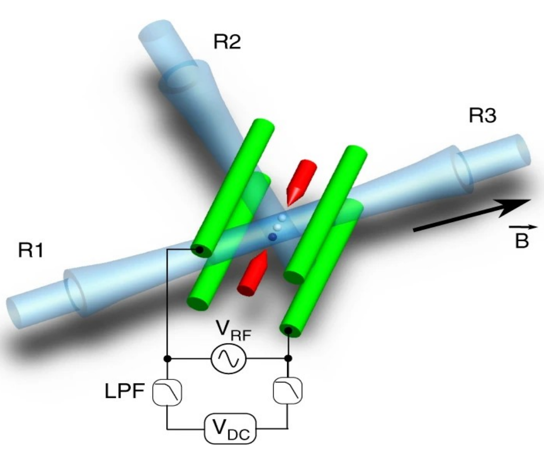


J.M. Gambetta *et al.* npj Quantum Information (2017)

Quantum thermal machines:



Klatzow, *et al.* PRL (2019)



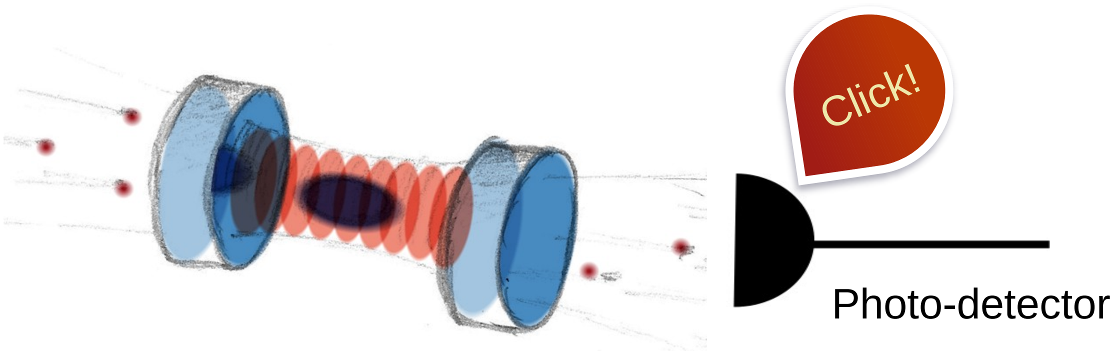
Maslennikov, *et al.* (2019)

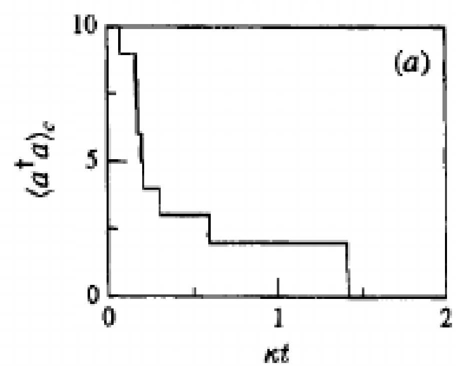
Assessing fluctuations

Two-point measurement (TPM) scheme

- Projective measurements of generic observables
- Markovian dynamics

Continuosly monitored quantum system





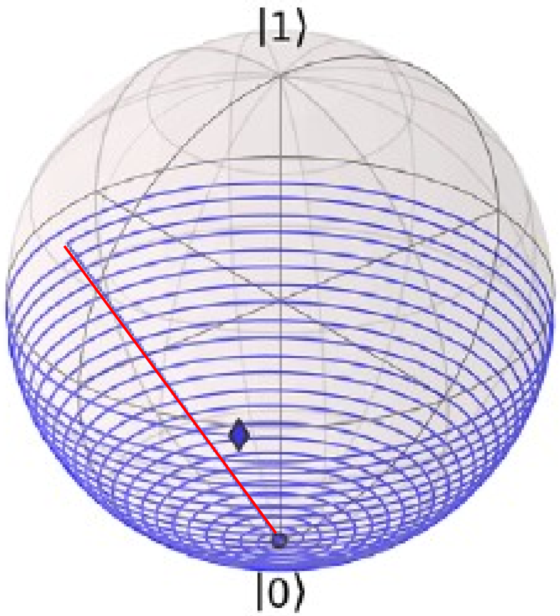
Pioneering works: Horowitz, PRE (2012) ; Hekking and Pekola PRL (2013)

Recent mini-review: Manzano and Zambrini, AVS Quantum Science **4** (2022)

Quantum (jump) trajectories:

Lindblad (GKS) equation: $\dot{\rho} = -i[H(\lambda_t), \rho] + \sum_k \mathcal{D}[L_k(\lambda_t)](\rho)$ ("average" evolution)

LDB $L_{\tilde{k}} = L_k^\dagger e^{-\Delta s_k/2}$ with: $\Delta s_k = q_k/T$



Stochastic master equation: (monitored evolution)

Evolution of the system *conditioned* on a measurement record:

$$d\rho_c = -i[H, \rho_c]dt + \sum_k \left(dt\mathcal{H}[L_k]\rho_c + dN_k \mathcal{J}[L_k]\rho_c \right)$$

Smooth (non-unitary) part

Jump of type k

Poisson increments $dN_k(t) = \{0, 1\}$ "signaling" jumps $dN_k^2 = dN_k$

Measurement record: $\Gamma_{[0,\tau]} = \{n_0, (k_1, t_1), (k_2, t_2), ..., (k_J, t_J), n_\tau\}$

Trajectory operator: $\mathcal{T}(\Gamma_{[0,\tau]}) = \mathcal{U}(\tau, t_J) L_{k_J} \dots L_{k_1} \mathcal{U}(t_1, 0)$ $\rho_c(\tau) = \frac{\mathcal{T}\rho_c(0)\mathcal{T}^\dagger}{p_\Gamma}$

Thermodynamic process:

Initial distribution: $\rho_0 = \sum_n p_n^{(0)} \Pi_n^{(0)}$ —Lindblad→ Final distribution: $\rho_\tau = \sum_n p_n^{(\tau)} \Pi_n^{(\tau)}$

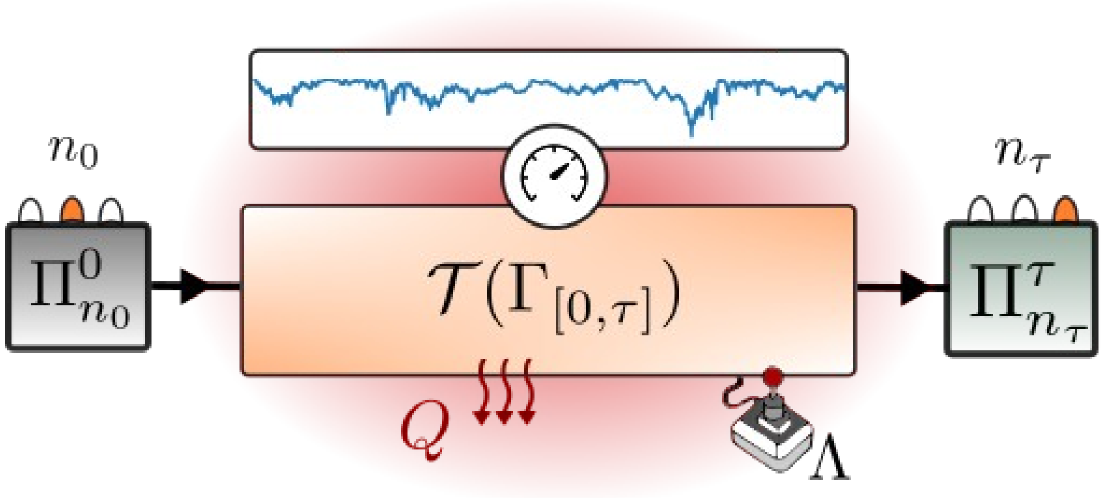
Extrernal driving:

Hamiltonian and Lindblad operators may depend on an external parameter:

$$H(\lambda) \qquad L_k(\lambda)$$

Driving protocol:

$$\Lambda = \{\lambda(t); 0 \leq t \leq \tau\}$$



Forward process

Trajectory in the TPM scheme: $\Gamma_{[0,\tau]} = \{n_0, (k_1, t_1), (k_2, t_2), ..., (k_J, t_J), n_\tau\}$

(Path) Probability of a trajectory: $\mathbb{P}(\Gamma_{[0,\tau]}) = p_{n_0}^{(0)} \mathrm{Tr}[\Pi_{n_\tau}^{(\tau)} \mathcal{T}_\Lambda \Pi_{n_0}^{(0)} \mathcal{T}_\Lambda^\dagger]$

Energetics and the first law:

Energy change during a trajectory:

$$\Delta E(\Gamma_{[0,\tau]}) = \mathrm{Tr}[H(\tau)\Pi^{(\tau)}_{n_\tau}] - \mathrm{Tr}[H(0)\Pi^{(0)}_{n_0}]$$

Expected final energy in trajectory

Expected initial energy in trajectory

NOTE: measurements not necessarily in energy basis

$$\Pi^{(\tau)}_{n_\tau} = |\psi\rangle\langle\psi|_{n_\tau}$$

$$|\psi_{n_\tau}\rangle = \alpha_1 |E_1\rangle + \alpha_2 |E_2\rangle$$

Stochastic First law : $$\Delta E(\Gamma_{[0,\tau]}) = W_\Lambda(\Gamma_{[0,\tau]}) + \sum_r Q_r(\Gamma_{[0,\tau]})$$

Heat: $$Q_r(\tau) = k_B T_r \int_0^\tau \sum_{k\in J(r)} dN_k \Delta s_k(t)$$ energy associated to entropy exchange

Driving work: $$W^{\mathrm{drive}}_\Lambda(t) := \int_0^\tau dt \mathrm{Tr}[\dot{H}(\lambda_t)\rho_c(t)]$$ external input work (Hamiltonian)

Chemical work: $$W^{\mathrm{chem}}_\Lambda(t) := \int_0^\tau \sum_r \sum_{k\in J(r)} dN_k \mu_r \frac{\mathrm{Tr}[N\mathcal{D}[L_k](\rho_c)]}{\langle L_k^\dagger L_k\rangle}$$ fluxes of particles (e.g. electric current)

However it happens that:

$$\Delta E(\Gamma_{[0,\tau]}) \neq W_\Lambda^{\text{drive}}(\Gamma_{[0,\tau]}) + W_\Lambda^{\text{chem}}(\Gamma_{[0,\tau]}) + \sum_r Q_r(\Gamma_{[0,\tau]})$$

Measurement energy (work): (mismatch energy)

$$W_\Lambda^{\text{meas}}(t) := \int_0^\tau dt \sum_k \mathrm{Tr}[\mathcal{H}_k(\rho_c)] + \sum_k dN_k \left(\frac{1}{2} \frac{\mathrm{Tr}[\{H, L_k^\dagger L_k\}\rho_c]}{\langle L_k^\dagger L_k \rangle} - E(t) \right)$$

+ It is **stochastic** and non-zero during both **no-jump periods and jumps**

+ Only fluctuations (average is zero): $\langle \dot{W}_\Lambda^{\text{meas}} \rangle = 0$

+ Becomes zero when the monitored system is always an eigenstate of $H(\lambda)$ or $L_k^\dagger L_k$

+ Extra contribution also due to the **final measurement** in the TPM

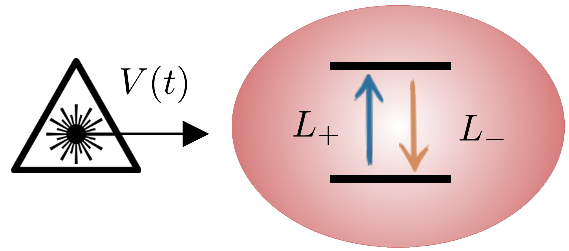
$$W_{\text{TPM}}(\gamma_{[0,\tau]}) = \mathrm{Tr}[H(\lambda_\tau)(\Pi_{n_\tau}^{(\tau)} - \rho_\gamma(\tau))]$$ (similar properties)

Dissipative-driven qubit:

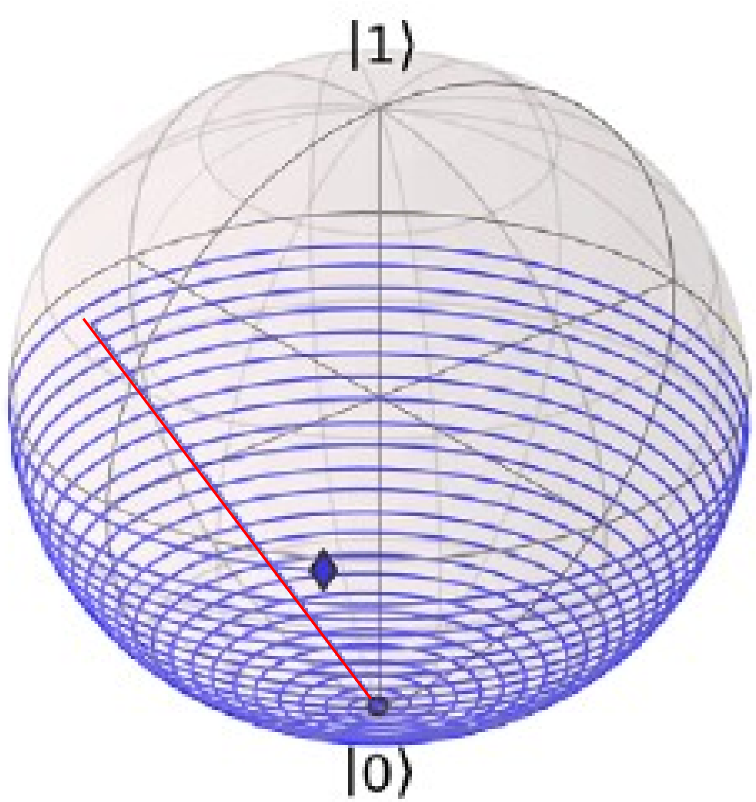
$$H(t) = \omega\,|1\rangle\langle 1| + V(t) \quad \text{with}$$

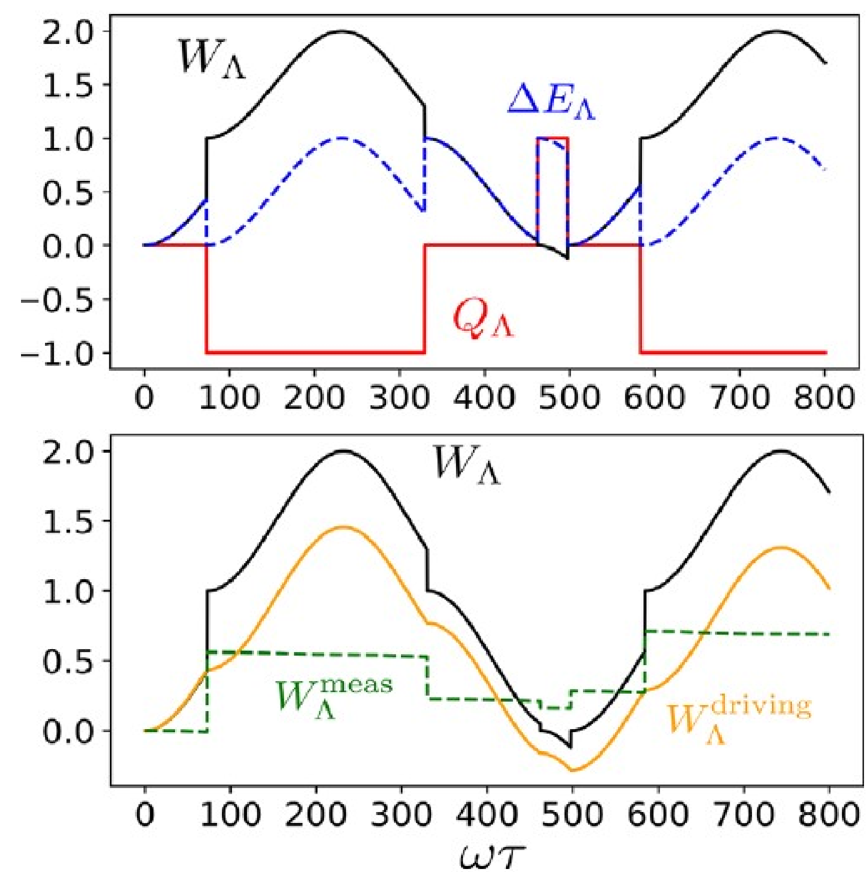
$$V(t) = \varepsilon(e^{-i\omega t}\sigma_+ + e^{i\omega t}\sigma_-)$$

$\epsilon \ll \omega$ (weak periodic driving)



$$L_k = \{\sqrt{\Gamma_-}\sigma_-, \sqrt{\Gamma_+}\sigma_+\} \qquad \Delta s_k = \mp\omega/T$$





Time-reversed process:

Θ Anti-unitary time-reversal operator

Initial distribution: $\tilde{\rho}_0 = \Theta \rho_\tau \Theta^\dagger$ —Lindblad→ Final distribution: $\tilde{\rho}_\tau \neq \Theta \rho_0 \Theta^\dagger$

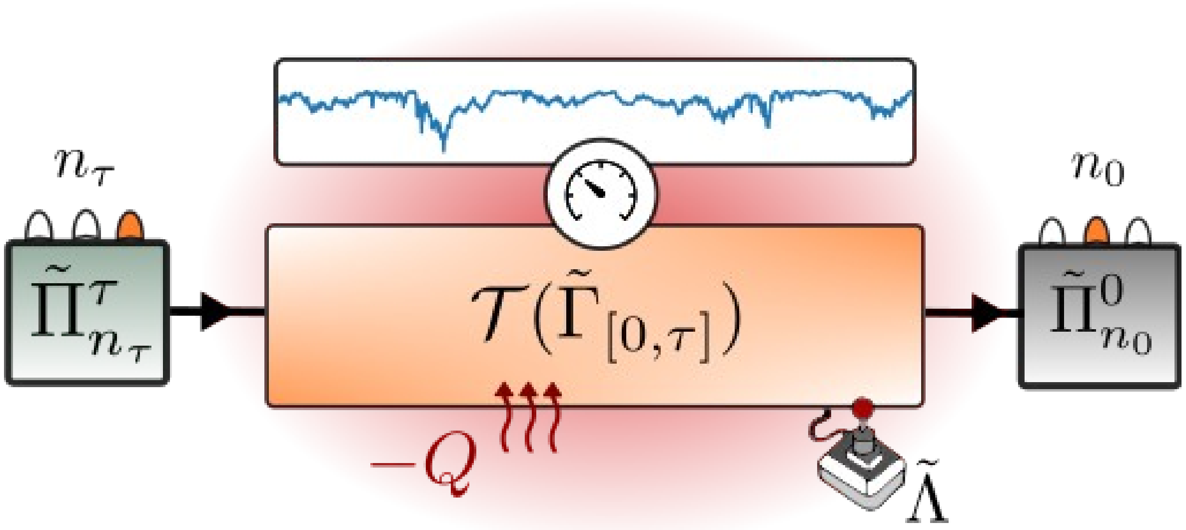
Time-reversed driving:

Hamiltonian and Lindblad ops:

$$\tilde{H}(\lambda) = \Theta H(\lambda) \Theta^\dagger$$
$$\tilde{L}_k(\lambda) = \Theta L_k(\lambda) \Theta^\dagger$$

Time-reversed driving protocol:

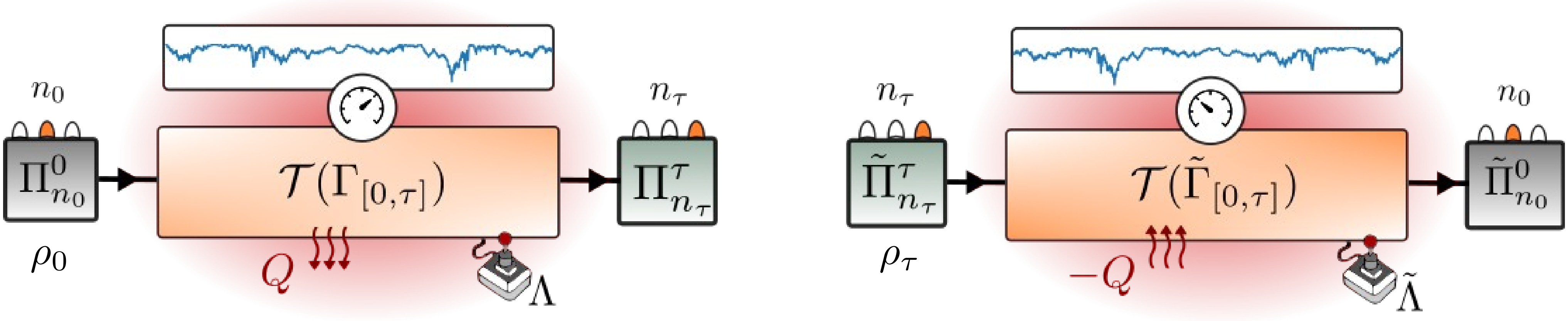
$$\tilde{\Lambda} = \{\lambda(\tau - t); 0 \leq t \leq \tau\}$$



Backward process

Reversed trajectory: $\tilde{\Gamma}_{[0,\tau]} = \{n_\tau, (\tilde{k}_J, \tau - t_J), ..., (\tilde{k}_2, \tau - t_2), (\tilde{k}_1, \tau - t_1), n_0\}$

Probability of the reversed trajectory: $\mathbb{P}(\tilde{\Gamma}_{[0,\tau]}) = p^{(\tau)}_{n_\tau} \mathrm{Tr}[\tilde{\Pi}^{(0)}_{n_0} \mathcal{T}_{\tilde{\Lambda}} \tilde{\Pi}^{(\tau)}_{n_\tau} \mathcal{T}^\dagger_{\tilde{\Lambda}}]$



Forward process   Backward process

Stochastic Entropy Production

$$S_{\text{tot}}(\Gamma_{[0,\tau]}) = \log\left(\frac{\mathbb{P}(\Gamma_{[0,\tau]})}{\tilde{\mathbb{P}}(\tilde{\Gamma}_{[0,\tau]})}\right) \stackrel{!}{=} \Delta S_{\text{sys}}(\tau) + \sum_r \beta_r Q_r(\tau)$$

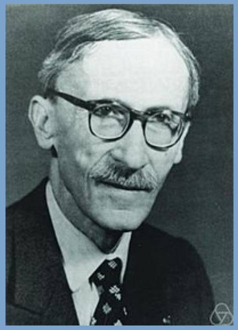
system entropy — entropy flow

System entropy change: $$\Delta S_{\text{sys}}(\gamma_{[0,\tau]}) = -\log p^{(\tau)}_{n_\tau} + \log p^{(0)}_{n_0}$$

Entropy flows (heat): $$\beta_r Q_r(\tau) = \sum_{k\in J(r)} \int_0^\tau dN_k \Delta s_k(t)$$

IFT and Second-law inequality: $$\langle e^{-S_{\text{tot}}}\rangle = 1 \qquad \langle S_{\text{tot}}\rangle \geq 0$$

Martingale theory:

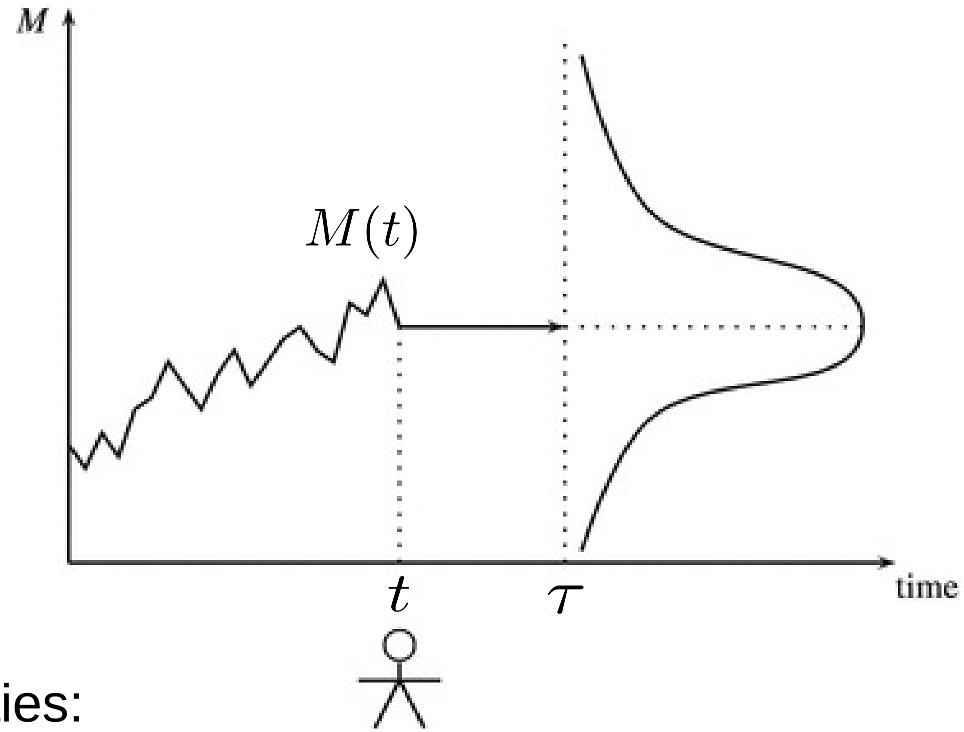


Martingale process

Introduced in prob. theory by Paul Lévy in 1934

$$\langle M(\tau)|\mathbf{X}_t\rangle = M(t)$$

for $0 \leq t \leq \tau$



Martingales verify many interesting properties:

- Doob's optional stopping theorem: $\langle M(t)\rangle_{\mathcal{T}} = \langle M(0)\rangle$ at stochastic times $\mathcal{T}$
- Doob's maximal inequality: $\Pr[M_{\max}(\tau) \geq m] \leq \langle M(\tau)\rangle/m$ maximum in an interval

In classical NESS $\langle e^{-S_{\rm tot}(\tau)}|\mathbf{X}_t\rangle = e^{-S_{\rm tot}(t)}$

Recent review:

Roldán *et al. Advances in Physics* ***72****, 1-258 (2023)*

Quantum Martingale theory

Split of entropy production in quantum/classical parts

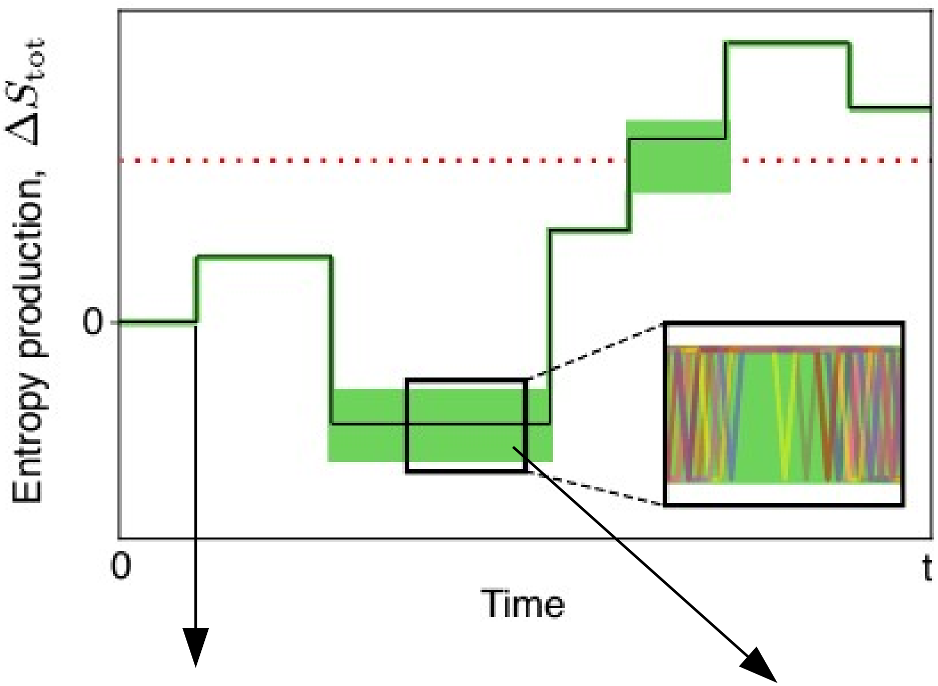
Martingale / Uncertainty entropy production

Manzano, Fazio, Roldán, PRL **122**, 220602 (2019)

$$S_{\rm tot}(t) = S_{\rm unc}(t) + S_{\rm mar}(t)$$

→ "martingale" entropy production (classicalization)

→ "uncertainty" entropy production (superposition of eigenstates)

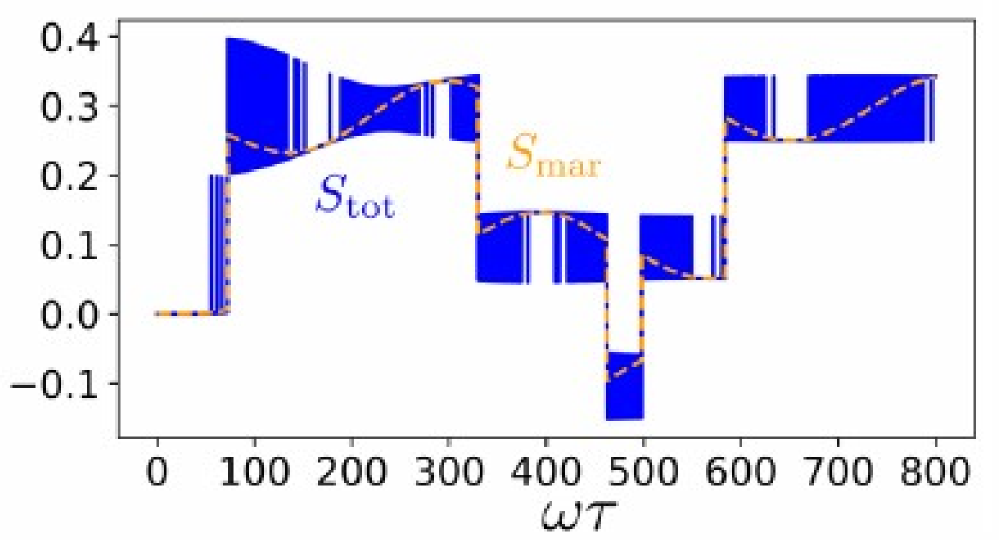


- Entropy production needs measurements on the system.
- System entropy is (sometimes) not well defined at intermediate times
- How to define stopping times / conditions ?
- Thermodynamic consequences?

$|\psi(t)\rangle = |n(t)\rangle$ in a eigenstate

$|\psi(t)\rangle$ superposition of eigenstates

Introduce an entropy for superpositions: $S_\psi := -\log\langle\psi_t|\rho|\psi_t\rangle$ (quantum fidelity)



Uncertainty entropy production:

$$S_{\rm unc}(t) = -\log p_{n_t}^{(t)} - S_\psi(t)$$

Martingale ("classicalised") entropy production:

$$S_{\rm mar}(t) = S_\psi(t) + \log p_{n_0}^{(0)} - \sum_r \frac{Q_r(t)}{T_r}$$

All the "good" properties (IFT, second-law-inequality) are recovered for both.

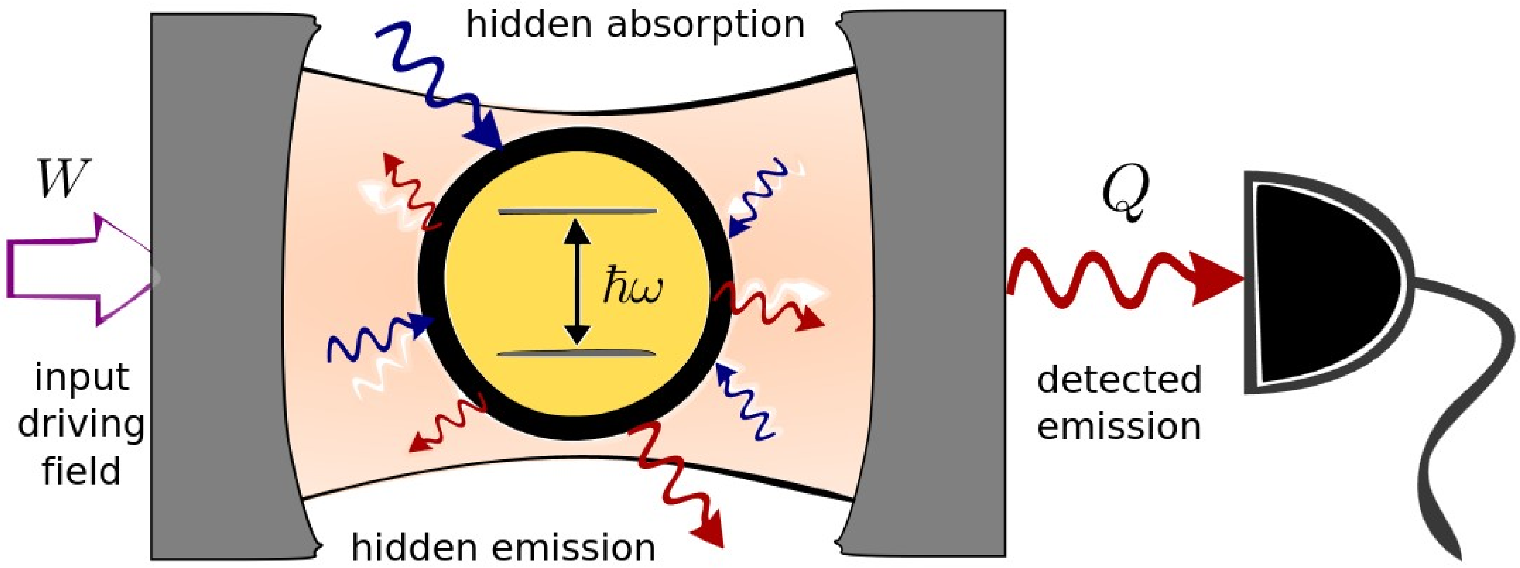
More importantly, we obtain fluctuation theorems **at stopping times**:

$$\langle e^{-S_{\rm mar}(t)} | \Gamma_{\{0,\tau\}} \rangle = e^{-S_{\rm mar}(\tau)} \Rightarrow \begin{array}{c} \langle e^{-S_{\rm mar}(\mathcal{T})}\rangle = 1 \\ \langle S_{\rm tot}(\mathcal{T})\rangle \geq \langle S_{\rm unc}(\mathcal{T})\rangle \end{array}$$

For NESS !

Extension to transient dynamics and driven systems: G. Manzano, et al. PRL **126**, 080603 (2021)

What happens if we do not detect all the jumps? Or not monitor all the channels?



Modified evolution with detector efficiencies $\{\eta_k\}$

$$d\sigma_c = -i[H,\sigma_c]dt + \sum_k \left(dt(1-\eta_k)\mathcal{D}[L_k]\sigma_c + dt\eta_k\mathcal{H}[L_k]\sigma_c + dN_k\mathcal{J}[\sqrt{\eta_k}L_k]\sigma_c \right)$$

hidden dissipation part — Smooth (non-unitary) part — Jump of type k

Can we still formulate a second law at the level of fluctuations?

arXiv > quant-ph > arXiv:2308.08491

Quantum Physics

[Submitted on 16 Aug 2023 (v1), last revised 11 Apr 2024 (this version, v2)]

Entropy production and fluctuation theorems for monitored quantum systems under imperfect detection

Mar Ferri-Cortés, Jose A. Almanza-Marrero, Rosa López, Roberta Zambrini, Gonzalo Manzano

Phys. Rev. Research 7, 013077 (2025)




Split ideal monitoring record: $\Gamma_{[0,\tau]} = \gamma_{[0,\tau]} \cup h_{[0,\tau]}$

Visible trajectory: $\gamma_{(0,\tau)} = \{n_0, (k'_1, t'_1), (k'_2, t'_2), ..., (k'_V, t'_V), n_\tau\}$ $\quad L'_k = \sqrt{\eta_k} L_k$

Hidden jumps: $h_{(0,\tau)} = \{(k^*_1, t^*_1), (k^*_2, t^*_2), ..., (k^*_V, t^*_V)\}$ $\quad L^*_k = \sqrt{1-\eta_k} L_k$

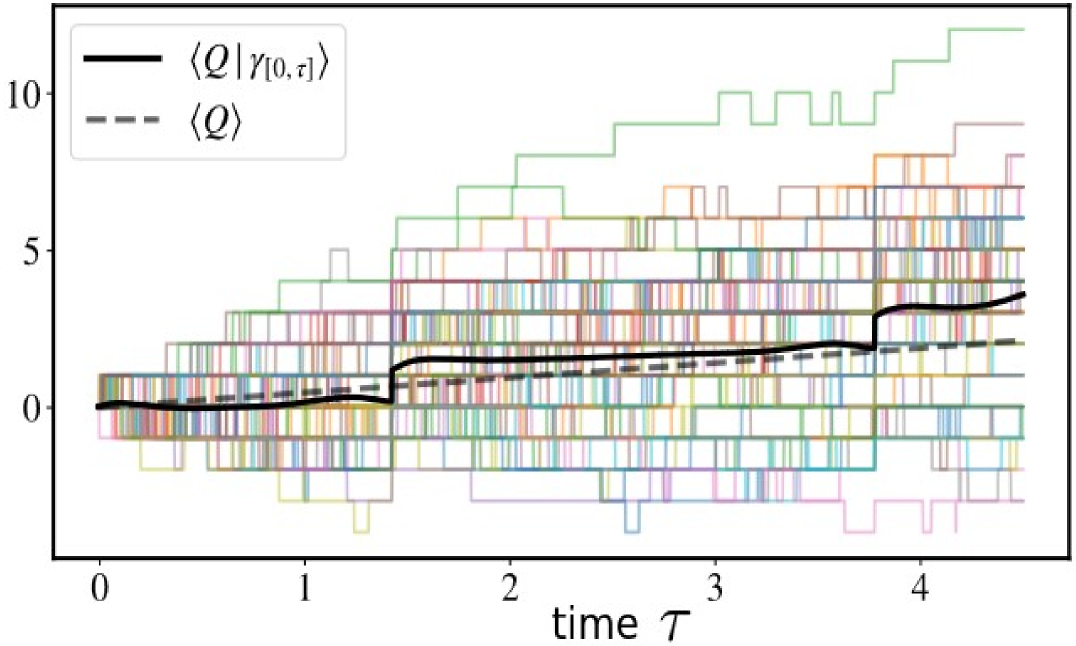
What can be said about the dissipation only from the visible jumps?

Conditional entropy production:

$$\langle S_{\rm tot}|\gamma_{[0,\tau]}\rangle = \Delta S_{\rm sys}(\tau) + \sum_r \beta_r \langle Q(\tau)|\gamma_{[0,\tau]}\rangle$$

conditional average:

$$\langle A[\Gamma]|\gamma_{[0,\tau]}\rangle = \sum_{h_{[0,\tau]}} P(h_{[0,\tau]}|\gamma_{[0,\tau]}) A[\Gamma]$$



Estimator of irreversibility:

$$\Sigma(\tau) = \log\left(\frac{P(\gamma_{[0,\tau]})}{\tilde{P}(\tilde{\gamma}_{[0,\tau]})}\right)$$

Constructed from marginalized path probabilities

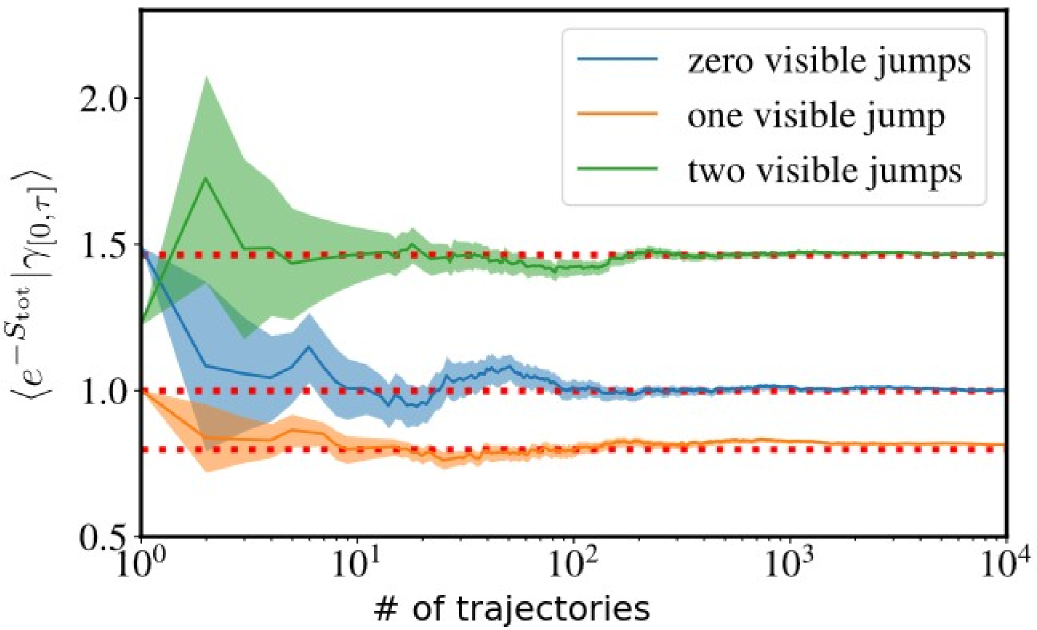
$$P(\gamma_{[0,\tau]}) = \sum_{h_{[0,\tau]}} \mathbb{P}(\Gamma_{[0,\tau]}) \qquad \tilde{P}(\tilde{\gamma}_{[0,\tau]}) = \sum_{h_{[0,\tau]}} \tilde{\mathbb{P}}(\tilde{\Gamma}_{[0,\tau]})$$

which verifies the FT $\langle e^{-\Sigma(\tau)}\rangle = 1$ and we know that: $\langle S_{\rm tot}(\tau)\rangle \geq \langle \Sigma(\tau)\rangle$

Kawai, Parrondo, Van der Broeck PRL (2007); Gomez-Marín, Parrondo, Van der Broeck PRE (2008)

Main result: there still exists a link between irreversibility and stochastic EP

$$\langle e^{-S_{\rm tot}} | \gamma_{[0,\tau]} \rangle = e^{-\Sigma(\tau)}$$



Corollaries:

$$\langle S_{\rm tot}(\tau) | \gamma_{[0,\tau]} \rangle \geq \Sigma[\gamma_{[0,\tau]}]$$

Bound for single trajectories!

Bounds on the EP distribution tails:

$$\Pr(S_{\rm tot} - \Sigma < -\xi) \leq e^{-\xi}$$

Overestimation of stochastic EP is exponentially unlikely

$$\Pr(S_{\rm tot} - \Sigma \geq \xi) \leq e^{-q\xi} \langle e^{q(S_{\rm tot} - \Sigma)} | \gamma_{[0,\tau]} \rangle$$

Understimation of stochastic EP

Bounds for even moments of EP: $\langle S_{\rm tot}^k \rangle \geq \langle \Sigma^k \rangle \qquad k = 2, 4, 6, \ldots$

Extra comments

Can be reformulated as a bound for the heat dissipated: $\langle Q(\tau) | \gamma_{[0,\tau]} \rangle \geq \phi[\gamma_{[0,\tau]}]$

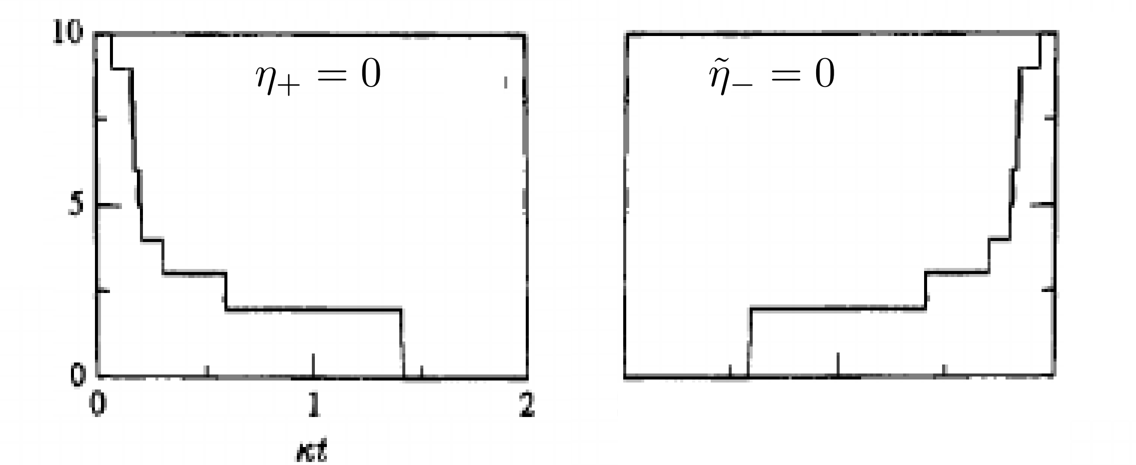
How the time-reversal process for imperfect monitoring looks like?

In general, coarse-graining and taking the time-reversal do no commute!

$$\tilde{P}(\tilde{\gamma}_{[0,\tau]}) = \sum_{h_{[0,\tau]}} \tilde{\mathbb{P}}(\tilde{\Gamma}_{[0,\tau]}) \longrightarrow$$

+ Time-reversed driving

+ Exchange of efficiencies in comp. channels



$$\tilde{L}'_- = \eta_+ L_-$$

$$\tilde{L}'_+ = \eta_- L_+$$

Allows to describe processes for which one of the efficiency is zero but also pose limitations for estimation in some situations (time-reversed may be difficult to implement)

Conclusions:

+ **Stochastic thermodynamics can be formulated along quantum trajectories** with quantum contributions on both the energetics and on the irreversibility

+ **Fluctuation theorems at stopping times** with quantum correction term

+ **Conditional fluctuation relation** that links entropy production and irreversibility under imperfect monitoring

+ **Lower bounding dissipation from irreversibility along single trajectories** (no overstimation, bounded understimation)

+ Both imperfect (inefficient) and partial detection (only some transitions)

+ Can be used to handle (apparent) unidirectional transitions

+ Applicable to quantum and classical jump processes alike.

Outlook

+ Testing in experiments !

THANK YOU

for your attention

Manzano, Horowitz, Parrondo PRE **8**, 032129 (2015)

Manzano, Horowitz, Parrondo PRX **8**, 031037 (2018)

Manzano, Fazio, Roldán PRL **122**, 220602 (2019)

Manzano, et al. PRL **126**, 080603 (2021)

Manzano, Zambrini, AVS Quantum Science **4**, 025302 (2022) **(review)**

Ferri-Cortés, et al. PRR **7**, 013077 (2025)

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Quantum Martingale and fluctuation relations at stopping times:

$$\langle e^{-\Delta S_{\mathrm{mar}}(\tau)-\delta_{\mathrm{q}}(\tau)}|\gamma_{\{0,t\}}\rangle = e^{-\Delta S_{\mathrm{mar}}(t)-\delta_{\mathrm{q}}(t)}$$

Quantum version of stochastic reversibility: $$\delta_{\mathrm{q}}(t) = \log\left(\frac{\langle\psi(t)|\rho(t)|\psi(t)\rangle}{\langle\psi(t)|\tilde{\rho}(\tau - t)|\psi(t)\rangle}\right)$$

Using Doob's optional stopping theorem:

$$\langle e^{-\beta[W-\Delta F]-\delta_{\mathrm{q}}+\Delta S_{\mathrm{unc}}}\rangle_{\mathcal{T}} = 1$$

$$\langle W\rangle_{\mathcal{T}} - \langle\Delta F\rangle_{\mathcal{T}} \geq -k_B T[\langle\delta_{\mathrm{q}}\rangle_{\mathcal{T}} - \langle\Delta S_{\mathrm{unc}}\rangle_{\mathcal{T}}]$$

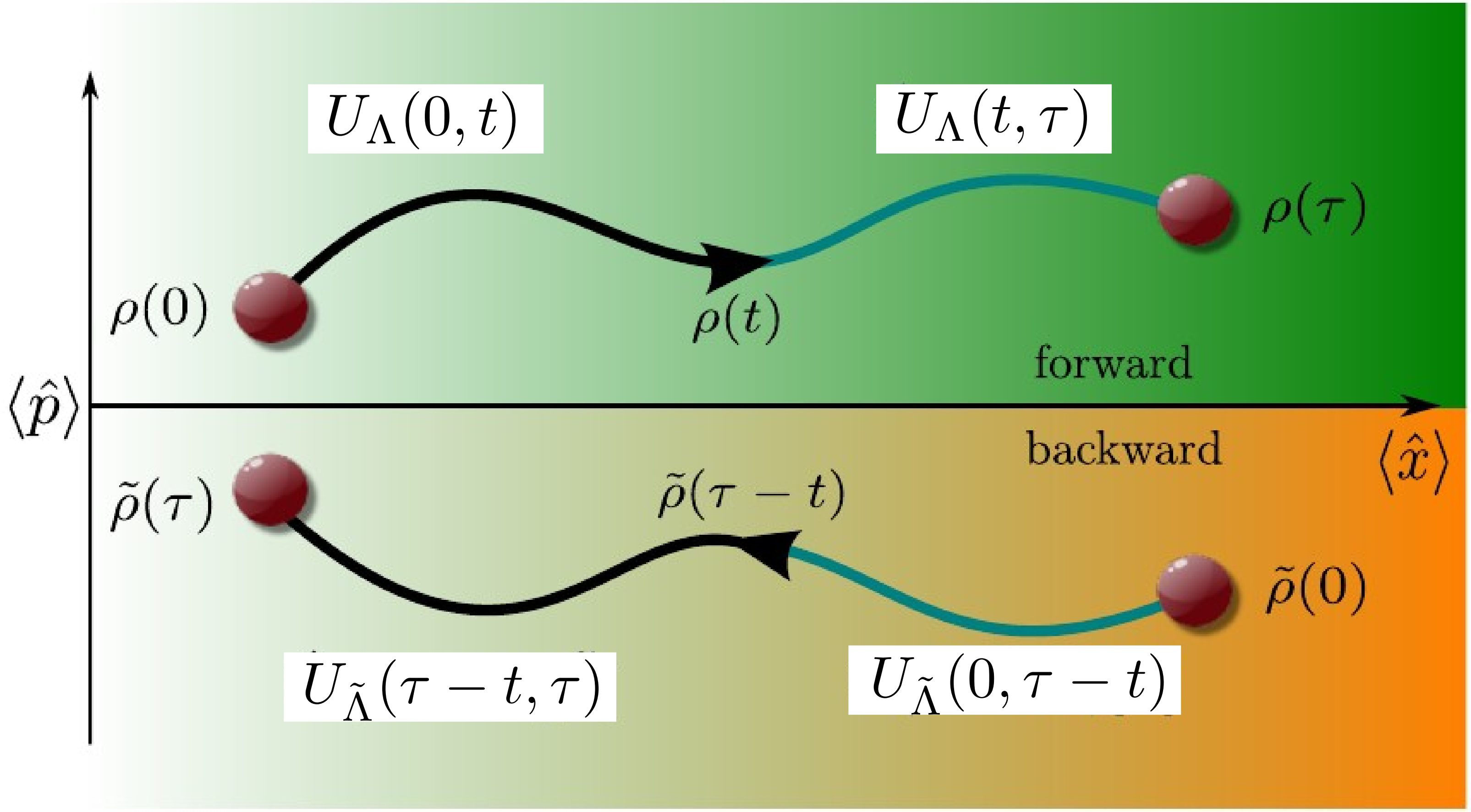
- Beneficial if $\langle\Delta S_{\mathrm{unc}}\rangle_{\mathcal{T}} < 0$
- Detrimental if $\langle\Delta S_{\mathrm{unc}}\rangle_{\mathcal{T}} > 0$

In classical systems: $\delta_{\mathrm{q}}(t) \to \delta(t)$ $\qquad \Delta S_{\mathrm{unc}}(t) \to 0$

Micro-reversibility:

In order to establish a consistent thermodynamic framework we extend micro-reversibility:

$$\Theta^\dagger U^\dagger_{\tilde{\Lambda}}(\tau - t, \tau)\Theta = U_\Lambda(0,t)$$ (isolated non-autonomous systems)



See also: M. Campisi. P. Talkner and P. Hänggi, Rev. Mod. Phys. (2011)

Micro-reversibility in open systems:

$$\Theta^\dagger \mathcal{T}^\dagger_{\tilde{\Lambda}}(\tilde{\Gamma}_{(0,\tau)})\Theta = \mathcal{T}_\Lambda(\Gamma_{(0,\tau)}) e^{-\sigma_\Lambda(\Gamma_{(0,\tau)})/2}$$ (open monitored systems)

Entropy flow to the environment (entropy of the medium): $\sigma_\Lambda(\Gamma_{[0,\tau]}) = \sum_r \beta_r Q_r(\tau)$

For quantum jumps adjoint set of operators $\{L_{k+}, L_{k-}\}$ where $L^\dagger_{k+} \propto L_{k-}$

$\Rightarrow \; L_{k+} = L^\dagger_{k-} e^{\Delta s_{k+}(\lambda)/2}$ with $\Delta s_{k\pm} = \pm \log(\Gamma_+/\Gamma_-)$ **(LDB)**

$$\sigma_\Lambda(\gamma_{[0,\tau]}) = \sum_k \int_0^\tau dN_k \Delta s_k(\lambda_t)$$ sum of entropy exchanged with environment in each jump during the trajectory

"Measurement work" vs. "Quantum heat"

May these extra energy fluctuations be considered work or heat?

Quantum heat: stochastic, entropy of the system can change ...

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ARTICLE OPEN

The role of quantum measurement in stochastic thermodynamics

Cyril Elouard[1], David A. Herrera-Martí[1], Maxime Clusel[2] and Alexia Auffèves[1]

Measurement work: not related to entropy flow

deterministic $\Rightarrow$ work non-deterministic $\not\Rightarrow$ heat

Example 1: chemical work

Example 2: throw a coin and apply a different (deterministic) driving depending on the result

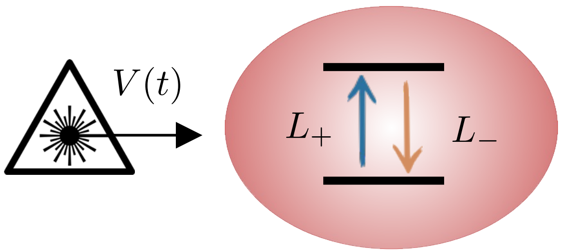
Testing with example

Dissipative-driven qubit:

$$H(t) = \omega \left|1\right\rangle \left\langle 1\right| + V(t) \quad \text{with}$$

$$V(t) = \varepsilon(e^{-i\omega t}\sigma_+ + e^{i\omega t}\sigma_-)$$

$\epsilon \ll \omega$ (weak periodic driving)



Single equilibrium reservoir:

$$L_k = \{\sqrt{\Gamma_-}\sigma_-, \sqrt{\Gamma_+}\sigma_+\}$$

$$\Delta s_k = \mp\omega/T$$

Efficiencies: η_- η_+