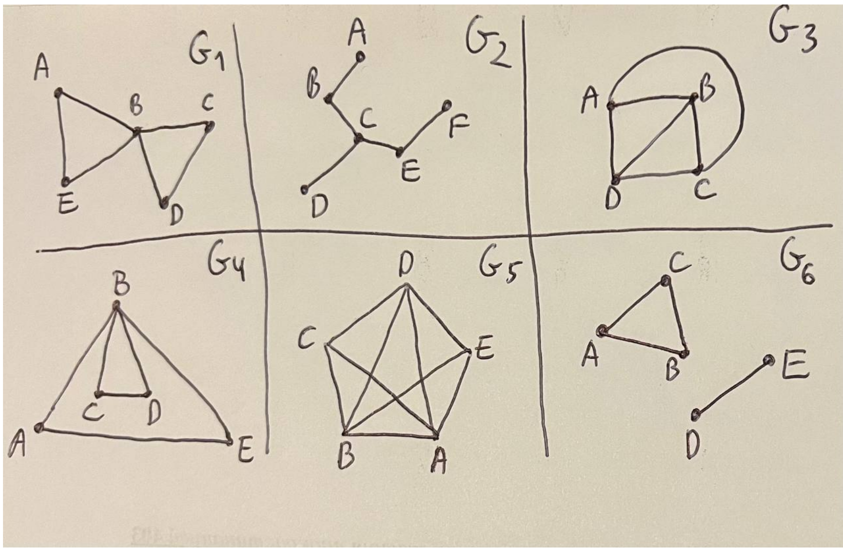
Graphs

1 Introduction



There are six graphs above, we name these graphs $G_1, G_2, G_3, G_4, G_5, G_6$.

Definition. *Graph* is a set of several *nodes*, some of which are connected by the *edges*.

For example, graph G_1 above consist of 5 nodes A, B, C, D, E and 6 edges. We can give names to edges, or just denote each edge by the pair of nodes it connects. So we can say that graph G_1 has an edge (A, B) and does not have an edge (E, D).

Connection to real world Many real-life structures can be seen as a graph:

- Molecule as a graph: atoms are nodes, connections between atoms are edges
- Social network as a graph: users are nodes, friendship connections are edges
- Road system as a graph: crossroads are nodes, roads are edges

Drawing Usually we draw an edge as a straight line, but we can draw edge as any curve as long as it connects two nodes (see edge (A, C) in graph G_3). If for some reason two edges intersect outside of nodes, we pretend that nothing happened and do not count the intersection as a new node (see edges (A, C) and (B, E) in graph G_5). All that matters is what set of nodes graph has and which pairs of nodes are connected by edges, so for instance G_1 and G_4 is actually the same graph.

2 Definitions

Definition. *Degree* of a node is the number of edges connected to that node.

For instance, the degree of node C in graph G_2 is 3, and degree of node A in graph G_2 is 1.

Problem 1. Prove that in any graph the sum of degrees of all nodes is even (see the end of the next page for a hint)

Problem 2. Prove that in any graph the number of nodes with odd degrees is even. Does there exist a graph with 8 nodes whose degrees are $5, 4, 4, 4, 3, 2, 2, 1$?

Definition. In graph G_1 the sequence of edges $(A, B), (B, C), (C, D)$ forms a *path* from node A to node D. Give a formal definition of a *path*.

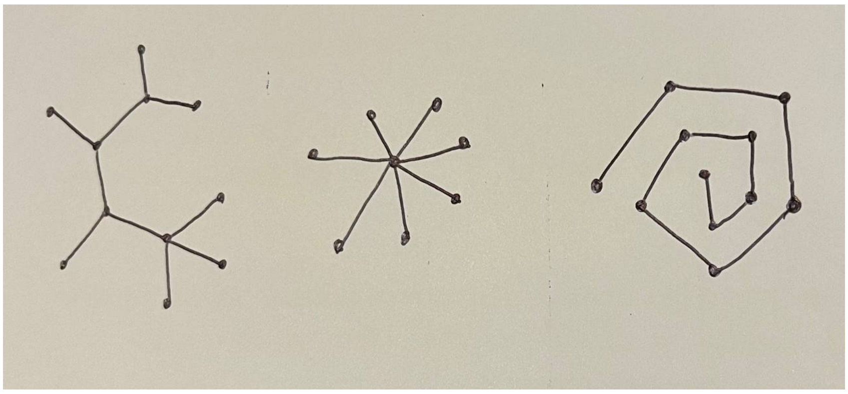
Definition. Graphs G_1, G_2, G_3, G_4, G_5 are called *connected* graphs, and graph G_6 is called *disconnected* graph (it consist of two separated parts). Give a formal definition of what does word *connected* mean (see the end of this page for a hint).

Definition. In graph G_5 the edges $(A, B), (B, D), (D, E), (E, A)$ form a *cycle*. Give a formal definition of a *cycle*.

3 Trees

Definition. Connected graph without cycles is called a *tree*.

Of the six graphs in the picture on the first page only G_2 is a tree, since all other graphs either have cycles or are disconnected. There are three more trees in the picture below, check that they are indeed trees:



Problem 3. Is it true that any tree with at least two nodes has at least one node of degree 1? Such node is called a *leaf*.

Problem 4. What is the connection between number of nodes and number of edges in a tree? (see the end of this page for hints)

Problem 5. Prove that in any connected graph G we can delete some edges to get a tree. Such tree is called a *spanning tree*.

For example, in graph G_1 we can delete edges (A, E) and (B, D), and the graph which is left is a tree.

Suppose we have n nodes $V_1, V_2, \ldots, V_n$. What is the number of different trees that can be constructed on these nodes? The Cayley's formula answers this question, we formulate it below.

Theorem (Cayley's formula)**.** *For every positive integer n the number of trees on n nodes is n^{n-2}.*

The proof of Cayley's formula will be the main result of session on November 9th. Check that the Cayley's formula is true for $n = 2, 3, 4$.

4 Hints

Hint for Problem 1 Suppose we delete an edge in a graph. How do degrees of nodes change? How does the sum of all degrees change?

Hint for definition of connected graph For each pair of nodes consider a path between them.

Hints for Problem 4 Find these numbers for several trees to guess the formula. Also result of Problem 3 about the leaves can be useful.