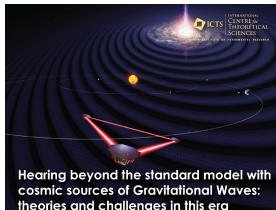


Probing non-minimal coupling through super-horizon instability and secondary gravitational waves

Ayan Chakraborty

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Hearing Beyond the Standard Model with Cosmic Sources of Gravitational Waves



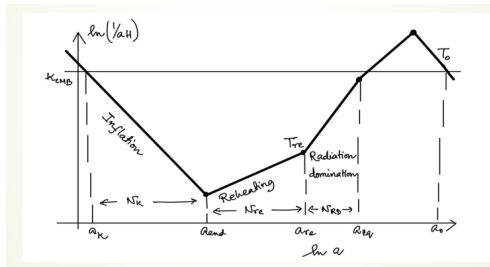
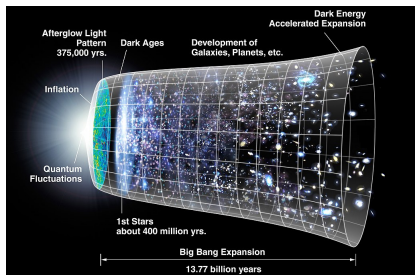
- 1 Motivation
- 2 Spectrum of non-minimally coupled scalar fluctuations
- 3 Generation of secondary gravitational waves by the scalar field source
- 4 Constraining non-minimal coupling strength(ξ) based on observational bound
- 5 Important findings

- Scalar fluctuations, non-minimally coupled to gravity, can be treated as a potential source of secondary gravitational waves.

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- Constraining non-minimal coupling through *Planck* bound on tensor-to-scalar ratio and ΔN_{eff} .

Cosmic Evolution



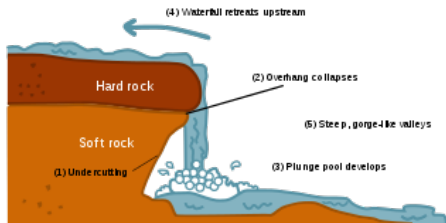
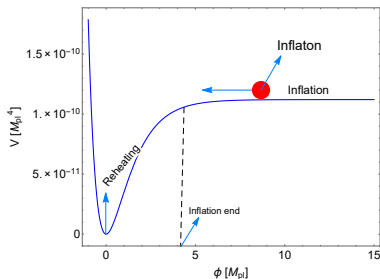
- Cosmic evolution and dynamics of Hubble horizon through modified expansion history.

Why do we need reheating phase?

- ▶ At the end of early accelerated expansion (Inflation), universe was left in a super cold state of vanishing entropy, and particle no. density.
- ▶ To achieve successful nucleosynthesis, universe must transit to a hot, thermalized radiation-dominated phase.

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- ▶ Inflaton \rightarrow SM+BSM \rightarrow hot thermal bath \rightarrow reheating



General set up of non-minimally coupled scalar field(χ) system

► Lagrangian of the system:

$$\mathcal{L}_{[\phi,\chi]} = - \underbrace{\sqrt{-g}}_{a^4(\eta)} \left(\frac{1}{2} \partial_\mu \phi \partial^\mu \phi + V(\phi) + \frac{1}{2} \partial_\mu \chi \partial^\mu \chi + \frac{1}{2} m_\chi^2 \chi^2 + \frac{1}{2} \xi R \chi^2 \right)$$

$a \rightarrow$ scale factor; $R \rightarrow$ Ricci scalar; $\xi \rightarrow$ non-minimal coupling

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- ▶ **Fourier decomposition:**

$$\chi(\eta, \vec{x}) = \int \frac{d^3 k}{(2\pi)^3} \chi_k(\eta) e^{i\vec{k} \cdot \vec{x}}$$

- ▶ **EoM of rescaled field mode($X_k = a(\eta)\chi_k(\eta)$):**

$$X_k'' + \left[k^2 + a^2 m_\chi^2 - \frac{a''}{a} (1 - 6\xi) \right] X_k = 0 \quad (1)$$

$$R = (6a''/a^3)$$

Dynamical equation and appearance of IR instability (Tachyonic instability)

► Form of scale factor:

$$a(\eta) = a_{\text{end}} \left(\frac{1+3w_\phi}{2|\eta_{\text{end}}|} \right)^{\frac{2}{1+3w_\phi}} \left(\eta - \eta_{\text{end}} + \frac{2|\eta_{\text{end}}|}{1+3w_\phi} \right)^{\frac{2}{1+3w_\phi}} ; \eta_i < \eta \leq \eta_{\text{end}}$$

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- ▶ We are interested in IR modes ($k < a_{\text{end}} H_{\text{end}} = k_{\text{end}}$) of very low mass case, $m_\chi \approx 0$

- ▶ Inflationary evolution: $X_k'' + \underbrace{\left[k^2 - \frac{2(1-6\xi)}{\eta^2} \right]}_{\omega_k^2 < 0 \text{ (Instability)} \rightarrow \text{for } \xi < 1/6} X_k = 0$

- ▶ Post-inflationary evolution:

$$X_k'' + \underbrace{\left[k^2 - \frac{2(1-3w_\phi)(1-6\xi)}{(1+3w_\phi)^2 \left(\eta + \frac{3(1+w_\phi)}{a_{\text{end}} H_{\text{end}} (1+3w_\phi)} \right)^2} \right]}_{\omega_k^2 < 0 \rightarrow \text{for } w_\phi > 1/3, \xi > 1/6, \text{ for } w_\phi < 1/3, \xi < 1/6} X_k = 0$$

Inflationary and post-inflationary vacuum solution

Adiabatic vacuum solution

Inflationary vacuum solution: $X_k^{(\text{inf})} = \frac{\sqrt{\pi|\eta|}}{2} e^{i(\pi/4 + \pi\nu_1/2)} H_{\nu_1}^{(1)}(k|\eta|)$

Post-inflationary vacuum solution:

$$X_k^{(\text{reh})} = \sqrt{\frac{\bar{\eta}}{\pi}} \exp\left[\frac{3ik\mu}{a_{\text{end}}H_{\text{end}}} + \frac{i\pi}{4}\right] K_{\nu_2}(ik\bar{\eta})$$

► **EoS and ξ dependent indices:** $\nu_1 = \sqrt{9 - 48\xi}/2$; $\mu = \frac{(1+w_\phi)}{(1+3w_\phi)}$;

$$\nu_2 = \frac{\sqrt{3(1+w_\phi)\left(3(1-w_\phi)^2 + 16\xi(3w_\phi - 1)\right)}}{2\sqrt{1+3w_\phi}\sqrt{1+4w_\phi+3w_\phi^2}}; \bar{\eta} = (\eta + 3\mu/a_{\text{end}}H_{\text{end}})$$

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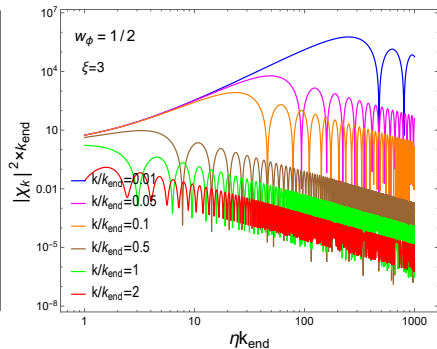
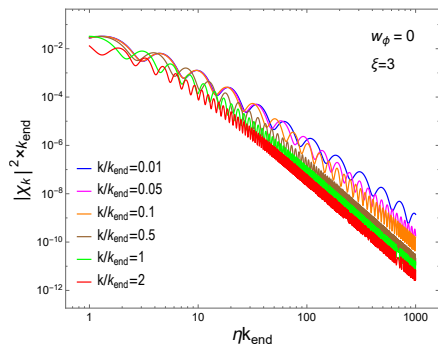
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► **General reheating field solution:** $X_k(\eta) = \alpha_k X_k^{(\text{reh})} + \beta_k X_k^{*(\text{reh})}$

$\alpha_k, \beta_k \rightarrow$ Bogoliubov coefficients

Time-evolution of long-wavelength(IR) modes of scalar fluctuations



¹ A.chakraborty, S.Maiti, and D.Maity [arxiv: 2408.07767]

Defining field power spectrum and energy-density spectrum

- ▶ Field power spectrum: $\mathcal{P}_\chi(k, \eta) = \frac{k^3}{2\pi^2 a^2} |X_k|^2$

Defining field power spectrum and energy-density spectrum

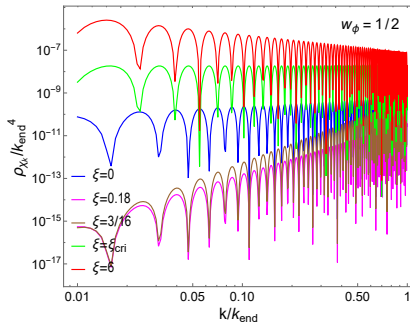
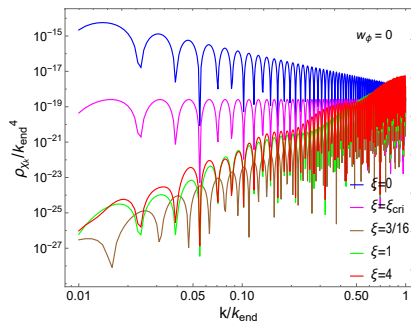
- ▶ Field power spectrum: $\mathcal{P}_\chi(k, \eta) = \frac{k^3}{2\pi^2 a^2} |X_k|^2$
- ▶ Field energy-density spectrum:

$$\rho_{\chi k}(\eta) = \frac{k^3}{4\pi^2 a^4} (|X_k'|^2 + k^2 |X_k|^2) = (k^2/a^2) \mathcal{P}_\chi(k, \eta)$$

Energy spectrum of IR modes for $1/3 < w_\phi \leq 1$

$$\rho_{\chi k}(\eta > \eta_{\text{end}}) \propto \begin{cases} (k/k_{\text{end}})^{2(2-\nu_1-\nu_2)} & \text{for } 0 \leq \xi < 3/16 \\ (k/k_{\text{end}})^{2(2-\nu_2)} & \text{for } \xi = 3/16 \\ (k/k_{\text{end}})^{2(2-\nu_2)} & \text{for } \xi > 3/16 \end{cases} \quad (2)$$

Behavior of energy-density spectrum



▶ $w_\phi = 0 \rightarrow \xi_{\text{cri}} \approx 5/48$

$w_\phi = 1/2 \rightarrow \xi_{\text{cri}} \approx 4.073$

¹ A.chakraborty, S.Maiti, and D.Maity [arxiv: 2408.07767]

Model independent definition of reheating parameters ($N_{\text{re}}, T_{\text{re}}$)

Reheating point: $\rho_{\text{R}}(a_{\text{re}}) = \rho_{\phi}(a_{\text{re}})$

Reheating e-folding number: $N_{\text{re}} = \frac{1}{3(1+w_{\phi})} \ln \left(\frac{90H_{\text{end}}^2 M_{\text{pl}}^2}{\pi^2 g_{\text{re}} T_{\text{re}}^4} \right)$

Defining k_{end} and k_{re} :

$(k_{\text{end}}/a_0) = \left(\frac{43}{11g_{\text{re}}} \right)^{1/3} \left(\frac{\pi^2 g_{\text{re}}}{90} \right)^{\alpha} \frac{H_{\text{end}}^{1-2\alpha} T_{\text{re}}^{4\alpha-1} T_0}{M_{\text{pl}}^{2\alpha}}$, $(k_{\text{end}}/k_{\text{re}}) = \exp\left(\frac{N_{\text{re}}(1+3w_{\phi})}{2}\right)$, $\alpha = 1/3(1+w_{\phi})$, $a_0 \rightarrow$ present scale factor, and $T_0 = 2.725$ K is the present CMB temperature

¹ L. Dai, M. Kamionkowski and J. Wang, PRL. 113, 041302 (2014)

² J. L. Cook, et al. JCAP 04 (2015) 047

Generation of secondary(induced) gravitational wave(SGW)

▶ **Perturbed FLRW metric:**

$$ds^2 = a^2(\eta) [-d\eta^2 + (\delta_{ij} + h_{ij})dx^i dx^j], \text{ transverse-traceless tensor}$$
$$\rightarrow \partial^i h_{ij} = h^i_i = 0$$

▶ **anisotropic stress tensor:** $\Pi_{ij} \sim (1 - 2\xi)\partial_i\chi\partial_j\chi - 2\xi\chi\partial_i\partial_j\chi + \xi\chi^2 G_{ij}$

▶ **Evolution equation:**

$$h_{\mathbf{k}}^{\lambda\prime\prime} + 2\frac{a'}{a}h_{\mathbf{k}}^{\lambda\prime} + k^2 h_{\mathbf{k}}^{\lambda} = \frac{2}{M_{pl}^2} e_{\lambda}^{ij}(k) P_{ij}^{lm}(\hat{k}) T_{lm}(k, \eta), \quad P_{ij}^{lm}(\hat{k}) \rightarrow$$

transverse-traceless projector

▶ outline of evolution Equation

Defining Gravitational wave(GW) energy spectrum

▶ GW energy spectrum:

$$\Omega_{\text{gw}}(k, \eta) = (\Omega_{\text{gw}}^{\text{pri}} + \Omega_{\text{gw}}^{\text{sec}}) = \frac{(1+k^2/k_{\text{re}}^2)}{24} (\mathcal{P}_{\text{T}}^{\text{pri}}(k, \eta_{\text{re}}) + \mathcal{P}_{\text{T}}^{\text{sec}}(k, \eta_{\text{re}}))$$

▶ Energy spectrum for today:

$$\Omega_{\text{gw}}(k)h^2 \approx \left(\frac{g_{r,0}}{g_{r,\text{eq}}} \right)^{1/3} \Omega_R h^2 \Omega_{\text{gw}}(k, \eta), \quad \Omega_R h^2 = 4.3 \times 10^{-5}$$

For $w_\phi > 1/3$, $\xi > 3/16$

$$\Omega_{\text{gw}}^{\text{pri}}(k < k_{\text{re}})h^2 \propto (k/k_{\text{re}})^0 \quad (3)$$

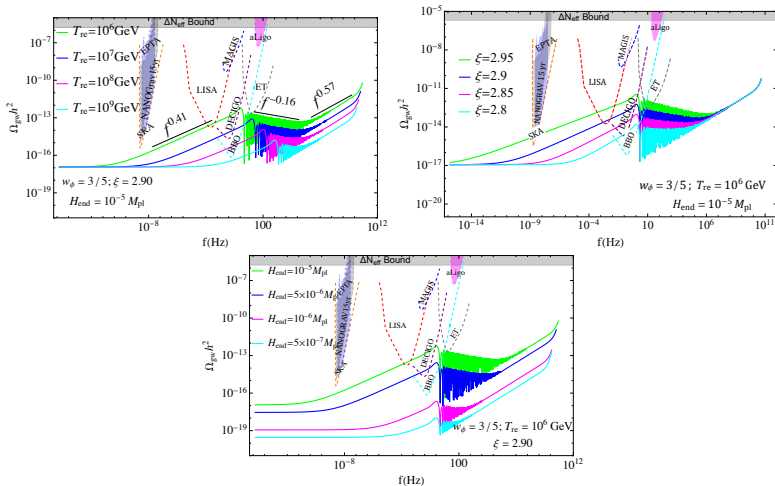
$$\Omega_{\text{gw}}^{\text{pri}}(k > k_{\text{re}})h^2 \propto (k_{\text{end}}/k_{\text{re}})^{n_w} (k/k_{\text{end}})^{n_w}; \quad n_w = 2(3w_\phi - 1)/(1 + 3w_\phi) \quad (4)$$

$$\Omega_{\text{gw}}^{\text{sec}}(k < k_{\text{re}})h^2 \propto (k_{\text{end}}/k_{\text{re}})^{4-2\delta} (k/k_{\text{end}})^{2(4-2\nu_2)}; \quad \delta = 4/(1 + 3w_\phi) \quad (5)$$

$$\Omega_{\text{gw}}^{\text{sec}}(k > k_{\text{re}})h^2 \propto (k_{\text{end}}/k_{\text{re}})^{2-\delta} (k/k_{\text{end}})^{6+\delta-4\nu_2} \quad (6)$$

▶ Defining secondary tensor power spectrum

GW spectrum for varying ξ , T_{re} and inflationary energy scale (H_{end})



Constraining ξ through tensor-to-scalar ratio ($r_{0.05}$) and ΔN_{eff} (for the scalar field)

- ▶ For $w_\phi > 1/3$, in the regime $k < k_{\text{re}}$; $r_{0.05} \propto$

$$\left(\frac{90H_{\text{end}}^2 M_{\text{pl}}^2}{\pi^2 g_{\text{re}} T_{\text{re}}^4} \right)^{\frac{2(3w_\phi - 1)}{3(1+w_\phi)}} \left(\frac{k_*}{k_{\text{end}}} \right)^{4(2-\nu_2)} \leq 0.036; (k_*/a_0) = 0.05 \text{ Mpc}^{-1}$$

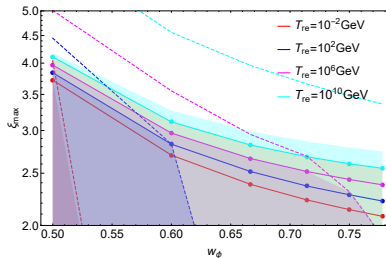
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- ▶ This massless scalar, possible candidate for dark radiation, solely contributes to $\Delta N_{\text{eff}} \rightarrow$

$$\left(\frac{g_{r,0}}{g_{r,\text{eq}}} \right)^{1/3} \Omega_{\text{R}} h^2 \Omega_{\chi}(\eta_{\text{re}}) \simeq 1.6 \times 10^{-6} \left(\frac{\Delta N_{\text{eff}}}{0.284} \right)$$



Contribution of GWs to ΔN_{eff}

- ▶ If GWs(PGW+SGW) solely contributes to ΔN_{eff} then

$$\Omega_{\text{gw}} h^2 \leq 1.6 \times 10^{-6} \left(\frac{\Delta N_{\text{eff}}}{0.284} \right), \quad \Omega_{\text{gw}} h^2 = \int_{k_{\text{min}}}^{k_{\text{end}}} \frac{dk}{k} \Omega_{\text{gw}}(k) h^2$$

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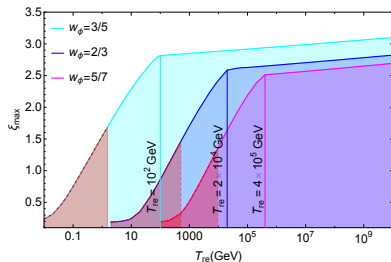
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- ▶ Minimum bound on T_{re} (avoiding overproduction of extra degrees of freedom):

$$T_{\text{re}}^{\text{min}} \geq \left(\frac{90 H_{\text{end}}^2 M_{\text{pl}}^2}{\pi^2 g_{\text{re}}} \right)^{1/4} \beta^{\frac{3(1+w_\phi)}{4(3w_\phi-1)}} \left(\frac{0.284}{\Delta N_{\text{eff}}} \right)^{\frac{3(1+w_\phi)}{4(3w_\phi-1)}}, \quad \beta \rightarrow$$

$$(1.43 \times 10^{-11} / n_w) (H_{\text{end}} / 10^{-5} M_{\text{pl}})^2$$



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Important outcomes

- Post-inflationary instability effect is dominant for higher EoS $w_\phi > 1/3$. The longer the wavelength, the more the enhancement owing to prolonged instability for a larger coupling strength ξ .

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- Post-inflationary instability effect is dominant for higher EoS $w_\phi > 1/3$. The longer the wavelength, the more the enhancement owing to prolonged instability for a larger coupling strength ξ .
- For $w_\phi > 1/3$, significant IR instability beyond a certain large ξ leaves a visible imprint on the SGW spectrum overcoming the PGW strength at the low and intermediate frequency ranges.
- Combining two strong observational bounds, $r_{0.05}$ and ΔN_{eff} , to prevent the overproduction of tensor fluctuations at the CMB scale and the overproduction of extra relativistic degrees of freedom, we have found a tight constraint on coupling strength. We find that $\xi_{\text{max}} \lesssim 4$ for any $w_\phi \geq 1/2$ for a wide range of reheating temperatures. Unlike $w_\phi > 1/3$, for $w_\phi < 1/3$, we put lower bound on ξ . For $w_\phi = 0$, we find the lower bound $\xi_{\text{min}} \gtrsim 0.02$.

Thank you!

- **Bogoliubov coefficients** (α_k, β_k): Making the adiabatic vacuum solutions $X_k^{(\text{inf})}$ and $X_k^{(\text{reh})}$, and their first derivatives continuous at the junction $\eta = \eta_{\text{end}}$, we compute the Bogoliubov coefficients as follows ¹:

$$\begin{aligned}\alpha_k &= i \left(X_k^{(\text{inf})'}(\eta_{\text{end}}) X_k^{(\text{reh})*}(\eta_{\text{end}}) - X_k^{(\text{inf})}(\eta_{\text{end}}) X_k^{(\text{reh})*'}(\eta_{\text{end}}) \right) \\ \beta_k &= -i \left(X_k^{(\text{inf})'}(\eta_{\text{end}}) X_k^{(\text{reh})}(\eta_{\text{end}}) - X_k^{(\text{reh})'}(\eta_{\text{end}}) X_k^{(\text{inf})}(\eta_{\text{end}}) \right) \quad (7)\end{aligned}$$

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- ▶ **Action with anisotropic stress:**

$$S_{GW} = \int dx^4 \sqrt{-g} \left[-\frac{g^{\mu\nu}}{64\pi G} \partial_\mu h_{ij} \partial_\nu h^{ij} + \frac{1}{2} \Pi^{ij} h_{ij} \right]$$

- ▶ **Fourier decomposition:**

$$h_{ij}(\eta, \mathbf{x}) = \sum_{\lambda=(+, \times)} \int \frac{d^3\mathbf{k}}{(2\pi)^{3/2}} e_{ij}^\lambda(\mathbf{k}) h_{\mathbf{k}}^\lambda(\eta) e^{i\mathbf{k}\cdot\mathbf{x}}, \quad e_{ij}^\lambda(\mathbf{k}) \rightarrow \text{polarization tensor}$$

Defining secondary tensor power spectrum

◀ Back

- **Tensor power spectrum:** $\mathcal{P}_T(k, \eta) = 4 \frac{k^3}{2\pi^2} |h_{\mathbf{k}}(\eta)|^2$, $h_{\mathbf{k}}(\eta) = h_{\mathbf{k}}^{\text{vac}} + \frac{2e^{ij}(\mathbf{k})}{M_{pl}^2} \int d\eta_1 \mathcal{G}_k(\eta, \eta_1) \Pi_{ij}^{\text{TT}}(\mathbf{k}, \eta_1)$

Defining secondary tensor power spectrum

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► **Secondary tensor power spectrum:**

$$\mathcal{P}_T^{\text{sec}}(k, \eta_{\text{re}}) \propto \frac{1}{M_{\text{pl}}^4} \left(\int_{x_e}^{x_{\text{re}}} dx_1 \frac{\mathcal{G}_k^{\text{re}}(x_{\text{re}}, x_1)}{a^2(x_1)} \right)^2 \times \int_{k_{\text{min}}}^{k_{\text{end}}} \frac{dq}{k} \int_{-1}^1 d\gamma (1 - \gamma^2)^2 \times \frac{(q/k)^3 \mathcal{P}_X(q, \eta_1) \mathcal{P}_X(|\mathbf{k} - \mathbf{q}|, \eta_1)}{|1 - q/k|^3}, \quad x = k\eta, \quad \cos\gamma = \hat{\mathbf{k}} \cdot \hat{\mathbf{q}}$$

For $w_\phi > 1/3$, $\xi > 3/16$

$$\mathcal{P}_T^{\text{sec}}(k < k_{\text{re}}, \eta_{\text{re}}) \propto \left(\frac{k_{\text{end}}}{k_{\text{re}}} \right)^{4-2\delta} \left(\frac{k}{k_{\text{end}}} \right)^{4(2-\nu_2)}; \delta = 4/(1 + 3w_\phi) \quad (9)$$

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