* Irreducible components of Zariski closure

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*We prove Density theorem out of Black Box theorem a general theory of Zariski closure in a pro-étale variety of an infinite set of close points, and we prepare some notation and geometric lemmas to prove the density theorem. As before, let $\alpha_{\mathfrak{l}} = \text{diag}[1, \varpi_{\mathfrak{l}}]$. Choose a large power $\mathfrak{l}^m$ which is generated by $\varpi = \varphi\varphi^c$ ($\varphi \in R$) and put $\alpha = \text{diag}[1, \varpi]$. We may assume $\alpha/\alpha_{\mathfrak{l}}^m \in K$ and hence pretend $\alpha = \alpha_{\mathfrak{l}}^m$ to have the same action on $Sh_K = Sh/K$. α preserves each irreducible component of $Sh_{K/\mathbb{F}}$ as long as $\det(K) = \widehat{O}^\times$. Set $N := \{\alpha(u) | u \in O_{\mathfrak{l}}\}$, and assume $K \supset N$. For a variety $Y_{/\mathbb{F}}$, write $\text{Irr}_d(Y)$ for the set of irreducible components of dimension d of Y. Set $\text{Irr}(Y) := \bigsqcup_d \text{Irr}_d(Y)$ and $\text{Irr}_+(Y) := \bigsqcup_{d>0} \text{Irr}_d(Y)$.

§0. The idea.

For a proper R_n-ideal $\mathcal{A}_n$ prime to $p\mathfrak{l}$, define a proper $R_{n'}$-ideal by $\widehat{\mathcal{A}}_{n'} := \widehat{\mathcal{A}}_n^{(\mathfrak{l})} \times R_{n',\mathfrak{l}}$. We have three identities:

$$\alpha_{\mathfrak{l}}(x(\mathcal{A}_n)) = x(\mathcal{A}_{n+1}) \quad \text{for } \alpha_{\mathfrak{l}} = \mathrm{diag}[1, \varpi_{\mathfrak{l}}],$$

$$O_{\mathfrak{l}}/\mathfrak{l}^i \cong \Gamma_n[\mathfrak{l}^i] \quad \text{by } u \mapsto \alpha(u/\varpi_{\mathfrak{l}}^i)x(R_n) =: x(\mathcal{A}_{u,n}) \quad \text{if } n \geq i,$$

For $\xi \in \mathcal{T}_j := (M^\times \cap R_{j,\mathfrak{l}}^\times) \subset \mathrm{GL}_2(F_{\mathbb{A}}^{(p\mathfrak{l}\infty)})$, $\xi(x(\mathcal{A})) = x((\xi)\mathcal{A})$.

We show, if $\underline{n}$ is an arithmetic progression, the semi-group $\alpha^{\mathbb{N}}$ generated by α^i $i = 1, 2, \ldots$ for the m-th power α of $\alpha_{\mathfrak{l}}$ acts on $\mathrm{Irr}_0(X_K)$ for $X_K = \overline{\Xi}_{\underline{n},j,K} \subset V_K^{\mathcal{Q}}$. Since each orbit of $\alpha^{\mathbb{N}}$ is infinite, for the image X_K in Sh/K,

$$\mathrm{Irr}_0(X_K) \neq \emptyset \Rightarrow |\mathrm{Irr}_0(X_K)| = \infty,$$

a contradiction against noetherian property of $X_K \subset V_K$, and hence $\mathrm{Irr}_0(X) = \varinjlim_K \mathrm{Irr}_0(X_K) = \emptyset$. Since $\alpha \in \mathrm{Aut}(V)$ (automorphisms) and by the above identity, $\alpha^{-1}(\Xi) - \Xi$ is a finite set, which we need later. We may replace $\alpha^{\mathbb{N}}$ by $\mathcal{S} = \alpha^{\mathbb{N}} \cdot \mathcal{T}_j$ to make the action on Ξ transitive.

§**1. Basics.** Write $\mathcal{V} := V^{\mathcal{Q}}$, and adding the subscript K implies the image in $(Sh/K)^{\mathcal{Q}}$. Take K sufficiently small so that $\mathcal{V}_{/\mathcal{V}_K}$ is étale. Since for proper R_n-ideals $\mathcal{A}$ and $\mathcal{A}'$

$$X(\mathcal{A}) \cong X(\mathcal{A}') \Leftrightarrow [\mathcal{A}]_n = [\mathcal{A}']_n \in Cl_n,$$

a. $\pi_* : C \cap \mathcal{V} \cong C_K \cap \mathcal{V}_K$ by projection $\pi : \mathcal{V} \to \mathcal{V}_K$.

b. For the chosen infinite subset $\Xi \subset C \cap \mathcal{V}$, let X (resp. X_K) be the Zariski closure of Ξ in $\mathcal{V}$ (resp. Ξ_K in $\mathcal{V}_K$). Then X_K is the reduced image of X in $\mathcal{V}_K$.

c. For the image $[\delta_n] \in Cl_n^-$ of $\delta \in \mathcal{Q}$ with δ_n prime to $p\mathfrak{l}$, $Cl_n \ni [\mathcal{A}]_n \mapsto [\mathcal{A}\delta_n]_n \in Cl_n^-$ commutes with the action of $\mathcal{S}$ and $\alpha(u)$ for $u \in F_{\mathfrak{l}}$, since the action of $\mathcal{S}$ and $\alpha(u)$ is concentrated at $\mathfrak{l}$ and the multiplication by $[\delta_n]_n$ is outside $\mathfrak{l}$-action. So the diagonal action of $\mathcal{S}$ on $\mathcal{V}$ preserves $C \cap \mathcal{V}$.

§**2. Irr(X).** Let $\pi_*(\mathrm{Irr}(X_U)) := \{\pi(Z')|Z' \in \mathrm{Irr}(X_U)\}$ for a closed subgroup U of K and the projection $\pi : \mathcal{V}_U \twoheadrightarrow \mathcal{V}_K$ for the projection, where $\pi(Z') \subset X_K$ is the reduced image. Then

1 (Going up theorem). For $Y \in \mathrm{Irr}(X_K)$, if $Z \in \mathrm{Irr}(\pi^{-1}(Y))$ is contained in X_U, we have $Z \in \mathrm{Irr}(X_U)$, where $\pi^{-1}(Y) = Y \times_{\mathcal{V}_K} \mathcal{V}_U$.

2. The image $\pi_*(\mathrm{Irr}(X_U))$ contains $\mathrm{Irr}(X_K)$; so, for $Y \in \mathrm{Irr}(X_K)$, we have $Z' \in \mathrm{Irr}(X_U)$ such that $\pi_*(Z') = Y$, because any closed irreducible subvariety is contained in an irreducible component.

3. We have a unique section $\mathrm{Irr}_0(X_K) \hookrightarrow \mathrm{Irr}_0(X_U)$ of $\mathrm{Irr}_0(X_U) \twoheadrightarrow \pi_*(\mathrm{Irr}_0(X_U)) \subset X$ and $\mathrm{Irr}_0(X_U) \subset \Xi_U$. Moreover

$$\mathrm{Irr}_0(X_U) = \varinjlim_{U'} \mathrm{Irr}_0(X_{U'}) \subset \Xi$$

for U' running over all open subgroups of K containing U.

I will give a proof of some of these assertions later if time allows.

§**3. Correspondence action of $\alpha^{\mathbb{N}}$ on** $\mathrm{Irr}_d(X_K)$**.** Let $\beta = \alpha^i$. For an irreducible component $Y_K \in \mathrm{Irr}_d(X_K)$, let $Y_U = \bigcup_{Z \in \mathrm{Irr}_d(\pi_{U,K}^{-1}(Y_K)) \cap \mathrm{Irr}_d(X_U)} Z$. Consider the diagram for $U = K \cap K^\beta$ for $K^\beta := \beta^{-1} K \beta$ (so, $UU^{\beta^{-1}} \subset K$):

$$\begin{array}{ccc} X_U \supset Y_U & \xrightarrow{v \mapsto \beta(v)} & \beta(Y_U) \subset X_{U^{\beta^{-1}}} \\ \pi_{U,K} \downarrow & & \downarrow \pi = \pi_{U^{\beta^{-1}},K} \\ X_K \supset Y_K & \longrightarrow & \pi(\beta(Y_U)) \subset X_K. \end{array}$$

We define the correspondence action of β by

$$[\beta](Y_K) := \{\pi\beta(Z) | Z \in \mathrm{Irr}_d(Y_U)\}.$$

This set $[\beta](Y_K)$ can be shown to be a subset of $\mathrm{Irr}_d(X_K)$. As we only need the case of $d = 0$, we prove this fact assuming $d = 0$. Then $[\beta](Y_K)$ is a singleton made of $\beta(Y_K)$.

§**4.** β**-action on** $\mathrm{Irr}_0(X_K)$**.** Suppose $d = 0$, and write $U' := U^{\beta^{-1}}$. By Property 3, $x_K = Y_K \in \mathrm{Irr}_0(X_K)$ falls in the image Ξ_K in $\mathcal{V}_K$ of Ξ. Since $\Xi \cong \Xi_U \cong \Xi_K$, $p_{U,K}^{-1}(x_K) \cap X_U = \{x_U\} \subset \Xi_U$ is a singleton. Therefore $Y_U = \{Z := x_U\}$ is a singleton. Take an irreducible component Y'_K of X_K containing $\beta(x)_K = \beta(x_{K^\beta})$ such that $\beta(Z) \subset Z'$ for an irreducible component Z' of $Y'_{U'}$ (so, $\beta(x_U) \in Z'$). Such a Y'_K exists by Property 2. So $\dim Z' = \dim Y'_K \geq 0$. We want to prove $\dim Y'_K = 0$. Since $\mathrm{Irr}_+(\beta^{-1}(X)_U) = \mathrm{Irr}_+(X_U)$ by $|\beta^{-1}(\Xi) - \Xi| < \infty$, if $\dim Z' > 0$, we have $\dim \beta^{-1}(Z') > 0$ and $\beta^{-1}(Z')$ is an irreducible component of X_U. Since $\beta^{-1}(Z') \supset Z = x_U$ by construction and the two are irreducible components of X_U (by going-up), we find that $\beta^{-1}(Z') = Z = x_U$, a contradiction against $\dim Z' > 0$. Hence $\dim Z' = 0$ and $Z' = \beta(Z) = \beta(x_U)$, and $Y'_K = p_{U',K}(Z') = \beta(x)_K$. This implies that $[\beta]$ brings $\mathrm{Irr}_0(X_K)$ into $\mathrm{Irr}_0(X_K)$, and $x_K \mapsto [\beta](x_K) = \beta(x)_K$ is really an action (not a correspondence action) of $\alpha^{\mathbb{N}}$ on $\mathrm{Irr}_0(X_K)$, and the action is compatible with the action of $\alpha^{\mathbb{N}}$ on Ξ as $\mathrm{Irr}_0(X_K) \subset \Xi_K \cong \Xi$.

§5. Proof of density theorem.

Density Theorem. *Assume* $\mathcal{Q} \hookrightarrow Cl_{\infty}^{-}/Cl^{alg}$. *Let* $\underline{n} \subset \mathbb{Z}_{+}$ *be the sequence defining* Ξ. *If* $\underline{n}$ *contains an arithmetic progression, then* $X \cap \Xi \neq \emptyset$ *and* Ξ *is Zariski dense in* $V^{\mathcal{Q}}$.

Proof. We can replace $\underline{n}$ by an arithmetic progression of suitable difference so that $\alpha^{\mathbb{N}}$ preserve $\Xi_{\underline{n},r}$. Then $\mathcal{S} = \alpha^{\mathbb{N}} \cdot \mathcal{T}$ acts transitively on Ξ with all orbits are infinite. If $\mathrm{Irr}_0(X_K) \neq \emptyset$, by the action of $\mathcal{S}$ on $\mathrm{Irr}_0(X_K)$ described in §4, $\mathrm{Irr}_0(X_K)$ is infinite. This is a contradiction, as X_K is a noetherian scheme.

Thus $\mathrm{Irr}_0(X) = \varinjlim_K \mathrm{Irr}_0(X_K) = \emptyset$, therefore all irreducible components of X has positive dimension; so, we have an irreducible component Z of X with $x \in Z \cap \Xi$, By Black Box Theorem, $Z = X = V^{\mathcal{Q}}$ as desired. □

§**6. Proof of Property 3.** Since $\mathcal{V}_U \to \mathcal{V}_K$ is étale, it is affine; so, we may assume that $\mathcal{V}_U = \mathrm{Spec}(A')$ and $\mathcal{V}_K = \mathrm{Spec}(A)$ with $A'_{/A}$ finite. Write $\mathrm{Irr}_?(A) = \mathrm{Irr}_?(\mathrm{Spec}(A))$ and regard it as a set of minimal primes. Then $X_U = \mathrm{Spec}(B')$ and $X_K = \mathrm{Spec}(B)$ for $B' = A'/\bigcap_{P \in \Xi_U} P$ and $B = A/\bigcap_{P \in \Xi_U}(A \cap P)$ regarding Ξ_U a set of maximal A'-ideals. Pick $\mathfrak{m} \in \mathrm{Irr}_0(B)$. Then $B = B^{(\mathfrak{m})} \oplus B/\mathfrak{m}$ for a subring $B^{(\mathfrak{m})} \subset B$ as $\mathrm{Spec}(B/\mathfrak{m})$ is a connected component of $\mathrm{Spec}(B)$. Since $B' \supset B$, the above decomposition induces an algebra direct sum $B' = B'^{(\mathfrak{m})} \oplus B'/\mathfrak{m}B'$. Since B' is finite over B, $B'/\mathfrak{m}B'$ has dimension 0. By reducedness of B', the direct summand $B'/\mathfrak{m}B'$ of B' is a direct sum of fields. Then π induces a surjection $\mathrm{Irr}_0(B') \supset \pi_0(\mathrm{Spec}(B'/\mathfrak{m}B')) \overset{\pi^*}{\twoheadrightarrow} \{\mathfrak{m}\}$ for each $\mathfrak{m}$. Therefore $\pi_*(\mathrm{Irr}_0(B')) \supset \mathrm{Irr}_0(B)$. If $\mathfrak{m} \notin \Xi_K$, $\Xi_K \subset \mathrm{Spec}(B^{(\mathfrak{m})})$ as $\mathrm{Spec}(B) = \mathrm{Spec}(B/\mathfrak{m}) \sqcup \mathrm{Spec}(B^{(\mathfrak{m})})$. This implies $B = A/\bigcap_{P \in \Xi_K} P$ is equal to $B^{(\mathfrak{m})}$, a contradiction. Thus $\mathfrak{m} \in \Xi_K$, and $\mathrm{Irr}_0(B) \subset \Xi_K$. Since $\Xi \cong \Xi_K$, $\pi_* : \mathrm{Irr}_0(B') \to \pi_*(\mathrm{Irr}_0(B'))$ has a unique section $\pi^* : \mathrm{Irr}_0(B) \to \mathrm{Irr}_0(B')$.

§7. Proof of Property 1.

As $\mathcal{V} \twoheadrightarrow \mathcal{V}_K$ is étale, $\pi^{-1}(Y)$ is étale over Y; so, equi-dimensional. Suppose that $Z \subset X'$ for $Z \in \mathrm{Irr}(\pi^{-1}(Y))$. Then we find $Z' \in \mathrm{Irr}(X')$ such that $Z' \supset Z$; so, $\pi(Z') \subset X$. We are going to show $Z' = Z$. We have $X \supset \pi(Z') \supset Y$. Since $\pi(Z')$ is irreducible, $\pi(Z')$ containing $Y \in \mathrm{Irr}(X)$ implies $\pi(Z') = Y$. Thus $Z' \twoheadrightarrow Y$ is a integral dominant; so, $\dim Z' = \dim Z = \dim Y$. This shows $Z = Z' \in \mathrm{Irr}(X')$, as desired. Thus Property 1 follows.

§8. Proof of Property 2.

Pick $\mathfrak{p} \in \mathrm{Irr}(B)$ giving $Y \in \mathrm{Irr}(\mathrm{Spec}(B))$. Since B'/B is integral, we find a prime $P' \in \mathrm{Spec}(B')$ such that $P' \cap B = \mathfrak{p}$ by going-up theorem. For each $P' \in \mathrm{Spec}(B')$ with $P' \cap B = \mathfrak{p}$ (i.e., $P' \in \pi^{-1}(Y) = \mathrm{Spec}(B'/\mathfrak{p}B')$), take a minimal prime $\mathfrak{p}' \subset P'$ (i.e., $\mathfrak{p}' \in \mathrm{Irr}(B')$). Then $\mathfrak{p}' \cap B$ is a prime ideal of B and $\mathfrak{p} \supset \mathfrak{p}' \cap B$; so, by minimality of $\mathfrak{p}$, we have $\mathfrak{p} = \mathfrak{p}' \cap B$. Thus $\mathfrak{p}$ is in the image of $\mathrm{Irr}(B')$. This proves Property 2.