A new Weyl group action and a cluster structure for representations of shifted quantum groups

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November 20, 2023 - ICTS Conference

Classical Theory

- $\mathfrak{g}$ complex finite-dimensional simple Lie algebra of rank n.
- Simple finite dimensional modules parametrized by dominant weights or monomials in
$$\mathbb{Z}[y_i]_{1\leq i\leq n},$$
where the y_i correspond to fundamental weights.
- Character morphism :
$$\chi : \mathrm{Rep}(\mathfrak{g}) \to \mathbb{Z}[y_i^{\pm 1}]_{1\leq i\leq n}$$
- Image of the character morphism :
$$\mathrm{Im}(\chi) = (\mathbb{Z}[y_i^{\pm 1}]_{1\leq i\leq n})^W$$
W : Weyl group W generated by the simple reflexions s_i
$$s_i(y_j) = y_j a_i^{-\delta_{ij}} \text{ where } a_i = \prod_{k\in I} y_k^{C_{ji}}.$$
a_i corresponds to a simple root (C is the Cartan matrix of $\mathfrak{g}$).

Classical Theory

- Example, for $\mathfrak{g} = sl_2$:

$$a_1 = y_1^2$$

$$s_1(y_1) = y_1 a_1^{-1} = y_1^{-1},$$

$$s_1^2(y_1) = y_1$$

$$s_1(y_1 + y_1^{-1}) = y_1 + y_1^{-1}.$$

$$\mathrm{Im}(\chi) = (\mathbb{Z}[y_1^{\pm 1}])^W = \mathbb{Z}[y_1 + y_1^{-1}].$$

Quantum affine algebra

- $\hat{\mathfrak{g}}$: affine Kac-Moody algebra.
- (Central extension of the loop algebra $\mathfrak{g} \otimes \mathbb{C}[t, t^{-1}]$).
- $q \in \mathbb{C}^*$: quantum parameter (not root of unity).
- Quantum affine algebra : $\mathcal{U}_q(\hat{\mathfrak{g}})$.
- $\mathcal{U}_q(\hat{\mathfrak{g}})$: Hopf algebra, q-deformation of $\mathcal{U}(\hat{\mathfrak{g}})$.
- For simplicity of the notations : we assume $\mathfrak{g}$ simply-laced (most results will be for general types).

Quantum affine algebra

- $\mathcal{C}$: Category of finite-dimensional representations of $\mathcal{U}_q(\hat{\mathfrak{g}})$: very interesting (and intricated) category.
- $\mathcal{C}$: tensor category, but not semi-simple and not braided.

Theorem (Chari-Pressley)

Simple finite-dimensional representations of $\mathcal{U}_q(\hat{\mathfrak{g}})$ (of type 1) are parameterized by n-tuple of rational fractions of the form

$$q^{deg(P_i)} \frac{P_i(zq^{-1})}{P_i(zq)}$$

where $P_i(z) = \prod_{a\in\mathbb{C}^*}(1 - za)^{u_{i,a}} \in \mathbb{C}[z]$ and $P_i(0) = 1$ (Drinfeld polynomials).

- Monomial notation : $m = \prod_{i\in I, a\in\mathbb{C}^*} Y_{i,a}^{u_{i,a}}$.

q-characters

- Analogue of character morphism : q-character (Frenkel-Reshetikhin) :

$$\chi_q : \mathrm{Rep}(\mathcal{U}_q(\hat{\mathfrak{g}})) \to \mathcal{Y} = \mathbb{Z}[Y_{i,a}^{\pm 1}]_{1 \leq i \leq n, a \in \mathbb{C}^*}.$$

- Injective ring morphism on the Grothendieck ring of $\mathcal{C}$:

$$\mathrm{Rep}(\mathcal{U}_q(\hat{\mathfrak{g}})).$$

- Recovers χ by forgetting the spectral parameters a.
- Example : fundamental representations of $\mathcal{U}_q(\hat{\mathfrak{sl}_2})$:

$$\chi_q(V_1(a)) = Y_{1,a} + Y_{1,aq^2}^{-1}.$$

- Weyl-group symmetry ?

Symmetry of q-characters : braid group action

- Braid group approach : Chari defined ring automorphisms of $\mathcal{Y}$

$$T_i(Y_{j,a}) = Y_{j,a}A_{i,aq}^{-\delta_{i,j}},$$

$$A_{i,a} = Y_{i,aq^{-1}}Y_{i,aq}\prod_{j|C_{j,i}=-1} Y_{j,a}^{-1}.$$

- T_i operator of infinite order.

Theorem (Chari 2002)

The operators T_i define a braid group action.
The q-characters are "partially" preserved by these operators

- Example : $\mathfrak{g} = sl_2$,

$$T_1(Y_{1,a}) = Y_{1,aq^2}^{-1} \ , \ T_1(Y_{1,aq^2}^{-1}) = Y_{1,aq^4}.$$

Partial symmetry of $\chi_q(V_1(a)) = Y_{1,a} + Y_{1,aq^2}^{-1}$.

Symmetry of q-characters

- Different operators (Frenkel-H. 2022) :

$$\Theta_i(Y_{j,a}) = Y_{j,a} A_{i,aq^{-1}}^{-\delta_{i,j}} \frac{\Sigma_{i,aq^{-3}}^{\delta_{i,j}}}{\Sigma_{i,aq^{-1}}^{\delta_{i,j}}}$$

- Here $\Sigma_{i,a}$ is the solution of the q-difference equation

$$\Sigma_{i,a} = 1 + A_{i,a}^{-1} \Sigma_{i,aq^{-2}}$$

in a sum

$$\Pi = \bigoplus_{w \in W} \tilde{\mathcal{Y}}^w$$

of completions $\tilde{\mathcal{Y}}^w$ of $\mathcal{Y}$.

- $\tilde{\mathcal{Y}}^w$: topological completions of $\mathcal{Y}$ with respect to the partial ordering on weights $\prec_w$ twisted by $w \in W$.

Symmetry of q-character

- Example : $\mathfrak{g} = sl_2$,

$$\Theta_1(Y_{1,a}) = Y^{-1}_{1,aq^{-2}} \frac{\Sigma_{1,aq^{-3}}}{\Sigma_{1,aq^{-1}}}.$$

- Here $\Sigma_{1,a}$ is the couple

$$(1 + A^{-1}_{1,a}(1 + A^{-1}_{1,aq^{-2}}(1 + \cdots), -A_{1,aq^2}(1 + A_{1,aq^4}(1 + \cdots))).$$

It belongs to

$$\Pi = \mathcal{Y}^e \oplus \mathcal{Y}^{s_1}.$$

Symmetry of q-characters

- $\mathcal{Y}$ embeds in Π diagonally.
- We establish that the Θ_i define involutions of Π.
- And then :

Theorem (Frenkel-H. 2022)

The Θ_i define a Weyl group action and

$$\mathcal{Y}^W = Im(\chi_q).$$

- For $w \in W$, we have Θ_w well-defined.

Symmetry of q-character

- Example : $\mathfrak{g} = sl_2$,

$$\Theta_1(Y_{1,a} + Y_{1,aq^2}^{-1}) = Y_{1,aq^{-2}}^{-1} \frac{\Sigma_{1,aq^{-3}}}{\Sigma_{1,aq^{-1}}} + Y_{1,a} \frac{\Sigma_{1,aq}}{\Sigma_{1,aq^{-1}}} = Y_{1,a} + Y_{1,aq^2}^{-1}.$$

- The Chari operator T_i can be recovered as a "leading term" of one component Θ_i.
- The Frenkel-Reshetikhin screening operators S_i can be recovered from the limit of another component of Θ_i.
- Formally, at "q root of unity", one component of Θ_i is related to an operator introduced by Inoue.

Shifted quantum affine algebras

- Representation theoretical interpretation of the new Weyl group action ?
- We use **shifted** quantum affine algebras.
- Algebras introduced by Finkelberg-Tsymbaliuk in the study of K-theoretical Coulomb branches (in the sense of Braverman-Finkelberg-Nakajima).
- $\mathcal{U}_q^\mu(\hat{\mathfrak{g}})$: variations of quantum affine algebras depending on a shift parameter : a coweight μ of $\mathfrak{g}$.
- The Coulomb branches are realized as quotient of $\mathcal{U}_q^\mu(\hat{\mathfrak{g}})$ (the truncated shifted quantum affine algebras).
- Rational analogues : shifted Yangians (Brundan-Kleshchev, Kamnitzer-Webster-Weekes-Yacobi, Nakajima-Weekes).

Shifted quantum affine algebras

- Construction : same (Drinfeld) generators as $\mathcal{U}_q(\hat{\mathfrak{g}})$, but certain relations are modified inside the Cartan-Drinfeld subalgebra :

$$\phi_i^-(z) = z^{\alpha_i(\mu)}\phi_{i,\alpha_i(\mu)}^- \exp\left((q^{-1}-q)\sum_{r>0} h_{i,-r}z^{-r}\right).$$

- $\mu = 0$: $\mathcal{U}_q^0(\hat{\mathfrak{g}})$ is (a central extension) of $\mathcal{U}_q(\hat{\mathfrak{g}})$.
- μ anti codominant : $\mathcal{U}_q^\mu(\hat{\mathfrak{g}})$ contains the Borel algebra

$$\mathcal{U}_q(\hat{\mathfrak{b}}) \subset \mathcal{U}_q^\mu(\hat{\mathfrak{g}})$$

  of the ordinary quantum affine algebra $\mathcal{U}_q(\hat{\mathfrak{g}})$.
- There are shift morphisms for μ' anti codominant :

$$\mathcal{U}_q^\mu(\hat{\mathfrak{g}}) \to \mathcal{U}_q^{\mu+\mu'}(\hat{\mathfrak{g}})$$

Representations of shifted quantum affine algebras

- Representations of $\mathcal{U}_q^\mu(\hat{\mathfrak{g}})$ very different that for $\mathcal{U}_q(\hat{\mathfrak{g}})$ in general.

Theorem (H. 2020)

$\mathcal{U}_q^\mu(\hat{\mathfrak{g}})$ has non-zero finite-dimensional representation if and only if μ is codominant.

- For any coweight μ, $\mathcal{U}_q^\mu(\hat{\mathfrak{g}})$ has an abelian category $\mathcal{O}^\mu$ of representations : non necessarily finite-dimensional, but with finite-dimensional weight spaces (and the usual cone condition on the weight of a representation).

Theorem (H. 2020)

The simple representations in $\mathcal{O}^\mu$ are parameterized by n-tuples of rational fractions $(\psi_i(z))_{1\leq i\leq n}$ regular at 0 and so that

$$deg(\psi_i(z)) = \alpha_i(\mu).$$

Representations of shifted quantum affine algebras

- $\mu = 0$: essentially same representations as for $\mathcal{U}_q(\hat{\mathfrak{g}})$. We have $\deg(\psi_i(z)) = 0$ as for Chari-Pressley rational fractions. $\mathcal{O}^0$ contains finite-dimensional fundamental representations.
- $\mu = \omega_i^\vee$, $a \in \mathbb{C}^*$. We have $L_{i,a}^+$ in $\mathcal{O}^{\omega_i^\vee}$ positive prefundamental representations where

$$\psi_j(z) = (1 - za)^{\delta_{i,j}}.$$

  It is of dimension 1 !
- $\mu = -\omega_i^\vee$, $a \in \mathbb{C}^*$: We have $L_{i,a}^-$ in $\mathcal{O}^{-\omega_i^\vee}$ negative prefundamental representations where

$$\psi_j(z) = (1 - za)^{-\delta_{i,j}}.$$

  Infinite dimensional, simple as a $\mathcal{U}_q(\hat{\mathfrak{b}})$-module, extends $\mathcal{U}_q(\hat{\mathfrak{b}})$-representations of H.-Jimbo (related to Baxter Q-operators) : limit of finite-dimensional Kirillov-Reshetikhin modules.

Grothendieck ring

- The sum of Grothendieck groups

$$K_0(\mathcal{O}) = \bigoplus_{\mu} K_0(\mathcal{O}^{\mu})$$

has a ring structure, induced from the fusion product obtained from the Drinfeld coproduct (topological coproduct).

- The structure constants on simple classes are positive.
- We have a notion of q-characters in the shifted context. Injective ring morphism

$$\chi_q : K_0(\mathcal{O}) \to \tilde{\mathcal{Y}}$$

completion of $\mathcal{Y}$.

- For simplicity of notations : in the following parameters $a \in q^{\mathbb{Z}}$.

Interpretation of the Weyl group action

Theorem (Frenkel-H.)

We have (up to one-dimensional invertible representations) :

$$Y_{i,a} = \chi_q(L^+_{i,aq^{-1}})/\chi_q(L^+_{i,aq}).$$

$$\textit{(the first component of)}\ \Theta_i(Y_{i,a}) = \chi_q(L^{s_i(\omega_i^\vee)}_{aq^{-1}})/\chi_q(L^{s_i(\omega_i^\vee)}_{aq}).$$

for a family of simple representations $L^{s_i(\omega_i^\vee=}_{i,a}$ *in* $\mathcal{O}^{s_i(\omega_i^\vee)}$.

- For a general $w \in W$, $i \in I$, we introduce

$$Q_a^{w(\omega_i^\vee)} \in \tilde{\mathcal{Y}}^e$$

so that the first component of $\Theta_w(Y_{i,a})$ is

$$Q_{aq^{-1}}^{w(\omega_i^\vee)}/Q_{aq}^{w(\omega_i^\vee)}.$$

Interpretation of the Weyl group action

- We also introduce a family of simple representations $L_a^{w(\omega_i^\vee)}$ (with explicit parameter) and explicit conjectural q-character formula :

$$\chi_q(L_a^{w(\omega_i^\vee)}) = Q_a^{w(\omega_i^\vee)}$$

- We prove the leading term of $Q_a^{w(\omega_i)}$ can be recovered from Chari braid group action by a change of variables.
- Interpretation of Θ_w associated to $w \in W$: the simple representation $L_a^{\omega_i^\vee}$ is replaced by the simple representation $L_a^{w(\omega_i^\vee)}$.
- Finite-dimensional representations of $\mathcal{U}_q(\hat{\mathfrak{g}})$ are invariant by this substitution.

QQ-system

- Additional properties of the $Q_a^{w(\omega_i^\vee)}$:

Theorem (Frenkel-H. 2023)

The series $Q_a^{w(\omega_i^\vee)}$ satisfy the QQ-system

$$Q_{aq}^{(ws_i)(\omega_i^\vee)} Q_{aq^{-1}}^{w(\omega_i^\vee)} - Q_{aq^{-1}}^{(ws_i)(\omega_i^\vee)} Q_{aq}^{w(\omega_i^\vee)} = \prod_{j | C_{i,j} = -1} Q_a^{w(\omega_j^\vee)}.$$

- Motivations : QQ-system in the context of affine opers (Masoero-Raimundo-Valeri, Mukhin-Varchenko).
- Example ($\mathfrak{g} = sl_2$) : Quantum Wronskian relation :

$$\tilde{Q}_{aq} Q_{aq^{-1}} - \tilde{Q}_{aq^{-1}} Q_{aq} = 1.$$

where $\tilde{Q}_a = Q_a^{-\omega_1^\vee}$, $Q_a = Q_a^{\omega_1^\vee}$.

Cluster algebras (quick reminder)

- Cluster algebra $\mathcal{A}_Q$ attached to a quiver Q.
- Q_0 : set of vertices of Q.
- Start from $\mathcal{F} = \mathbb{Q}(X_i)_{i\in Q_0}$.
- $\mathcal{A}_Q$ is the commutative subalgebra of $\mathcal{F}$ generated by cluster variables.
- Initial cluster variables : X_i $(i \in Q_0)$.
- New cluster variables : obtained from the initial by mutations (exchange relations controlled by the quiver Q).
- The cluster variables are grouped into overlapping subsets : the clusters.
- Cluster monomials : monomials in the cluster variables from the same cluster.

Cluster structure - finite-dimensional representations

- H.-Leclerc : realize Grothendieck rings of representations of quantum groups as cluster algebras ?
- Many developments for the category $\mathcal{C}$ of ordinary $\mathcal{U}_q(\hat{\mathfrak{g}})$: H.-Leclerc, Nakajima, Qin, Kashiwara-Kim-Oh-Park, Bittmann, Brito-Chari...
- Now : $\mathcal{C}^{sh} \subset \mathcal{O}$ category of finite-dimensional representations of shifted quantum affine algebras.

Theorem (H.-Leclerc 2016, Kashiwara-Kim-Oh-Park 2020, H. 2020)

$K_0(\mathcal{C}^{sh})$ is isomorphic to a cluster algebra $\mathcal{A}_{\Gamma_\infty}$ (explicit quiver Γ_∞). Initial cluster variables : classes of positive prefundamental representations. The cluster monomials correspond to certain classes of simple modules.

Cluster structure - finite-dimensional representation

- Example : $\mathfrak{g} = \mathfrak{sl}_2$. Infinite linear quiver :

$$\cdots \longrightarrow \bullet \longrightarrow \bullet \longrightarrow \cdots$$

- Initial seed of 1-dimensional representations

$$\cdots \longrightarrow L^+_{1,q^{-2}} \longrightarrow L^+_{1,1} \longrightarrow L^+_{1,q^2} \longrightarrow \cdots$$

- First step mutations are given by Baxter TQ-relations

$$[L^+_{1,a}][V_1(a)] = [L^+_{1,aq^2}] + [L^+_{1,aq^{-2}}]$$

where V_1 : 2-dimensional fundamental representation.

Cluster structure - category $\mathcal{O}$

- Work in progress (with Geiss and Leclerc) :
- New families of quivers Γ'_∞ attached to each $\mathfrak{g}$.

Theorem (Geiss-H.-Leclerc 2023)

The Grothendieck ring $K_0(\mathcal{O})$ is isomorphic to (a completion of) $\mathcal{A}_{\Gamma'_\infty}$.

- Crucial ingredients :
- There is an initial seed with cluster variables of the form $Q_{i,q^r}^{w(\omega_i^\vee)}$ for various $1 \leq i \leq n$, $w \in W$, $r \in \mathbb{Z}$.
- The QQ-systems [FH] are identified with distinguished exchange relations.
- "Periodicity" property.

Cluster structure - category $\mathcal{O}$

- Example : $\mathfrak{g} = \mathfrak{sl}_2$

$$\cdots \longrightarrow \bullet \longrightarrow \bullet \longleftarrow \bullet \longrightarrow \bullet \longrightarrow \cdots$$

- Initial seed :

$$\cdots \longrightarrow L^+_{1,q^2} \longrightarrow L^+_{1,1} \longleftarrow L^-_{1,q^{-2}} \longrightarrow L^-_{1,q^{-4}} \longrightarrow \cdots$$

- Mutation at $L^+_{1,1}$:

$$\cdots \longrightarrow L^+_{1,q^2} \longleftarrow L^-_{1,1} \longrightarrow L^-_{1,q^{-2}} \longrightarrow L^-_{1,q^{-4}} \longrightarrow \cdots$$

- Mutation at $L^-_{1,q^{-2}}$:

$$\cdots \longrightarrow L^+_{1,q^2} \longrightarrow L^+_{1,1} \longrightarrow L^+_{1,q^{-2}} \longleftarrow L^-_{1,q^{-4}} \longrightarrow \cdots$$

Conjecture

- We conjecture : all cluster monomials in $\mathcal{A}_{\Gamma'_\infty}$ correspond to classes of simple objects in $\mathcal{O}$ through our isomorphism.

Theorem (Geiss-H.-Leclerc 2023)

The conjecture is true for $\mathfrak{g} = sl_2$.

- Complete list of cluster variables for $\mathfrak{g} = sl_2$:

$$L^+_{1,a}\ ,\ L^-_{1,a}\ ,\ W^{(k)}_{1,a}$$

where the $W^{(k)}_{1,a}$ are finite-dimensional Kirillov-Reshetikhin modules.