A Dialogue between the Mathematics and Physics of Discrete Spacetime

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Quantum Gravity: the arc of physics is long but it bends toward unity

Causal Set Theory is an approach to quantum gravity from the Relativity (rather than the Particle Physics) tradition.

Two pillars of the the approach:

- the spacetime causal order from General Relativity
- the path integral from quantum theory

The third pillar is a novel hypothesis:

- fundamental discreteness of spacetime

IMPORTANT

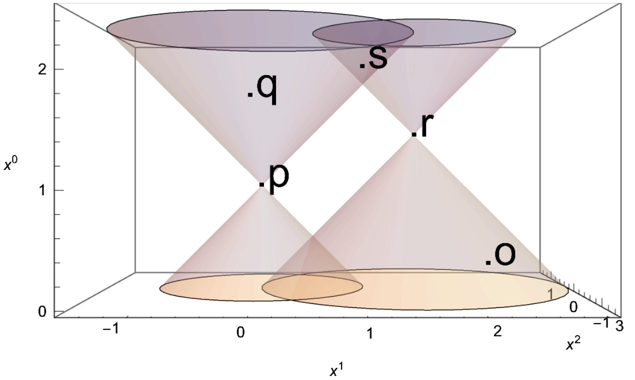
The continuum is recovered as an **approximation** and NOT as a **limit**. The continuum is an approximation to the discrete and not the other way around!

In **physics** we think of kinetic theory of fluids. In **mathematics** .....

Question

What mathematics exists in which a continuum approximates a combinatorial entity?

What is spacetime causal order?



- In General Relativity the geometry of spacetime is **Lorentzian** not Riemannian.
- The so-called spacetime *causal order* is a precedence order: before and after
- $o < r$, $r < s$, $o < s$, $p < q$, $p < s$. No physical order between p and r.
- The spacetime causal order is a **partial order** on spacetime points
- In GR this order relation is central to the physics e.g. the physics of black holes.

Causal order marries discreteness

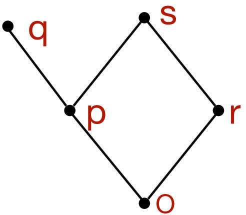


Figure: Hasse diagram. Mathematicians say "*s* is above *o*", "*p* and *q* are incomparable" and "*s* covers *p*

- Pragmatism: defining the path integral
- Philosophy: abhor any physical infinity
- Physics: the finite black hole entropy
- Spacetime is a **discrete order**
  = transitive directed acyclic **graph**
  = causal set

- Physically, elements = spacetime atoms = smallest, indivisible **events**
- Planck scale: $\approx 10^{240}$ spacetime atoms make up the observable universe
- Consequence: The Path Integral for Quantum Gravity is a SUM over causal sets:

$$\text{“}\quad Z = \sum_{M \in \mathcal{M}} \int_{g \in \mathcal{G}} [dg]\, e^{iS[g]} \longrightarrow Z(N) = \sum_{\text{causal sets with N elements}} e^{iS(C)} \quad \text{”}$$

In the sum-over-causal sets there must be some that are well-approximated by Lorentzian manifolds.

Causal sets that are well-approximated by Lorentzian manifolds

The Discrete-Continuum Correspondence for causal sets:

(M, g) approximates causal set $(C, \prec)$ if C **faithfully embeds** in (M, g) *i.e.* there's an embedding $C \to M$ such that (Bombelli *et al.*, 1987)

- Number $\approx$ Volume (in fundamental Planckian units) in any large nice region
- The order $\prec$ of C equals the continuum causal order of the embedded points.
- The geometry (M, g) varies slowly on the Planck scale

The Poisson process of sampling the causal order of (M, g) according to the spacetime volume measure dV provides as many examples of faithfully embeddable causal sets as one could want:

- Each typical outcome of a Poisson process of "sprinkling" into (M, g) at Planckian density is faithfully embedded in (M, g) by definition.
- A continuum (M, g) encodes the structure of **many** different discrete manifolds: cf. a fluid encodes the large scale properties of many different molecular configurations.

Precisely formulate and prove the Hauptvermutung

If $(C, \prec)$ faithfully embeds in both (M, g) and (M', g') then $(M, g) \approx (M', g')$.

Evidence for Hauptvermutung

Let $<$ denote the causal relation for a causal spacetime.

Kronheimer-Penrose-Hawking-Malament (no Riemannian analogue)

Let (M, g), $(\tilde{M}, \tilde{g})$ be distinguishing spacetimes of dimension ≥ 3. Let $\phi : M \to \tilde{M}$ be a causal isomorphism, i.e, a bijection s.t. $x < y \Leftrightarrow \phi(x) < \phi(y)$. Then ϕ is a smooth conformal isometry: ORDER + VOLUME = LORENTZIAN GEOMETRY.

The Hauptvermutung: ORDER + NUMBER $\approx$ LORENTZIAN GEOMETRY

Direct evidence: properties of (M, g) counted out from a Poisson sprinkled C, *e.g.*

Volume measure; Dimension; Timelike geodesics and timelike geodesic distance; Scalar curvature; Homology of spacelike hypersurface; Trace of extrinsic curvature of spacelike hypersurface; co-dimension 2 volume of area of causal horizon

Improve these and find more

Some of these are only known for Minkowski space or approximately flat spacetimes, or in the continuum limit of the mean over sprinklings. For many, we still need to know (i) the finite density corrections to the limiting value and (ii) the variance.

Dynamics of discrete spacetime: Random Graph Models

- A discrete order can **grow** in a stochastic process. (Rideout & Sorkin, 2000).
- The ground set is $\mathbb{N}$, each element n is born in sequence and decides– according to a set of probabilities that defines the model– to be above some subset of the existing set $\{0, 1, 2, \ldots n-1\}$.
- The simplest Sequential Growth models are known in maths as random graph orders (section 3 of (Brightwell & Luczak, 2016)) and in physics as Transitive Percolation.
- Transitive percolation is the Lorentzian analogue of the Erdös-Rényi random graph $G(n, p)$ on n vertices where each pair of vertices is an edge with probability p.
- For any $0 < p < 1$ the Erdös-Rényi random graph, $G(\infty, p)$, is almost surely isomorphic to the Rado graph.
- Transitive Percolation: take the ground set of the random graph $G(n, p)$ to be $\{0, 1, 2 \ldots n-1\}$. For each pair $i < j$, if there is an edge (i, j) in G(n,p), then put j above i and take the transitive closure to form the random order $T(n, p)$.
- Very different to the ER random graph: $T(\infty, p)$ has a nontrivial distribution which depends on the model parameter p.

Classical Sequential Growth = Generalised Percolation

- In Transitive Percolation, each new element n chooses to be above each $i \in \{0, 1, \ldots n-1\}$ independently with probability p.
- Another way to express this: n chooses which subset A of $\{0, 1, \ldots n-1\}$ to be above according to

$$\mathbb{P}(A) = p^{|A|}(1-p)^{n-|A|}$$

  which only depends on $|A|$.
- In a CSG model, there is a set of non-negative parameters $\{t_n\}$, $n = 0, 1, 2, \ldots$ and the relative probability that n chooses subset A of $\{0, 1, \ldots n-1\}$ to be above is $t_{|A|}$.
- Now, Transitive Percolation is a model with "cycles" of expansion and collapse: there are a.s. infinitely many *posts* (Alon *et al.*, 1994).

Theorem (Brightwell, unpublished)

Let s_n be the expected cardinality of the set chosen by n and let $p_n = s_n/n$.
(a) Suppose that, for some $c > \pi^2/3$, and some $n_0 \in \mathbb{N}$, a CSG model satisfies $p_n \geq c/\log n$ for all $n > n_0$. Then there are a.s. infinitely many posts.
(b) Suppose that, for some $c > \pi^2/3$, and some $n_0 \in \mathbb{N}$, a CSG model satisfies $p_n \leq c/\log n$ for all $n \geq n_0$. Then there are a.s. finitely many posts.

Problem: find necessary and sufficient conditions for infinitely many posts

Indications of Causal Set Cosmology: why we're so interested in posts (Martin *et al.*, 2001; Sorkin, 2000)

- Space of CSG models is space of sets $\{t_n\}$, $n = 0, 1, 2, \ldots$. (A projective space since multiplying by a positive constant gives same model.)
- After a post, the post-post dynamics is given by a set of renormalised constants $\{\tilde{t}_n\}$.
- If there are infinitely many posts, then this process has no end.
- Transitive Percolation is a line of fixed points of this "cosmic renormalisation" and there is an extensive "basin of attraction" of these fixed points.
- Holds out the possibility of a class of models in which each epoch between two adjacent posts in the sequence gets closer to TP with a smaller parameter p, so that each epoch grows bigger and lasts longer than the previous: a Self-Tuning universe that could explain why our universe (since the last post/Big Bang) is so big and old.

Problem: find a specific example of such a dynamics and study its approach to TP

To Do List

- What mathematics exists in which a continuum approximates a combinatorial entity?
- Precisely formulate and prove the Hauptvermutung
- Improve existing examples of "counting out topology and geometry" and find more
- Find necessary and sufficient conditions for infinitely many posts
- Find a specific example of a dynamics with infinitely many posts and study its approach to TP
- Construct Quantum Sequential Growth Models .....

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