Lecture 1: Two out of three ain't bad. (And an abundance of curves)

Connections on a bundle E over $\mathbb{R}^4$. (Conformally, S^4)

$$\begin{aligned}\nabla_i =& \partial_i + A_i(x)\\ \nabla : \Gamma(E) \to& \Gamma(E \otimes T^*\mathbb{R}^4)\end{aligned}$$

Curvature

$$\begin{aligned}F_{i,j} = [\nabla_i, \nabla_j] =& \partial_i A_j - \partial_j A_i + [A_i, A_j]\\ F \in& \Gamma(E \otimes \Lambda^2(T^*\mathbb{R}^4))\end{aligned}$$

Hodge operator $* : \Lambda^2(T^*\mathbb{R}^4) \to \Lambda^2(T^*\mathbb{R}^4)$

$$\begin{aligned}< \alpha, \beta > d(vol) =& \alpha \wedge *\beta\\ *^2 =& \mathbb{I}\end{aligned}$$

Split into eigenspaces: Self-dual and anti-self dual parts

$$\begin{aligned}F^+ =& *(F^+)\\ F^- =& - * (F^-)\end{aligned}$$

Energy ($||F||^2 = ||F^+||^2 + ||F^-||^2$)
Topological degree $||F^+||^2 - ||F^-||^2$)
Minimal energy are Self-Dual, or Anti-Self-Dual (ASD)

Interactions with a complex structure (or several)

$$z_1 = x_1 + ix_2, \quad z_2 = x_3 + ix_4$$

2-forms split into bitype:

$$\begin{aligned} 2,0: &\quad dz_i \wedge dz_j \\ 1,1: &\quad dz_i \wedge d\bar{z}_j; \\ 0,2: &\quad d\bar{z}_i \wedge d\bar{z}_j \end{aligned}$$

With this, SD:

- the 2,0 forms,
- the 0,2 forms,
- in the 1,1 forms, the multiples of the Kähler form $\frac{-1}{2i} dz_1 \wedge d\bar{z}_1 + dz_2 \wedge d\bar{z}_2$

ASD:

- the orthogonal complement of the Kähler form in the 1,1 forms

If the connection is ASD, no $0,2$ curvature, and so an integrable $\bar{\partial}$ operator: a **holomorphic bundle**

Flip this around

Take a holomorphic bundle, and choose a hermitian metric. The Chern connection

- Has the $\bar{\partial}$ as its 0,1 part, and gives back the holomorphic structure: no 0,2 curvature,
- Because it is unitary has no 2,0 curvature either.

Two out of three for free. How to get the third:

1) Twistors

$\mathbb{R}^4$ has three complex structures: I, J, K, in fact a whole $\mathbb{P}^1$'s worth: $aI + bJ + cK, a^2 + b^2 + c^2 = 1$

Changing the complex structures rotates the SD forms amongst themselves; the connection is ASD if holomorphic for all of the complex structures.

Twistor space:

Get here holomorphic bundles on $\mathbb{P}^3(\mathbb{C})$, with reality constraints and a few other constraints. You can get the solution for this in fairly explicitly (ADHM construction)

2) The direct approach:

Fix one complex structure, and minimise energy on the metrics, getting that third piece of the curvature to zero. Typically through a heat flow.

Need some form of stability to ensure that the minimising flow converges; here enforced by a trivialisation at infinity.

Does **not** give the actual solution (minimising flow is a black box) but gives moduli. Here, on $\mathbb{R}^4$ (Donaldson):

$$\{\text{Instantons}\} = \{\text{bundles on } \mathbb{P}^2(\mathbb{C}), \text{trivial on a fixed line at infinity})$$

In more generality, for general manifolds

- **Hyperkähler manifolds** have twistor spaces; in 4D get instantons

- **Kahler manifolds** (Kobayashi-Hitchin-Donaldson-Yau-Uhlenbeck-Simpson-...but Narasimhan-Seshadri!) : again get two out of three for free from a holomorphic structure, need stability to flow to ASD

$$\{\text{Stable holomorphic bundles }\} = \{\text{HYM connections}\}$$

Some reductions- non compact cases.

1. Monopoles on R^3: (Hitchin, Donaldson, Murray-H, Jarvis)

- Impose time translation invariance. get ∇_i, ($\mathbb{R}^3$ directions) and φ (time-direction);
- ASD equation becomes $F = *\nabla\varphi$ (in 3d)
- Different boundary conditions: A_i decay, φ tends to a unique orbit in $u(n)$

Fix one complex structure ($z = x_1 + ix_2, w = x_3 + it$); the $0, 2$ component of ASD becomes

$$[\nabla_{\bar{z}}, \nabla_{x_3} + i\varphi] = 0$$

So, solving a scattering equation $(\nabla_{x_3} + i\varphi) \cdot s = 0$ gives a holomorphic bundle on the z plane. Doing this for all of the complex structures gives a complex bundle on the twistor space.

$$T\mathbb{P}^1 \rightarrow \mathbb{P}^1$$

Which bundles?

How to distinguish? The boundary behaviour

$$\varphi \simeq idiag(\alpha_1, \alpha_2, ...\alpha_n)$$

gives a flag of decay rates to

$$(\nabla_{x_3} + i\varphi) \cdot s = 0,$$

as $x_3 \to \infty$ AND as $x_3 \to -\infty$.

These two flags are generically transverse, and fail to be at a set of points in the z-plane.

Varying the directions in $\mathbb{P}^1$ gives $n-1$ curves in $T\mathbb{P}^1$, and these (along with some sections of line bundles on the spectral curve) determine the monopole. All is in principle computable from this.

For moduli.

For moduli, just need one direction, say x_3. The behaviour at $+\infty$ can be thought of as fixing a basis, and the scattering of the flag from $-\infty$ then gives a holomorphic map into the flag manifold

$$\mathbb{P}^1 \to U(n)/T.$$

These maps determine the monopole:

Moduli of based monopoles = Based rational maps into the flag manifold

2. Monopoles on $\Sigma \times S^1$.(Charbonneau-H)

Same equation, but now lose the symmetry of four dimensions (except when Σ of genus one.) The $0,2$ part is

$$[\bar{\nabla}_\Sigma, \nabla_\theta - i\varphi] = 0$$

Compactness implies the need to allow fixed Dirac style singularities at fixed points (Singularities of Abelian type, with integer charges $(m_1, m_2, ..., m_n)$)

Scattering through the singularity (in the S^1 direction) gives

$$g(z) = h_+(z) diag(z^{m_1}, z^{m_2}, ..., z^{m_n}) h_-(z)$$

(Used by Kapustin Witten to model Hecke transform) but going all the way round the circle gives

$$\Phi : E \to E,$$

- E a bundle on Σ
- Φ a meromorphic automorphism with singularities of type $diag(z^{m^i_1}, z^{m^i_2}, ..., z^{m^i_n})$ at fixed points p^i

Moduli of monopoles = Stable pairs (E, Φ)

Stability: No Φ invariant subbundle of big degree.

Spectral curve S: $\det(\Phi - z\mathbb{I}) = 0$
Sheaf L on S: $\mathrm{coker}(\Phi - z\mathbb{I})$

Moduli also a **space of (curves S, sheaves L)**

3. Instantons on $\mathbb{R}\times S^1\times\Sigma$, Σ of genus one. (Charbonneau-H)

A reduction from $\mathbb{R}^4$ (discrete symmetries), but different boundary conditions.

Finite energy, gives asymptotically flat connection on three-tori $S^1 \times \Sigma$ at $\pm\infty$.

Holomorphically. Compactify $\mathbb{R} \times S^1 = \mathbb{C}^*$ to $\mathbb{P}^1$; get a bundle on $\mathbb{P}^1 \times \Sigma$,

How to analyse? On generic $\{z\} \times \Sigma$, E is a sum

$$L_1 \oplus ... \oplus L_n, \quad L_i \in \Sigma^*.$$

Pick-out by a Fourier-Mukai transform

$$\mathbb{P}^1 \times \Sigma \leftarrow \mathbb{P}^1 \times \Sigma \times \Sigma^* \rightarrow \mathbb{P}^1 \times \Sigma^*.$$

$$E \mapsto \pi_1^*(E) \otimes \text{Poincare} \mapsto F = (\pi_2)_*(\pi_1^*(E) \otimes \text{Poincare})$$

F is generically a line bundle over a curve S of bidegree $(n, k = c_2(E))$ in $\mathbb{P}^1 \times \Sigma^*$.

- *Monopoles on* $\Sigma \times S^1$ *for* $(g(\Sigma) = 1)$ and
- *Instantons on* $\mathbb{R} \times S^1 \times \Sigma^*$

.

are *Nahm transforms* of each other.

4. Nahm's equations.

Here reduce $\mathbb{R}^4$ by three translations, get

- Connection $\nabla = \partial_t + T_0(t)$ along a line,
- $T_i(t)$ skew hermitian matrices.

ASD equations become

$$\nabla T_i = \frac{1}{2} \sum \epsilon_{ijk} [T_j, T_k]$$

Rewrite in a way compatible with the twistor paradigm:

$$A(\zeta, s) = T_1 + \mathrm{i}T_2 - 2T_3\zeta - (T_1 - \mathrm{i}T_2)\zeta^2,$$
$$A_+(\zeta, s) = -\mathrm{i}T_3 - \mathrm{i}(T_1 - \mathrm{i}T_2)\zeta,$$

with the Lax equation

$$[\nabla + A_+(\zeta, s), A(\zeta, s)] = 0$$

Invariant is again a spectral curve:

$$\det(A(\zeta, s) - \eta\mathbb{I}) = 0$$

Can solve with the Krichever-Novikov approach. (Ercolani)

Holomorphic data:

ζ is the twistor parameter, set $\zeta = 0$ for one complex structure.

$$[\nabla - \mathrm{i}T_3, T_1 + \mathrm{i}T_2] = 0.$$

In other words $T_1 + \mathrm{i}T_2$ is covariant constant. Not much there...

Answer is in boundary conditions.

For example, for monopoles, get via the Nahm transform a solution on a sequence of intervals, with matrices of different sizes n_μ on the intervals. As a sample of the boundary conditions, for $n_\mu > n_{\mu-1}$, at the common boundary point $s = 0$, from the big side:

$$T_j(s) = \begin{pmatrix} a_j(s) & b_j(s) \\ c_j(s) & -\frac{\rho_j}{2s} + d_j(s) \end{pmatrix}.$$

Here, the top left block is $n_{\mu-1} \times n_{\mu-1}$, the bottom right block is $m \times m$, with $m = n_\mu - n_{\mu-1}$; we ask that a_j, b_j, c_j, d_j be analytic at $s = 0$, and $\{\rho_j\}_{j=1}^3$ be the components of the m-dimensional irreducible representation of $su(2)$ in its standard basis, that is the standard representation in m dimensions of the Pauli matrices. (Additional vanishing for b_j, c_j at $s = 0$)

Furthermore, the solutions on the two intervals should match by

$$a_j(\lambda_i) = T_j^{small}(\lambda_i).$$

Different boundary conditions for nilpotent orbits, instantons on a Taub-NUT, etc.

Why bother?

Holomorphic data usually easy to describe; on the other hand the hyperKähler reduction gives you a hyperKähler metric.

5. Integrable Hierarchies and ASD

A whole zoo of integrable pde and ode are reductions of ASD.:

(Mason, Ward, Woodhouse, Ablowitz, Clarkson) see 2003 paper of Ablowitz and Clarkson in J. Math Phys.

- NLS
- KdV
- Sine Gordon
- N-Wave
- Toda
- Tops
- Painlevé
- ...

Lecture 2. Higgs bundles

Now reduce SDYM by two translations, get equation on a plane; instead, put them on a genus $g > 2$ compact Riemann surface X. Fix a degree d and a rank n, and repackage:

1) ∇_A, unitary connection and 2) $\varphi \in End(E) \otimes \Omega^{1,0}(X)$

$$\bar{\partial}_A \varphi = 0$$
$$F_A = -[\varphi, \varphi^*]$$

or again,

$$[\lambda \partial_A + \phi, \bar{\partial}_A + \lambda \varphi^*] = 0, \forall \lambda$$

Again, implicitly, a whole sphere of complex structures. Now look at the Higgs moduli in the I-complex structure ($\lambda = 0$).

$$\mathcal{M} = \{(E, \varphi) | E \text{ a rank } n \text{ holomorphic bundle}, \varphi \in H^0(X, End(E) \otimes K_X\}$$

(Semi-)stability: Slope condition: for any φ-invariant sub-bundle E'

$$\frac{deg(E')}{rank(E')} < (\leqslant) \frac{deg(E)}{rank(E)}$$

Guaranteed by stability of E.

Symplectically

Holomorphic symplectic structure from hyperKähler; but more simply:

If $\mathcal{N}$ = moduli of stable bundles, then a large piece of $\mathcal{M}$ is

$$T^*(\mathcal{N})$$

Deformations of bundles:

$$T_E(\mathcal{N}) = H^1(X, End(E))$$

(deform transition functions as $T_{\alpha,\beta}(\epsilon) = T_{\alpha,\beta}(0)(1 + \epsilon t_{\alpha,\beta})$

Dually:

$$T_E^*(\mathcal{N}) = H^0(X, End(E) \otimes K_X),$$

where φ lives, making for

$$T^*\mathcal{N} \subset \mathcal{M}$$

Alternately, from deformation theory, the symplectic form:

$$H^0(X, End(E) \otimes K_X) \to T\mathcal{M} \to H^1(X, End(E))$$

Dually, get the same sequence:

$$H^0(X, End(E) \otimes K_X) \to T^*\mathcal{M} \to H^1(X, End(E))$$

The identity map between the two defines a symplectic form. Alternately, it can be defined in terms of a symplectic reduction of an infinite dimensional sum of two spaces dual to each other, one of $\bar{\partial}$- operators, and the other of (smooth) Higgs fields.

An integrable system.

We have

$$H^i = tr(\varphi^i) \in H^0(X, K_X^i)$$

fixing a basis of the d_i-dimensional space $H^0(X, K_X^i)$ and taking components $H^{i,j}, i = 1, .., n, j = 1, ...d_i$ gives a family of Hamiltonians.

Theorem. *(Hitchin) These Hamiltonians Poisson commute, and give a completely integrable system on $\mathcal{M}$*

Spectral curves.

Combine the Hamiltonians into one object, the spectral curve Σ, living in the total space $\pi : \mathcal{K} \to X$ of the canonical bundle and cut out by

$$\det(\varphi - z\mathbb{I}) = 0$$

where z is the tautological section over $\mathcal{K}$ of the lift of the canonical bundle. The spectral curve combines all the invariants of the Hamiltonian flows.

There is then an exact sequence of sheaves

$$0 \to \pi^*(E \otimes K_X^{-1}) \xrightarrow{\varphi - z\mathbb{I}} \pi^* E \to \mathcal{L} \to 0$$

The quotient sheaf $\mathcal{L}$ is supported on the spectral curve, and is in the generic situation (smooth curve, multiplicity one) a line bundle over X.

Parametrisation by spectral curves and sheaves

$$\mathcal{M} = \{(E, \varphi)\} \simeq \{(\Sigma, L)\}$$

The inverse map is $E = \pi_*(L), \varphi = \pi_*(\times z)$. Note automatic stability if Σ irreducible.

Thus the general picture is of a family of Picard varieties $Pic^k(\Sigma)$ over a vector space space $\oplus_{i=1}^n H^0(X, K_X^i)$.

Hitchin: the fibres are compact (The whole Picard variety in a fixed curve gets realized)

Abelianization of the symplectic form

One has variations of the curve given by $H^0(\Sigma, N_\Sigma) = H^0(\Sigma, K_\Sigma)$, and variation of line bundles by $H^1(\Sigma, \mathcal{O})$.

These are Serre dual, and indeed, choosing a base line bundle on a neighbourhood of the curve

Theorem. *(Abelianisation) The symplectic structure on $\mathcal{M}$ is given by the natural symplectic pairing*

$$H^0(\Sigma, K_\Sigma) \oplus H^1(\Sigma, \mathcal{O}) \to \mathbb{C}$$

Then:

- The differential of a Hitchin hamiltonian gives a section of the conormal bundle of the spectral curve.
- Since the canonical bundle of the surface $\mathcal{K}_X$ is trivial, this is dual to a one-form on the spectral curve
- By the symplectic pairing, this gets identified with an element of $H^1(\Sigma, \mathcal{O})$, given by a cocycle $w_{\alpha,\beta}$.
- The flow of line bundles by a flow of transition functions $exp(tw_{\alpha,\beta})$.

Generalise (allow poles)

Let D be a positive divisor. Now X can be of arbitrary genus.

$\mathcal{M}_D = \{(E, \varphi) | E \text{ a rank n bundle}, \varphi \in \mathrm{H}^0(\mathrm{X}, \mathrm{End}(\mathrm{E}) \otimes \mathrm{K}_\mathrm{X}(\mathrm{D})\}$

Theorem. *(Biswas, Bottacin, Markman) This is a Poisson manifold. Symplectic leaves: fix the adjoint orbit of φ over D.*

Have again a spectral curve Σ.

$$\det(\varphi - z\mathbb{I}) = 0$$

sitting in $\mathcal{K}_X(D)$, and a sheaf L:

$$0 \to \pi^*(E \otimes K_X(D)^{-1}) \xrightarrow{\varphi - z\mathbb{I}} \pi^* E \to \mathcal{L} \to 0$$

Casimirs: Intersections of D with Σ.

Again,

$$\mathcal{M}_D = \{(E, \varphi)\} = \{(\Sigma, L)\}$$

A special case: $X = \mathbb{P}^1$, E degree 0.

Restrict to the generic locus of trivial E. Let $D = \sum_i^k \alpha_i$, for simplicity α_i distinct, and $\neq \infty$

$$\mathcal{M}_D^0 = \{A(\lambda) = \sum_{i=1}^n \frac{A_i}{\lambda - \alpha_i}, \sum_i A_i = 0\}/Gl(n, \mathbb{C})$$

The A_i are matrices. The vanishing at infinity is imposed by the canonical bundle being $\mathcal{O}(-2)$. The symplectic leaves correspond to fixing the (co-)adjoint orbits of the A_i.

Theorem. *The symplectic leaves of $\mathcal{M}_D^0$ are the reductions at zero of the product of coadjoint orbits of the A_i by the simultaneous action of $Gl(n, \mathbb{C})$.*

Another way of looking at this:

- $\widetilde{\mathfrak{g}}$ = polynomial loop algebra of $gl(n)$;
- Via $< a, b >= res_\infty Tr(ab)$, the dual $\widetilde{\mathfrak{g}}^*$ is the negative Laurent series,
- (reduced) finite dimensional coadjoint orbits are basically $\mathcal{M}_D^0$ (varying D, and conjugacy classes)
- The Adler-Kostant-Symes theorem gives for the Hamiltonian flows of

$$f_{i,j} = res_\infty Tr(A(\lambda)^i \lambda^j)$$

the Lax equation

$$\dot{A}(\lambda) = [(iA(\lambda)^{i-1}\lambda^j)_+, A(\lambda)]$$

Metatheorem *Any classical algebraically integrable system that anyone is interested in somehow fits into this picture, or its variants for other Lie algebras. Also finite gap solutions of integrable pde.* False, of course.

A classical example: the Neumann oscillator.

This oscillator appeared in the work of C. Neumann in 1859. It treats the motion of a particle on the $(n-1)$-dimensional sphere under the influence of a quadratic potential. It gets realized as a (constrained) flow on $\mathbb{R}^{2n}$. If x_i, y_i are coordinates, set

$$\mathcal{N}(\lambda) = \frac{\lambda^{-1}}{2} \begin{pmatrix} -\sum_{i=1}^n \frac{x_i y_i}{\lambda-\alpha_i} & -1-\sum_{i=1}^n \frac{y_i^2}{\lambda-\alpha_i} \\ \sum_{i=1}^n \frac{x_i^2}{\lambda-\alpha_i)} & \sum_{i=1}^n \frac{x_i y_i}{\lambda-\alpha_i} \end{pmatrix}.$$

The Hamiltonian is

$$\begin{aligned} H_0(\boldsymbol{x}, \boldsymbol{y}) &= \frac{1}{2}\left[(\sum_i x_i^2)(\sum_i y_i^2) + \sum_i \alpha_i x_i^2 - (\sum x_i y_i)^2\right] \\ &= Res_\infty \lambda^2 \mathcal{N}^2(\lambda) \end{aligned}$$

The equations of motion for our Hamiltonian system are equivalent to

$$\frac{d\mathcal{N}}{dt} = [\mathcal{B}, \mathcal{N}], \tag{1}$$

where

$$\mathcal{B} = \begin{pmatrix} \sum_{i=1}^n x_i y_i & \lambda - \sum_{i=1}^n y_i^2 \\ -\sum_{i=1}^n x_i^2 & -\sum_{i=1}^n x_i y_i \end{pmatrix}. \tag{2}$$

How to solve? Darboux coordinates.

Invariantly, $\mathcal{M}, \mathcal{M}_D$ is a space of pairs (Σ, L) of curves and (generically) a line bundle on the curve.

Twist in some uniform way so that L has generically a single section; this occurs when $deg(L) = g(\Sigma)$. Then

$$L \leftrightarrow \text{ divisor } \sum_\mu p_\mu, \quad p_\mu \in \Sigma.$$

Now erase the curve Σ, so that $\sum_\mu p_\mu \in Hilb^g(\mathcal{K}), Hilb^g(\mathcal{K}_D)$.

Theorem. *(Adams, Harnad, H., ... but see Garnier (1919)!) The p_μ provide Darboux coordinates on $\mathcal{M}, \mathcal{M}_D$; that is local "birational" symplectomorphisms*

$$\mathcal{M} \simeq Hilb^g(\mathcal{K}), \qquad \mathcal{M}_D \simeq Hilb^g(\mathcal{K}_D)$$

A nineteenth century integration

Fix a coordinate λ on X, and so cotangent coordinate ζ; the symplectic form on $\mathcal{K}$ is then $d\zeta \wedge d\lambda$. Let $P_1, ..., P_g$ be the Hitchin Hamiltonians; then have

$$\zeta = \zeta(\lambda, P_1, ..., P_g)$$

since λ determines ζ along the curve. Define a generating function

$$G(\lambda_\mu, P_j) = \sum_{\mu=1}^{g} \int_{\lambda_0}^{\lambda_\mu} \zeta(\lambda, P_1, \ldots, P_g) d\lambda.$$

One has $\frac{\partial G}{\partial \lambda_\mu} = \zeta_\mu$; then (standard trick; $d^2 G = 0$) the coordinates $\frac{\partial G}{\partial P_i}$ linearize the flow. But then

$$\frac{\partial G}{\partial P_i} = \sum_{\mu=1}^{g} \int_{\lambda_0}^{\lambda_\mu} \frac{\partial}{\partial P_i} (\zeta(\lambda, P_1, \ldots, P_g)) d\lambda. \qquad (3)$$

are Abelian integrals; if these flow linearly, the line bundle is too.

Explicitly, on $\mathbb{P}^1$

The divisors are computable: in $\mathbb{P}^1$ case, the section of L is an image of a section of E, the trivial bundle; vanishing computable in terms of cofactors.

For our Neumann example: get ellipsoidal coordinates $\lambda_\mu, \mu = 1, \ldots, n-1$ by

$$\sum_{i=1}^{n} \frac{x_i^2}{\lambda - \alpha_i} = \frac{\prod_{\mu=1}^{n-1}(\lambda - \lambda_\mu)}{a(\lambda)}.$$

and

$$\zeta_\mu = \frac{1}{2}\sum_{i=1}^{n} \frac{x_i y_i}{\lambda_\mu - \alpha_i} = \sqrt{\frac{\mathcal{P}(\lambda_\mu)}{a(\lambda_\mu)}}.$$

We have that $\partial S/\partial\lambda_\mu = \zeta_\mu$; the canonically conjugate coordinates to the P_i, undergoing linear flow, are then:

$$Q_j := \frac{\partial S}{\partial P_j} = \frac{1}{2}\sum_{\mu=1}^{n-1} \int_0^{\lambda_\mu} \frac{\lambda^j d\lambda}{\sqrt{\mathcal{P}(\lambda)}} = b_j t, \qquad j = 0, \ldots, n-2,$$

The integrands form a basis for the holomorphic differentials on a hyperelliptic curve $\mathcal{S}$ cut out by the equation

$$z^2 - \mathcal{P}(\lambda) = 0.$$

Variants-

A cousin- the Sklyanin system (elliptic case)

In its "easier" version, requires a rigid bundle. Here, on an elliptic curve X, take the (unique) stable bundle E of degree 1 (fixing the determinant)Then $H^1(X, sl(E)) = 0$, and so, covering X by $U_+ =$ disk centred at p, $U_- = X - p$, a splitting

$$H^0(U_+ \cap U_-, sl(E)) = H^0(U_+, sl(E) \oplus H^0(U_-, sl(E))$$

$$L\mathfrak{g} = L\mathfrak{g}_+ \oplus L\mathfrak{g}_-$$

with projections P_+, P_-, and a difference $R = P_+ - P_-$. On the loop group, there are two derivatives $D, D' \in L\mathfrak{g}$, defined at g by left and right translation to the origin:

$$< Df, h >= \frac{d}{dt} f(exp(th) \cdot g)|_{t=0}, < D'f, h >= \frac{d}{dt} f(g \cdot exp(th))|_{t=0}$$

and the Sklyanin bracket:

$$\{f, g\} = \frac{1}{2}(< R(Df), Dg > - < R(D'f), D'g >$$

This gives a bracket on the space of $\varphi \in \Gamma(Aut(E))$; the space of sections we allow will be meromorphic ones. They will have spectral curves S given by $\det(\varphi(\lambda) - \zeta\ I) = 0$, and a sheaf L given as the cokernel of φ

Theorem. *1) The symplectic leaves for the Sklyanin bracket are given by fixing the intersection of the spectral curves with $z = 0, z = \infty$.*

2) Taking a suitable twist, so that the divisor (λ_μ, ζ_μ) *representing* L *has degree equal to the genus of* S*, the symplectic form is given by* $\sum_\mu d\lambda_\mu \wedge \frac{d\zeta_\mu}{\zeta_\mu}$*.*

So a Hilbert scheme of a surface again.

Classification of Poisson surfaces.

These are fairly restricted.

Theorem. *(Bartocci, Macri) The complex Poisson surfaces P are given by*

1) Abelian surfaces, and K3 surfaces.

2) (g arbitrary) Ruled surfaces, of the form $\mathbb{P}(\mathcal{O} \oplus K_X(D))$. The Poisson tensor has divisor $2 \cdot \mathbb{E} + D'$, with D' positive, equivalent to D, and E

3) ($g = 1, 0$) Ruled surfaces, of the form $\mathbb{P}(\mathcal{O} \oplus L)$, $deg(L) = 0$. For $g = 1$, the Poisson tensor has divisor $E + E'$ with E, E' representing the divisor of the inclusion of $\mathcal{O}, L$ into $\mathbb{P}(\mathcal{O} \oplus L)$. For $g = 0$, the Poisson tensor has divisor $E + E' + D$, with now D a sum of two fibres of the projection to the base.

4) ($g > 1$), Ruled surfaces $\mathbb{P}(V)$, where $V = J^1(L) \otimes L^*$ is the non-trivial extension

$$0 \longrightarrow K_X \longrightarrow V \longrightarrow \mathcal{O} \longrightarrow 0\,.$$

Which system corresponds to $Hilb(M)$?

1) themselves?

2) Hitchin systems

3) Sklyanin

4) (Biswas, H) Ruled surfaces $\mathbb{P}(V)$, where $V = J^1(L) \otimes L^*$ is the non-trivial extension

$$0 \longrightarrow K_X \longrightarrow V \longrightarrow \mathcal{O} \longrightarrow 0\,.$$

Get a moduli space of by deforming the Hitchin system,

$$\mathcal{M}^C = \{(E, \psi) | E \text{a bundle}, \psi \in \mathrm{H}^0(\mathrm{X}, \mathrm{End}(\mathrm{E}) \otimes \mathrm{V}), \pi(\psi) = \mathbb{I}\}$$

(ψ is an $End(E)$-valued connection on L.) Get spectral curves, integrable system, divisor coordinates in $\mathbb{P}(V)$

Multi-Hamiltonian systems. Trick for integrating a Hamiltonian system if there is a large compatible family of Poisson structures. A flurry of papers on these; all based on the systems being secretly $Hilb(M)$, and the fact that any two Poisson brackets on a surface are automatically compatible.

Lecture 3. Integrable systems of Abelian varieties

Hitchin system

$\mathcal{M} = \{(E,\varphi) | E$ a rank n holomorphic bundle, $\varphi \in H^0(X, End(E) \otimes K_X)\}$

- The spectral curve Σ, living in the total space $\pi : \mathcal{K} \to X$ of the canonical bundle and cut out by

$$\det(\varphi - z\mathbb{I}) = 0$$

- The quotient sheaf $\mathcal{L}$ is supported on the spectral curve, and is in the generic situation (smooth curve, multiplicity one) a line bundle over Σ.

$$0 \to \pi^*(E \otimes K_X^{-1}) \xrightarrow{\varphi - z\mathbb{I}} \pi^* E \to \mathcal{L} \to 0$$

Abelianization:

$$\mathcal{M} = \{(E,\varphi)\} \simeq \{(\Sigma, L)\}$$

Represent the line bundle by a divisor on $\Sigma \subset \mathcal{K}$; with this:

Theorem. *Symplectically,*

$$\mathcal{M} \simeq Hilb^g(\mathcal{K})$$

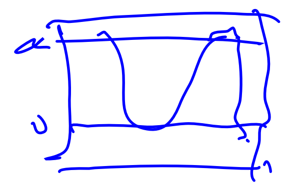
Sklyanin system (Sklyanin, H., Markman)

Base curve X of genus $1, 0$

Replace $End(E) \otimes K$ by $Aut(E)$, again get spectral curves in $\mathbb{P}^1 \times X$, line bundles,

$$\mathcal{N} = \{(E, \varphi)\} \simeq \{(\Sigma, L)\}$$

Again representing L by a divisor on $\Sigma \subset \mathbb{P}^1 \times X$



Theorem. *Symplectically,*

$$\mathcal{N} \simeq Hilb^g(\mathbb{P}^1 \times X)$$

Deformed Hitchin system(Biswas, H.)

Have rank 2 bundle V with $V \to \mathcal{O} \to 0$, $V = J^1(W) \otimes W^*$, W a l.b.

Replace $End(E) \otimes K$ by $\pi^{-1}(\mathbb{I}) \subset End(E) \otimes V$, again get spectral curves in $\mathbb{P}(V)$, line bundles L,

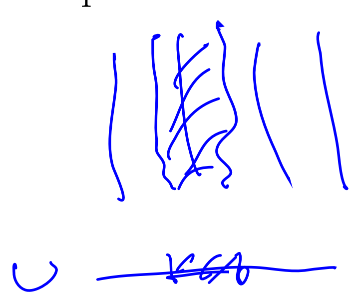
$$\mathcal{M}' = \{(E, \varphi)\} \simeq \{(\Sigma, L)\}$$

Again representing L by a divisor, get an element of $Hilb^g(\mathbb{P}(V))$

Theorem. *Symplectically*

$$\mathcal{M}' \simeq Hilb^g(\mathbb{P}(V))$$

All of these isomorphisms "semi-local", in the neighbourhood of a fixed Jacobian (almost the symmetric product of the curve)



Integrable systems of Jacobians- a local picture.

$$\mathcal{H} : \mathbb{J} \to U.$$

Here $U = U^g$ is a ball in $\mathbb{C}^g$, and $\mathbb{J} = \mathbb{J}^{2g}$ is $2g$-dimensional, symplectic (with form Ω). The fibration $\mathcal{H}$ is assumed to be Lagrangian, with fibers that are Jacobians of smooth genus g curves. Corresponding to $\mathbb{J}^{2g}$ there is a family of curves $\mathbb{S} = \mathbb{S}^{g+1}$, with

$$\mathcal{H}' : \mathbb{S} \to U. \tag{1.9}$$

When is $\mathbb{J}$ symplectic?

$\mathbb{J}$ is $\mathbb{C}^g \times U$, quotiented by A-periods $A_{i,j}(u), B_{i,j}(u)$. Get action-angle coordinates: t_i on the fibers, and H_i on U, with

$$\Omega = \sum_i dt_i \wedge dH_i$$

Normalise the A periods to $\mathbb{I}$, for the B periods, the invariance of the symplectic form under translation gives (Donagi-Markman)

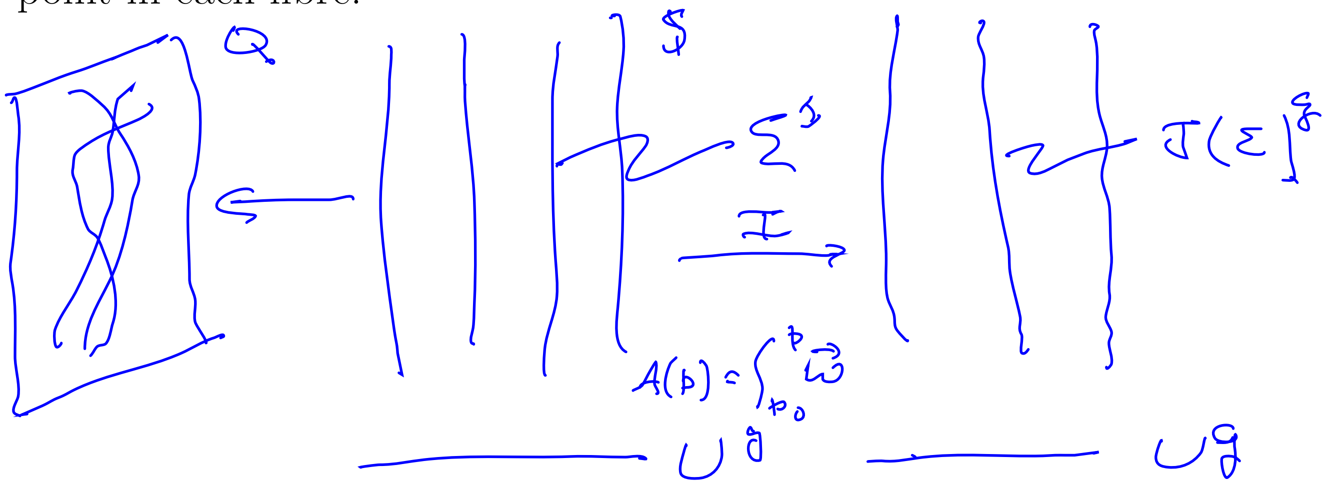
$$\boxed{B_{ij}(u) = \frac{\partial^2 F(u)}{\partial H_i \partial H_j}}$$

for a suitable F.

The Abel map gives us an embedding

$$I : \mathbb{S} \hookrightarrow \mathbb{J} \, .$$

This map is not unique, but depends on the choice of a base-point in each fibre.



Darboux $I^*\Omega = du \wedge dv$.

Theorem. *Let* $I^*\Omega \wedge I^*\Omega = 0$ *("rank 2 condition")*

(i) Under the embedding I*, the variety* $\mathbb{S}$ *is coisotropic. Quotienting by the null foliation, one obtains, restricting* U *if necessary, a surface* Q *to which the form* $I^*\Omega$ *projects, defining a symplectic form* ω *on* Q*. The curves* S_h *all embed in* Q*.*

(ii) If I*,* $\tilde{I}$ *are two Abel maps with* $I^*\Omega \wedge I^*\Omega = 0$*,* $\tilde{I}^*\Omega \wedge \tilde{I}^*\Omega = 0$*, then* $I^*\Omega = \tilde{I}^*\Omega$*, when* $g \geqslant 3$*, and so* Q *depends only on* $\mathbb{S}$ *and not on the particular Abel map chosen. For* $g = 2$*,* $I^*\Omega \wedge I^*\Omega = 0$ *always.*

(iii) There is a symplectic isomorphism

$$\Phi : Hilb^g(Q, \omega) \rightarrow \mathbb{J},$$

defined over a Zariski open set, between $\mathbb{J}$ *and* $Hilb^g(Q, \omega)$ *The symmetric product* $SP^g(S_h) = Hilb^g(S_u)$ *of the curves is Lagrangian in* $Hilb^g(Q, \omega)$*, and the restriction of* Φ *to* $SP^g(S_U)$ *is the Abel map*

$$SP^g(S_u) \rightarrow J_u \; .$$

Can turn this around

Given a symplectic surface Q, and a smooth curve S in Q.

- Sections of the normal bundle are differential forms on the curve.
- Deformations give a family of curves $\mathcal{S}$ over U, and so a fiberwise symmetric product $\mathcal{S}^g$ over U,
- and then to the Jacobian.

The surface is an invariant of the system

Not quite complete- note shift of degrees. Gerbe structure. (Donagi, Gaitsgory, Pantev)

What about other groups?

The Hitchin integrable system exists for arbitrary reductive groups.

- P_G a principal G-bundle
- φ a section of $ad(P_G) \otimes K_X$

Moduli space $\mathcal{M}_G(X)$*.*

Commuting Hamiltonians: for F an invariant function of degree i on $\mathfrak{g}$,

$$F(\varphi) \in H^0(X, K_X^i)$$

Taking components, get an integrable system on $\mathcal{M}_G(X)$.

What is the geometry of this? .

Back to $Gl(n)$; instead of individual eigenvalues of M diagonalisable, think of all ordered sequences $(\lambda_{i_1}, ...\lambda_{i_n})$ of eigenvalues, in other words all possible diagonalisations. This gives in general $n!$ points in $\mathbb{C}^n$, invariant under the action of the symmetric group.

For general G, generic φ, think at a generic $x \in X$ of all possible conjugations of φ to $\mathfrak{h} \otimes K_X$, with $\mathfrak{h}$ the Cartan subalgebra; at non-generic X, to a Borel subalgebra containing $\mathfrak{h}$; projecting to the Cartan, get a *cameral* curve $\Sigma \in \mathfrak{h} \otimes K_X$. It is Weyl invariant.

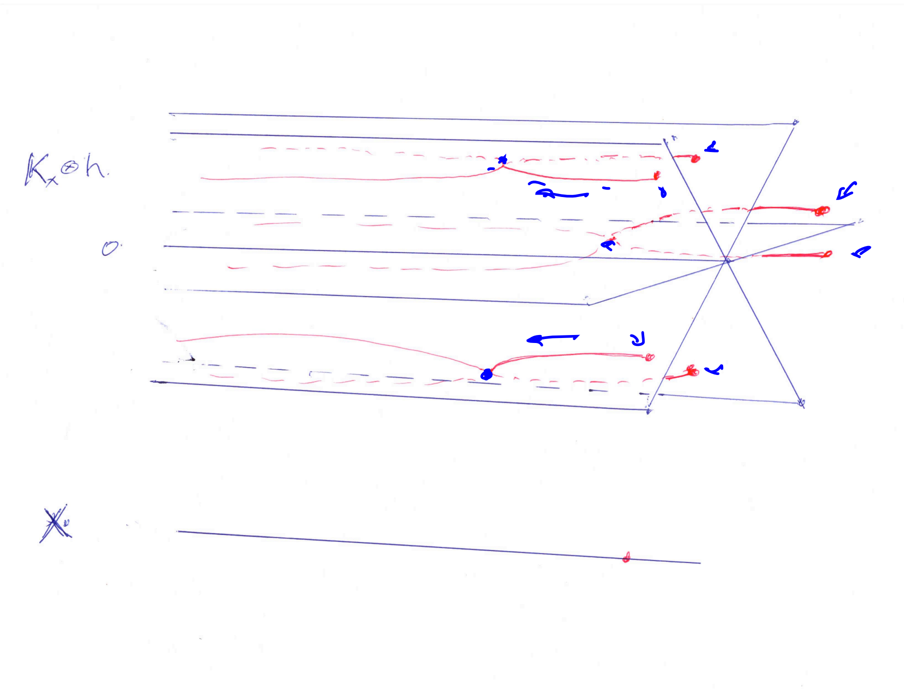


Figure 1: The cameral curve.

Analogue of the line bundle?

- The bundle lifted to the spectral curve comes with a natural reduction to the Borel subgroup.
- Projecting to the Cartan subgroup H, get an H bundle (think transition functions).
- It is not W-invariant, but a twist (computable in terms of roots and the branch points of the curve (Donagi, Scognamillo), is), and the result can be normalised to degree zero. Call the result P_H.

Note the action of W intertwines the action on H and the action on the curves.

H bundles: $J_\Sigma \otimes \chi = J^r$, where J_Σ is the Jacobian of X, and χ the lattice $exp^{-1}(1)$ in $\mathfrak{h}$

Invariant H-bundles. $(J_\Sigma \otimes \chi)^W$ = Prym

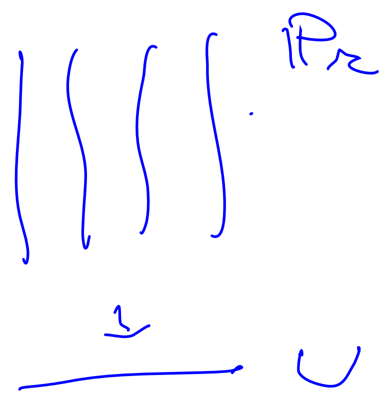
So: Abelianization (For smooth spectral curves)

$$\mathcal{M}_G(X) \simeq \{(\Sigma, P_H)\}$$

with Σ a W-invariant curve in $\mathcal{K}_X \otimes \mathfrak{h}$, $P_H \in (J_\Sigma \otimes_Z \chi)^W$.

The variety of $(J_\Sigma \otimes_Z \chi)^W$ is called a (generalized) Prym variety.

Geometry then, a fibration of Prym varieties, over a family U of W-invariant curves, parametrised by the Hitchin hamiltonians



Hitchin Hamiltonians give linear flows on $(J_\Sigma \otimes_{\mathbb{Z}} \chi)^W$.

So... any Darboux coordinates? (Markman, H.)

Deforming the H-bundle: $H^1(\Sigma, \mathcal{O} \otimes \mathfrak{h})^W$

Deforming the curve sitting in $K_X \otimes \mathfrak{h}$: $H^0(\Sigma, N_\Sigma)^W = H^0(\Sigma, K_\Sigma \otimes \mathfrak{h})^W$

Proposition 1. *(Abelianization) With an appropriate splitting, the symplectic form is the natural one on*

$$H^0(\Sigma, N_\Sigma)^W \oplus H^1(\Sigma, \mathcal{O} \otimes \mathfrak{h})^W$$

Where next?

There is an equivalence:

- The duality pairing W-invariant 2-form on

$$H^0(\Sigma, N_\Sigma)^W \oplus H^1(\Sigma, \mathcal{O} \otimes \mathfrak{h})^W$$

- The duality pairing W-invariant 2-form on

$$H^0(\Sigma, N_\Sigma)^W \oplus H^1(\Sigma, \mathcal{O} \otimes \mathfrak{h})$$

- The duality pairing W-invariant $\mathfrak{h}^*$-valued 2-forms on

$$H^0(\Sigma, N_\Sigma)^W \oplus H^1(\Sigma, \mathcal{O})$$

To get the second from the first, extend by zero (Schur's lemma). The second is basically a linear algebra equivalence.

In our case, our symplectic form on the family of Pryms on U becomes a $\mathfrak{h}^*$-valued symplectic form on the corresponding family of Jacobians.

Now pull back this form to the family of curves, embedded in $K_\Sigma \otimes \mathfrak{h}^*$, via the Abel map. Note that the bundle $K_\Sigma \otimes \mathfrak{h}$ has a natural W invariant $\mathfrak{h}^*$-valued 2-form Ω.

Proposition 2. *(Darboux coordinates) The pulled back form under the Abel map is Ω.*

Can allow poles in φ at a divisor D.

Essentially, same theorems.

- Have a Poisson moduli space $\mathcal{M}_G(X, D)$. Symplectic leaves fix conjugacy class over D. Have a log-symplectic form Ω_D on the bundle $K_\Sigma(D) \otimes \mathfrak{h}^*$.

- Hitchin Hamiltonians given by the invariant polynomials; Casimirs are values at D.

- $K_\Sigma \otimes \mathfrak{h}^*$ has a natural W-invariant $\mathfrak{h}^*$-valued log-symplectic form Ω, with poles at D.

- Abel map gives Darboux coordinates.

G-Sklyanin systems (Markman, H.)

On an elliptic curve X (and, degenerating, on rational curves), can define a Sklyanin system for arbitrary G.

(Etinghof- dynamical R-matrix. Geometrically- generic bundles reduce to the torus.)

Spaces of pairs

(Principal bundle P_G, meromorphic section of $Aut(P_G)$)

Recall that the $Sl(n)$ moduli space had symplectic leaves obtained by fixing the locus and the type of the singularities. At $x = 0$:

$$g(x)diag(x^{m_1}, x^{m_2}, ..., x^{m_n})h(x)$$

The general group case is given in the same way, fixing the locus of the singularities, and at each point of these, a co-character. Let $\mathbb{O}$ denote this data, analogous to a divisor.

- Have a symplectic moduli space $\mathcal{N}_G(X, \mathbb{O})$.
- Hitchin Hamiltonians given by the invariant polynomials; Casimirs are the locations of the singularities.
- A W-invariant spectral curve living in a $(\mathbb{P}^1)^r$-bundle V over X, compactifying $X \times \mathfrak{h}^*$.
- Build on V a natural W-invariant $\mathfrak{h}^*$-valued log-symplectic form Ω, with poles along $0, \infty$
- Pull-back via via Abel map gives Ω.

Local set-up

Suppose:

- A family U of W-invariant curves Σ_u
- An integrable system of Prym varieties $\mathbb{P}r = (\mathbb{J} \otimes_{\mathbb{Z}} \chi)^W$ over U, with a symplectic form ω.

Remark: W does not have to be a Weyl group. Works for "classical Pryms" $W = \mathbb{Z}/2$, χ = sign representation, curves with involution.

Set $V = \mathbb{C} \otimes_{\mathbb{Z}} \chi$ and $r = \dim(V)$. On the associated family $\mathbb{J}$ of Jacobians, the associated V^*-form $\omega_{\mathbb{J}}$. Embed the family of curves $\mathbb{S}$ into $\mathbb{J}$ via the Abel map A^*.

Definition 1. The system is of rank two iff $A^*(\omega_V) \wedge A^*(\omega_V) = 0$ (i.e. the components $A^*(\omega_V)_i \wedge A^*(\omega_V)_j = 0$)

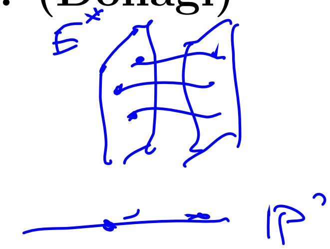
Theorem. *Under reasonable genericity assumptions:*

1) There is an $r+1$ dimensional W-invariant variety X into which the curves S_u embed. It has a natural V^* valued 2 form Ω_V, invariant under W.

2) For $r > 2$, there is an invariant codim 1 foliation of X such that the quotient by the foliation and by W is a curve Σ.

3) The curves S_u thus have a constant quotient $\Sigma = S_u/W$.

A $K3$ example- bundles on an elliptic $K3$. (Donagi)



K3 surface $Y \to \mathbb{P}^1$, elliptic fibers.

$\mathcal{M}_G = G-$ bundles on Y, degree zero.

On each elliptic curve E_z, reduce generically to the torus H, and give a W-orbit in $E^* \otimes_{\mathbb{Z}} \chi$.

Varying z, get a curve in the fiber product over $\mathbb{P}^1$ which replaces E_z with $E_z^* \otimes_{\mathbb{Z}} \chi$.

One gets, in addition an element of the Prym over the spectral curve., and an integrable system of Prym varieties.

(Roughly speaking, the curve tells you what the bundle is over each E_z, and the Prym element fits them together.)

Interesting- not given by a spectral construction, but by a Fourier-Mukai transform. (relative over $\mathbb{P}^1$).