Analytic geometric Langlands correspondence: Relations to conformal field theory and integrable models

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Part 1:
CFT-approach to the geometric Langlands correspondence

Geometric Langlands-correspondence

Schematically, the geometric Langlands-correspondence is often presented as relation

$$\boxed{{}^{\mathrm{L}}\mathfrak{g}\text{-local systems on } C} \quad \leftrightarrow \quad \boxed{\mathcal{D}\text{-modules on } \mathrm{Bun}_G(C)}$$

- C: Riemann surface, compact, or with punctures.
- G: complex group[1] with Lie algebra $\mathfrak{g}$.
- ${}^{\mathrm{L}}\mathfrak{g}$: Langlands dual Lie algebra of $\mathfrak{g}$.
- Local systems: pairs $(\mathcal{E}, \nabla)$, $\mathcal{E}$ holomorphic ${}^{\mathrm{L}}G$-bundle, ∇ connection on $\mathcal{E}$.
- $\mathcal{D}$-modules on $\mathrm{Bun}_G(C)$: Certain differential equations on Bun_G (here from flat connection, and eigenvalue equations, see later parts)

By now there exist stronger and more general versions, especially due to Arinkin and Gaitsgory. An important special case was first proven by Beilinson and Drinfeld, based on a key result of Feigin and Frenkel for the special cases where $(\mathcal{E}, \nabla)$ are opers.

[1] split, reductive

Geometric Langlands-correspondence – case of opers

The work of Beilinson and Drinfeld restricted attention to a special class of local systems $(\mathcal{E}, \nabla)$ called **opers**. Mostly discuss case $\mathfrak{g} = \mathfrak{sl}_2$ as an example. In this case:

- $\mathcal{E} = \mathcal{E}_{\mathrm{op}}$: Unique up to isomorphism extension

$$0 \to K^{\frac{1}{2}} \to \mathcal{E}_{\mathrm{op}} \to K^{-\frac{1}{2}} \to 0. \qquad \left[\text{transition functions } \sim \begin{pmatrix} \lambda_{ij} & \partial_{z_j}\lambda_{ij} \\ 0 & \lambda_{ij}^{-1} \end{pmatrix}\right].$$

- ∇ locally gauge equivalent to the form

$$\nabla = dz\left(\partial_z + \begin{pmatrix} 0 & t \\ 1 & 0 \end{pmatrix}\right), \quad \text{where } t \text{ transforms as } \textbf{projective connection},$$

$$t(z) = \big(\varphi'(z)\big)^2\, \tilde{t}\big(\varphi(z)\big) - \frac{1}{2}\{\varphi, z\}, \qquad \{\varphi, z\} := \frac{\varphi'''}{\varphi'} - \frac{3}{2}\left(\frac{\varphi''}{\varphi'}\right)^2.$$

Extension to generic local systems is possible by considering **meromorphic** opers with **apparent** singularities.

Results of Beilinson and Drinfeld still play an important role as foundations for many subsequent generalisations of the geometric Langlands program (Gaitsgory et. al.).

Conformal blocks – Key building blocks of conformal field theories.

Definitions customary in math and physics literature differ somewhat, roughly

M: spaces of (co-) invariants of VOAs like Virasoro algebra or affine Lie algebras,

P: functions satisfying differential equations called Ward identities.

We shall now briefly review the basic definitions and the relation between these points of view for WZNW-like CFTs, starting on the side of physics.

The CFT's of our interest are called WZW models. Start with **compact** WZW models associated to $G = SU(2)$, say. Basic objects:

$$\left[\begin{array}{l}\text{Space of states}\\ \mathcal{H} = \bigoplus_{m,n} R_m \otimes \bar{R}_n,\end{array}\right] \qquad \left[\begin{array}{ll}\text{vertex operators} & \mathcal{V}(v \otimes \bar{w}; z, \bar{z}), \quad v \in R_m,\\ \text{expectation values} & \big\langle\, \mathcal{V}(v \otimes \bar{w}; z, \bar{z})\,\big\rangle, \quad \bar{w} \in \bar{R}_n,\end{array}\right]$$

where R_m, $\bar{R}_n$ are representations of two copies of the affine algebra $\hat{\mathfrak{g}}_k = \widehat{\mathfrak{sl}}_{2,k}$, central extension of $\mathfrak{sl}_2 \otimes \mathbb{C}(\!(z)\!)$, generators $J_n^a \simeq T^a \otimes z^n$ and $\bar{J}_n^a \simeq T^a \otimes \bar{z}^n$.

$$\left[\begin{array}{l}\text{Expansion into conformal}\\ \text{blocks } \mathcal{F}_r(v, z, C)\end{array}\right]: \quad \big\langle\, \mathcal{V}(v \otimes \bar{w}; z, \bar{z})\,\big\rangle = \sum_{r,s} C_{rs} \mathcal{F}_r(v, z, C) \bar{\mathcal{F}}_s(\bar{w}, \bar{z}, C).$$

Translation to mathematics

Let $\hat{\mathfrak{g}}_k$ be the affine Lie algebra of level k. We shall consider a quadruple of data (C, P, z, R), where C is a Riemann surface, P a point on C, z a coordinate defined on a neighbourhood U_z of P which vanishes at P, and R a representation of $\hat{\mathfrak{g}}_k$.

A **conformal block** f is a linear map $f : R \to \mathbb{C}$ satisfying the invariance condition

$$f(J[\eta]v) = 0 \qquad \text{for all } \eta \in \mathfrak{g}_{\text{out}} = \mathfrak{g} \otimes \mathbb{C}[C \setminus P], \tag{1}$$

action of $\mathfrak{g}_{\text{out}}$ defined by Laurent expansion of η at P and replacing $z^n \eta_n$, $\eta_n \in \mathfrak{g}$ by the operators representing $\eta_n \otimes z^n \in \hat{\mathfrak{g}}_k$ on R.

Translation:

$$\boxed{\mathcal{F}_r(v, z, C) \equiv f(v).}$$

Space of conformal blocks: $\text{Conf}(C, R)$.

Twisted conformal blocks

For given data (C, P, z) as before and a complex Lie group G one can consider a holomorphic map $g : A_z \to G$ from the annulus $A_z = U_z \setminus P$ to G. The cover (U_0, U_z) with $U_0 = C \setminus P$, together with the transition function $g : A_z = U_0 \cap U_z$ define a holomorphic G-bundle $\mathcal{E} \equiv \mathcal{E}_g$. When G is semi-simple, all G-bundles can be represented in this way. This is the idea behind the Kac-Moody uniformisation

$$\mathrm{Bun}_G \simeq G_{\mathrm{out}} \setminus G(\!(t)\!) \,/\, G[\![t]\!].$$

Define **twisted** conformal blocks $f_{\mathcal{E}}$ by replacing $\mathfrak{g}_{\mathrm{out}}$ by $\mathfrak{g}_{\mathrm{out}}^{\mathcal{E}}$ in the condition (1).

Infinitesimal version: $\mathfrak{g}_{\mathrm{out}}^{\mathcal{E}} \setminus \mathfrak{g}(\!(t)\!) \,/\, \mathfrak{g}[\![t]\!] \rightsquigarrow$ map $\mathfrak{g}(\!(t)\!)$ to tangent vectors ξ to Bun_G, allowing one to define infinitesimal deformations of conformal blocks f via

$$\delta_\xi f_{\mathcal{E}}(v) := f_{\mathcal{E}}(J[\eta]v).$$

Representatives η of ξ can be chosen such that (i) variations δ_ξ preserve the defining invariance property (1), and (ii) $[\delta_{\xi_1}, \delta_{\xi_2}] = 0$. Then $\delta_\xi \rightsquigarrow$ flat connection on Bun_G.

Conformal blocks: $\mathcal{D}$-modules!

KZB equations I

Sugawara construction:

$$S(z) := \frac{1}{2} : J_a(z) J^a(z) := \sum_{n \in \mathbb{Z}} S_n z^{-n-2}, \qquad [S_n, J^a_m] = -m(k + h^\vee) J^a_{n+m},$$

$$[S_n, S_m] = (k + h^\vee)\big((n-m) S_{n+m} + \delta_{n,-m} \tfrac{k}{12} \dim(\mathfrak{g})\big),$$

which means that $L_n = (k + h^\vee)^{-1} S_n$ generate the Virasoro algebra inside $\mathcal{U}(\hat{\mathfrak{g}}_k)$.

Note the Virasoro uniformisation theorem,

$$\mathcal{T}(C) = \mathsf{Vect}(C \setminus P) \setminus \mathsf{Vect}(U_z \setminus P) \,/\, \mathsf{Vect}(U_z).$$

$\rightsquigarrow$ map from vector fields χ on $U_z \setminus P$ to complex structure deformations δ_ζ. Define

$$\delta_\zeta f_{\mathcal{E}}(v) := f_{\mathcal{E}}(T[\chi] v), \quad \text{where} \quad T[\chi] = \sum_n L_n \chi_n \ \text{ if } \ \chi(z) \frac{\partial}{\partial z} = \sum_n z^{n+1} \chi_n \frac{\partial}{\partial z},$$

a projectively flat connection on $\mathsf{Conf}(C, R)$ – now for variations of complex structure of C rather than bundle $\mathcal{E}$.

KZB equations II

Representing $T[\chi]$ in terms of J^a_m yields

$$\delta_\zeta f_{\mathcal{E}}(v) := \mathsf{K}_\zeta f_{\mathcal{E}}(v),$$

with K_ζ: Second order **differential operator** on a line bundle $\mathcal{L}^{\otimes k}$ on Bun_G.

Picking local coordinates $\mathbf{q} = (q_1, \ldots, q_h)$ on $\mathcal{M}(C)$ and $\mathbf{x} = (x_1, \ldots, x_d)$ on Bun_G, we can represent the KZB equations explicitly in the form

$$(k + h^\vee)\frac{\partial}{\partial q_r}\Psi(\mathbf{x},\mathbf{q}) = \mathsf{K}_r(\mathbf{x},\mathbf{q})\Psi(\mathbf{x},\mathbf{q}), \qquad \left[\,\mathsf{K}_r\,,\,\mathsf{K}_s\,\right] = 0.$$

Upshot: There is a relation between

a) spaces of affine algebra conformal blocks, and

b) spaces of solutions to the KZB-equations.

In good cases: One-to-one!

Remark: Precise relation can be subtle in general. May depend on types of representations. $\Psi(\mathbf{x},\mathbf{t})$ analytic where? Allow certain singularities?

Critical level I

Reconsider

$$S(z) := \frac{1}{2} : J_a(z) J^a(z) := \sum_{n\in\mathbb{Z}} S_n z^{-n-2} \qquad [S_n, J^a_m] = -m(k+h^\vee) J^a_{n+m}$$

$$[S_n, S_m] = (k+h^\vee)\big((n-m)S_{n+m} + \tfrac{k}{12}\dim\mathfrak{g}\,\delta_{n,-m}\big)$$

at the critical level $k = -h^\vee$. S_n generate large center $\mathfrak{z}(\mathfrak{g})$ inside of $\mathcal{U}(\mathfrak{g}_k)\big|_{k=-h^\vee}$.

$\rightsquigarrow$ $\exists$ Families of representations R_χ labelled by maps $\chi : \mathfrak{z}(\mathfrak{g}) \to \mathbb{C}$.

Such maps are characterised by their values $\chi(S_n) = t_n$, or the generating function $t(z) := \sum_{n\in\mathbb{Z}} z^{-n-2} t_n$ called classical energy-momentum tensor.

Notation $R_t \equiv R_\chi$. Generators S_n in R_t represented by multiplication with t_n.

Generalisation to general Lie algebra $\mathfrak{g}$: Theorem of Feigin and Frenkel:

> The center $\mathfrak{z}(\mathfrak{g})$ is canonically isomorphic (as Poisson-vertex algebra) to the algebra of ${}^{\mathrm{L}}\mathfrak{g}$-opers on the formal disc.

Critical level II

Reconsider definition of conformal blocks: Note that for $k \neq -h^\vee$ the invariance conditions defining conformal blocks $\Leftrightarrow$ expectation values

$$\langle\, J^a(z)\,\rangle := f(J^a(z)v_t), \qquad \langle\, T(z)\,\rangle := f(T(z)v_t),$$

have analytic continuations defining holomorphic $\mathfrak{g}$-connections and projective c-connections[2] on C, respectively (proof uses strong residue theorem).

For $k = -h^\vee$ one may consider $S(z)$ instead of $T(z)$, defining

$$\langle\, S(z)\,\rangle := f(S(z)v).$$

But $f(S(z)v_t) = t(z)f(v_t) \Rightarrow$ conformal blocks f can only exist if

$$\boxed{\; t(z) \text{ is the restriction to } U_z \text{ of a globally defined oper on } C. \;}$$

[2] modify the $\frac{1}{2}$ in transformation law of $t(z)$ to $\frac{c}{12}$

Critical level III

Putting things together:

- Given an oper $t(z)$ which is globally defined on C, there exist conformal blocks $f_{\mathcal{E}}$ of $\hat{\mathfrak{g}}_k$ at $k = -h^\vee$.
- Such conformal blocks naturally define a $\mathcal{D}$-module on Bun_G.

This is an essential part of the version of the geometric Langlands correspondence established by Beilinson and Drinfeld. Furthermore,

$$\begin{bmatrix} S(z) = \frac{1}{2} {:}J_a(z)J^a(z){:} \\ f_{\mathcal{E}}(J[\eta]v) = \delta_\xi f_{\mathcal{E}}(v) \end{bmatrix} \quad \Rightarrow \quad \begin{aligned} t(z) f_{\mathcal{E}}(v_t) &= f_{\mathcal{E}}(S(z)v_t) \\ &= \mathsf{S}(z) f_{\mathcal{E}}(v_t), \end{aligned}$$

where $\mathsf{S}(z)$: Second order differential operator on $\mathcal{L}^{-h^\vee} \simeq K_{\mathrm{Bun}}^{1/2}$ over Bun_G.
Picking a reference projective connection t_0, and a basis for $H^0(C, K^2)$ yields

$$\boxed{\mathsf{H}_r \, f_{\mathcal{E}}(v_t) = E_r \, f_{\mathcal{E}}(v_t).} \qquad \begin{aligned} t(z) - t_0(z) &= \sum_r E_r Q_r(z), \\ \mathsf{S}(z) - t_0(z) &= \sum_r \mathsf{H}_r Q_r(z). \end{aligned}$$

Critical level limit

Picking local coordinates $\mathbf{x} = (x_1, \ldots, x_d)$ on Bun_G, and setting $\Psi(\mathbf{x}, \mathbf{q}) = f_{\mathcal{E}_\mathbf{x}}(v_t)$ we can represent the **eigenvalue equations** explicitly in the form

$$\mathsf{H}_r(\mathbf{x}, \mathbf{q})\Psi(\mathbf{x}, \mathbf{q}) = E_r \Psi(\mathbf{x}, \mathbf{t}), \qquad \big[\,\mathsf{H}_r\,,\,\mathsf{H}_s\,\big] = 0.$$

Relation to the KZB-equations: $\quad (k + h^\vee)\dfrac{\partial}{\partial q_r}\Psi(\mathbf{x}, \mathbf{t}) = \mathsf{K}_r(\mathbf{x}, \mathbf{t}, k)\Psi(\mathbf{x}, \mathbf{q})$?

Observation (Reshetikhin-Varchenko): For $b^{-2} = -k - h^\vee \to 0$, there exist solutions as formal series in b^{-2} of the form

$$\Psi(\mathbf{x}, \mathbf{q}) = e^{-b^2 S(\mathbf{t})}\psi(\mathbf{x}, \mathbf{q})(1 + \mathcal{O}(b^{-2})), \qquad \left[\begin{array}{l} \dfrac{\partial}{\partial q_r} S(\mathbf{q}) = E_r(\mathbf{q}), \\ \mathsf{H}_r(\mathbf{x}, \mathbf{t})\psi = E_r(\mathbf{q})\psi. \end{array}\right]$$

Eigenvalue equations for commuting Hamiltonians $\mathsf{H}_r(\mathbf{x}, \mathbf{t}) \equiv \mathsf{K}_r(\mathbf{x}, \mathbf{t}, -h^\vee)$!

Relation to the Hitchin system

Hitchin moduli space $\mathcal{M}_{\mathrm{Hit}}(C)$:

Space of pairs $(\mathcal{E},\varphi)$, $\mathcal{E}$: holomorphic bundle on C, $\varphi \in H^0(C,\mathrm{End}(\mathcal{E})\otimes K)$.

$\mathcal{M}_{\mathrm{Hit}}(C)$ has canonical Poisson structure from Serre-duality between tangent space $H^1(C,\mathrm{End}(\mathcal{E}))$ to Bun_G at $\mathcal{E}$ and $H^0(C,\mathrm{End}(\mathcal{E})\otimes K)$.

Integrability: Let $\theta := \frac{1}{2}\mathrm{tr}(\varphi^2) \in H^0(C,K^2)$. We have

$$\theta(z) = \sum_{r=1}^{3g-3+n} H_r Q_r(z), \qquad \{H_r, H_s\} = 0.$$

$\Rightarrow$ Hamiltonians H_r define integrable structure on $\mathcal{M}_{\mathrm{Hit}}(C)$.

Observation: Differential operators H_r represent a **quantisation** of the Hamiltonians H_r!

Indeed, one may note that the symbols of the DOs H_r introduced before are global functions on Bun_G, homogenous in degree 2. Hitchin has proven that the algebra of such functions is generated by the Hamiltonians H_r.

Hecke functors – informally I

Crucial refinement provided by the Hecke functions. Key ingredients:

- **Hecke modifications**: Take bundle $\mathcal{E}$, open set U in a cover representing $\mathcal{E}$, $P' \in U$, replace U by $(U' = U \setminus P', D')$, with $D' \subset U$ open disk around P'. Specify new transition function on U', possibly singular at $P' \rightsquigarrow$ new bundle $\mathcal{E}'$.

- Space of Hecke modifications: Grassmannian $\mathrm{Gr} = G(\!(t)\!) \,/\, G[\![t]\!]$.

- Hecke modifications can relate two different parameterisations of Bun_G, labelled by points on $G[\![t]\!]$-orbits $\mathrm{Gr}_\lambda = G[\![t]\!] \cdot \lambda(t) \,/\, G[\![t]\!]$ in Gr, where λ: weight of ${}^{\mathrm{L}}G$.

- Due to the fact that Bun_G is a quotient of Gr, Hecke modifications induce transformations of differential equations (functors of $\mathcal{D}$-modules) on Bun_G,

- and therefore transformations between spaces of conformal blocks.

Key observation: The Hecke transformation of conformal blocks at a point P can be described by the insertion of a particular representation (vertex operator) at P – modifies the differential equations in the same way.

Hecke functors – informally II

The more precise description (Beilinson-Drinfeld) uses

- A realisation of $\hat{\mathfrak{g}}_{-h^\vee}$-modules as spaces of global sections $\Gamma(\mathrm{Gr}, \mathcal{S})$.
- The **geometric Satake** correspondence: Relation between cohomologies of certain sheaves IC_λ on Gr and representations of ${}^{\mathrm{L}}G$.

At the critical level one has (Beilinson-Drinfeld):

$$W_\lambda = \Gamma\big(\mathrm{Gr}, \mathrm{IC}_\lambda \otimes \mathcal{L}^{\otimes k}\big) \simeq V_\lambda \otimes V_{-h^\vee}(\mathfrak{g}).$$

Combining these results yields the crucial **Hecke eigenvalue property**:

$$\mathfrak{H}_{P'\lambda}\mathsf{Conf}(C, R) \simeq \mathsf{Conf}(C \setminus \{P'\}, R \otimes W_\lambda) \simeq V_\lambda \otimes \mathsf{Conf}(C, R).$$

Furthermore: Part of the $\mathcal{D}$-module structure yields pair $(\mathcal{E}_{\mathrm{op}}, \nabla_{\mathrm{op}})$ characterising infinitesimal changes of $P' \in C$ $\rightsquigarrow$ **recover local system!**

It will be useful to keep in mind that $\nabla_{\mathrm{op}}\big(\begin{smallmatrix}\chi_1\\ \chi_2\end{smallmatrix}\big) = 0$, $\nabla_{\mathrm{op}} = dz\big(\partial_z + \big(\begin{smallmatrix}0 & t\\ 1 & 0\end{smallmatrix}\big)\big)$, implies

$$(\partial_z^2 - t(z))\chi_2(z) = 0.$$

Part 2:

Analytic Langlands correspondence

Starting point: GL reinterpreted

Note that $\mathcal{D}$-modules appearing in GL $\sim$ eigenvalue equations for quantized Hitchin's Hamiltonians $\rightsquigarrow$ integrable model interpretation:

GL organises the "haystack" within which we can look for actual eigenfunctions.

Next step: Impose further conditions:

a) Single-valuedness?

b) Square-integrability?

These conditions will be called "quantisation conditions". Conditions of type b) define the spectrum im quantum mechanics. However, it turns out that conditions a)[3] and b)[4] will single out the **same** discrete subset of $\mathrm{Op}_{\mathfrak{g}}(C)$.

[3] J.T., arXiv:1707.07873; Troy Figiel and J.T., in preparation.

[4] Etingof, Frenkel, Kazhdan arXiv:1908.09677, arXiv:2103.01509, arXiv:2106.05243

Single-valuedness condition

For given pair $(\chi, \bar{\chi})$, where $\chi \in \mathrm{Op}(X)$, $\bar{\chi} \in \overline{\mathrm{Op}}(X)$, we may consider the pair of complex conjugate eigenvalue equations

$$\begin{aligned} \mathsf{H}\,\Psi &= \chi(\mathsf{H})\Psi, \quad \forall\, \mathsf{H} \in \mathcal{D}, \\ \bar{\mathsf{K}}\,\Psi &= \bar{\chi}(\bar{\mathsf{K}})\Psi, \quad \forall\, \bar{\mathsf{K}} \in \bar{\mathcal{D}}. \end{aligned} \tag{2}$$

The Hitchin Hamiltonian have singularities at **wobbly** bundles admitting nilpotent Higgs fields (cf. lectures by Ana Peon-Nieto). $\mathrm{Bun}_G^{\mathrm{vs}}$: moduli space of stable bundles which are not wobbly.

We may then look for smooth solutions on $\mathrm{Bun}_G^{\mathrm{vs}}$ locally of the form

$$\Psi(\mathbf{x}, \bar{\mathbf{x}}) = \sum_{r,s} C_{rs}\psi_r(\mathbf{x})\bar{\psi}_s(\bar{\mathbf{x}}), \qquad \begin{aligned} \mathsf{H}\,\psi_r(\mathbf{x}) &= \chi(\mathsf{H})\psi_r(\mathbf{x}), \quad \forall\, \mathsf{H} \in \mathcal{D}, \\ \bar{\mathsf{K}}\,\bar{\psi}_s(\bar{\mathbf{x}}) &= \bar{\chi}(\bar{\mathsf{K}})\bar{\psi}_s(\bar{\mathbf{x}}), \quad \forall\, \bar{\mathsf{K}} \in \bar{\mathcal{D}}, \end{aligned}$$

which are **single-valued**.

This requires that the nontrivial monodromies that the local sections $\psi_r(\mathbf{x})$ and $\bar{\psi}_s(\bar{\mathbf{x}})$ will generically have (e.g. around wobbly loci) cancel each other.

Separation of variables I – making GL explicit

For surface C with $g = 0, 1$ there exist[5] explicit integral transformations of the form

$$\Psi(\mathbf{x}, \bar{\mathbf{x}}) = \int d\mathbf{u} d\bar{\mathbf{u}} \; K(\mathbf{x}, \bar{\mathbf{x}} \,|\, \mathbf{u}, \bar{\mathbf{u}}) \Phi(\mathbf{u}, \bar{\mathbf{u}}), \tag{3}$$

$\mathbf{u} = (u_1, \ldots, u_D)$, $\bar{\mathbf{u}} = (\bar{u}_1, \ldots, \bar{u}_D)$, such that (2) is equivalent to

$$(\partial_{u_r}^2 - t(u_r))\phi(u_r, \bar{u}_r) = 0, \quad (\bar{\partial}_{\bar{u}_r}^2 - \bar{t}(\bar{u}_r))\phi(u_r, \bar{u}_r) = 0, \quad \forall r.$$

The proof uses a technique from the theory of quantum integrable models called Separation of Variables (SOV; Sklyanin). It follows that

$$\Phi(\mathbf{u}, \bar{\mathbf{u}}) = \prod_{r=1}^{D} \phi_r(u_r, \bar{u}_r), \qquad \begin{aligned} &(\partial_{u_r}^2 - t(u_r))\phi_r(u_r, \bar{u}_r) = 0, \\ &(\bar{\partial}_{\bar{u}_r}^2 - \bar{t}(\bar{u}_r))\phi_r(u_r, \bar{u}_r) = 0. \end{aligned} \tag{4}$$

The transformation (3) is invertible. Single-valuedness of the kernel K implies that the functions $\phi_r(u_r, \bar{u}_r)$ are single-valued. It will turn out that the single-valued solutions to (4) are unique up to normalisation, so $\phi_r(u, \bar{u}) = \phi(u, \bar{u})$ for all r.

[5] Sklyanin, Frenkel '95, Enriquez-Feigin-Rubtsov, Felder-Schorr, Ribault-J.T., Frenkel-Gukov-J.T.,....

Separation of variables II – making GL explicit

For $g > 1$: Frenkel ('95) has interpreted the first geometric construction of a Langlands correspondence by Drinfeld as a variant of the SOV, based on

$$\begin{array}{ccccccc} & & \mathcal{M}_{2,1}^n & & & & S^m X \\ & \swarrow^{i_n} & & \searrow^{j_n^\vee} & & \swarrow^{j_n} & \\ \mathcal{M}_2^n & & & & \mathrm{Jac}_n & & \end{array}$$

- $\mathcal{M}_{2,1}^n$: Space of pairs $(\mathcal{E}, \mathcal{L})$ defining extensions $0 \to \mathcal{O} \to \mathcal{E} \to \mathcal{L} \to 0$.
- Fibre of $j_n^\vee$: Space of extension classes $\mathbb{P}H^1(\mathcal{L}^{-1})$, fibre of j_n: $\mathbb{P}H^0(C, K \otimes \mathcal{L})$.
- (Serre-) duality between fibres $\sim$ map from $\mathcal{D}$-modules on $S^m X$ to $\mathcal{M}_2^n$.

Use to relate Hitchin-eigenvalue equations to equations (4):

- Represent solutions $\Psi_{\mathbf{E}}$ as homogenous functions $\Psi_{\mathbf{E}}(\mathbf{x}, \bar{\mathbf{x}})$ on $H^1(\mathcal{L}^{-1})$.
- Fourier-transformation w.r.t. $\mathbf{x} \rightsquigarrow$ functions $\Phi_{\mathbf{E}}(\mathbf{p}, \bar{\mathbf{p}})$ on $H^0(C, K \otimes \mathcal{L})$.
- Conjecture/claim: Expressing $\mathbf{p}$ in terms of zeros u_k of $\varphi_- \in H^0(C, K \otimes \mathcal{L})$ $\rightsquigarrow$ equations (4).

Single-valuedness

The separation of variables maps single-valued solutions of pairs of complex-conjugate Hitchin eigenvalue equations to

$$(\partial_u^2 - t(u))\phi(u,\bar{u}) = 0, \qquad (\bar{\partial}_{\bar{u}}^2 - \bar{t}(\bar{u}))\phi(u,\bar{u}) = 0.$$

It is possible to show that smooth solutions can be represented in the form

$$\phi(u,\bar{u}) = \chi(u) \cdot C \cdot \chi^\dagger(\bar{u}), \qquad \chi(u) = (\chi_1(u), \chi_2(u)), \qquad (\partial_u^2 - t(u))\chi(u) = 0.$$

The matrix C can be brought to diagonal form by a change of basis. It can be shown that up to changes of C induced by changes of basis the only choice of C which gives single-valued solutions $\phi(u,\bar{u})$ is $C = \mathrm{diag}(1,-1)$. This is single-valued if the monodromy is in $\mathrm{SU}(1,1)$, which is conjugate to $\mathrm{SL}(2,\mathbb{R})$.

Subtlety: May allow solutions defined up to a sign, corresponding to solutions $\phi(u,\bar{u})$ defined on a cover of C.

Real opers I

A **real** oper is an oper with real **monodromy**. Opers are in one-to-one correspondence to **projective structures** on C. Indeed, one may use the ratios

$$A(z) = \frac{\chi_1(z)}{\chi_2(z)}, \quad \chi_i(z): \text{ linearly independent solutions of } (\partial_z^2 - t(z))\chi_i(z) = 0,$$

to define new local coordinates $w = A(z)$ on C. Features:

- Oper represented by $\tilde{t}(w) = 0$,
- Changes of coordinates $w'(w)$: Möbius transformations

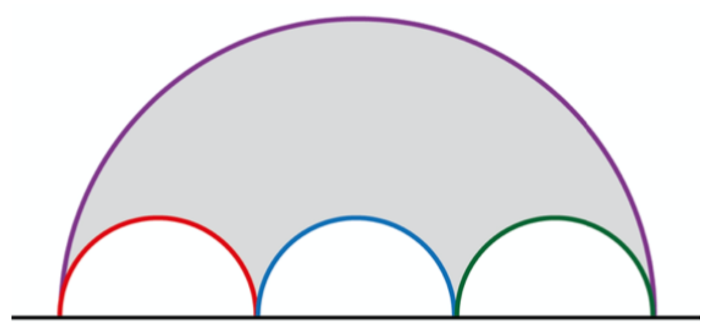
An atlas $\{U_\imath; \imath \in \mathcal{I}\}$ for C with transition functions represented by Möbius transformations defines a **projective structure**, allowing one to represent C as a cover of a fundamental domain in $\mathbb{C}$.

Example:

Monodromy Fuchsian

$\rightsquigarrow$ uniformisation

e.g. $C = C_{1,1}$:

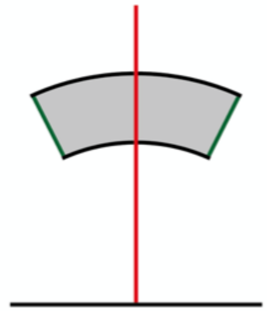


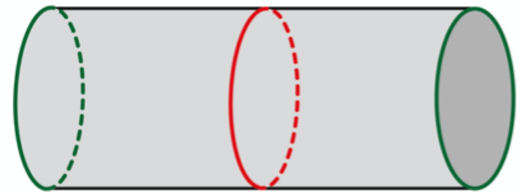
Real opers II – Grafting (Surgery on projective structures)

Grafting:

Cut figure on bottom

left open along γ,

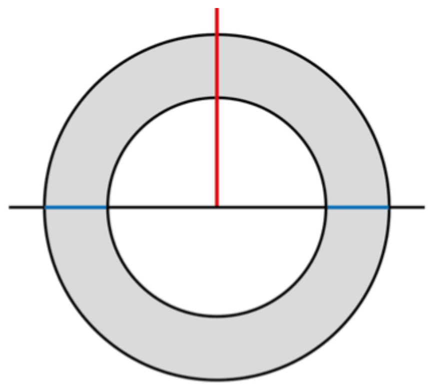


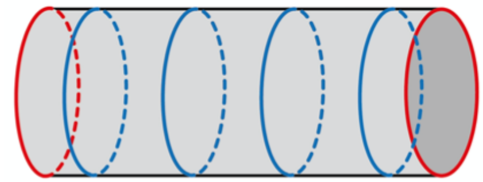


insert multiple copies

of the annulus,

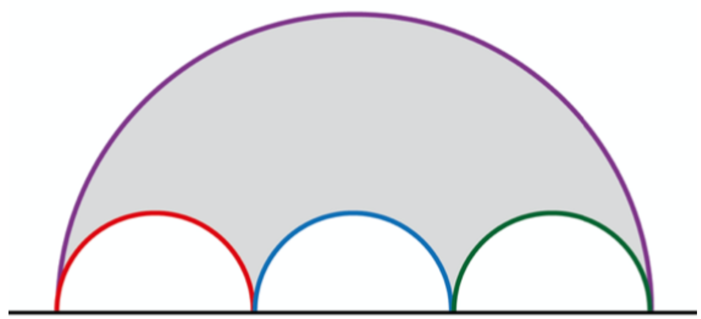
reglue along red curves.

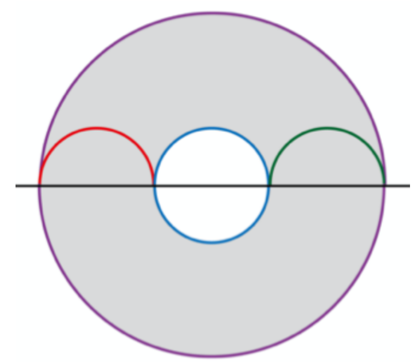




Result depicted in the

figure on bottom right.





Real opers III – Goldman's classification

All real opers can be constructed in this way:

Integer-measured laminations Λ: Homotopy classes of finite collections of simple non-intersecting noncontractible closed curves on S with integral weights, such that

- The weight of a curve is non-negative unless the curve is peripheral.
- A lamination containing a curve of weight zero is equivalent to the lamination with this curve removed.
- A lamination containing two homotopic curves of weights k and $l \sim$ a lamination with one of the two curves removed and weight $k+l$ assigned to the other.

The set of all such laminations on C is denoted as $\mathcal{ML}_C(\mathbb{Z})$. **Half**-integer measured laminations $\Lambda \in \mathcal{ML}_C(\frac{1}{2}\mathbb{Z})$ can be defined in an analogous way.

Theorem (Goldman '87): Real projective structures are in one-to-one correspondence to half-integer measured laminations $\Lambda \in \mathcal{ML}_C(\frac{1}{2}\mathbb{Z})$.

In other words: **All** real projective structures can be obtained from the uniformising projective structure by grafting along a $\Lambda \in \mathcal{ML}_C(\frac{1}{2}\mathbb{Z})$.

The work of Etingof, Frenkel and Kazhdan

The work[6] of Etingof, Frenkel and Kazhdan (EFK) deepens the story considerable by introducing harmonic analysis aspects into analytic GL.

Up to now: Solution to eigenvalue problem restricted to **single-valuedness** only.

EFK introduce a smooth algebraic moduli space $\mathrm{Bun}_G^{\mathrm{rs}}(C)$ by considering the stack of bundles $\mathrm{Bun}_G^{\circ}(C)$ with automorphisms in the center of G, and forgetting automorphisms. On $\mathrm{Bun}_G^{\mathrm{rs}}(C)$ introduce

- Line bundle of half-densities $\Omega_{\mathrm{Bun}}^{1/2} := |K_{\mathrm{Bun}}^{1/2}|$, where $|\mathcal{L}| = \mathcal{L} \otimes \bar{\mathcal{L}}$,
- $\mathcal{S}_G$ space of smooth compactly supported sections of $\Omega_{\mathrm{Bun}}^{1/2}$, and
- define a Hermitian form $\langle .,. \rangle$ by

$$\langle v, w \rangle := \int_{\mathrm{Bun}_G^{\mathrm{rs}}} v\, \bar{w}, \qquad v, w \in \mathcal{S}_G.$$

- **Hilbert space** $\mathcal{H}_G$: Completion of $\mathcal{S}_G$ with respect to $\langle .,. \rangle$.

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[6] arXiv:1908.09677, arXiv:2103.01509, arXiv:2106.05243

The work of Etingof, Frenkel and Kazhdan II

EFK formulate a set of conjectures and prove them in some cases, leading to the following picture:

- The eigenspaces $\mathcal{H}_{\chi,\bar{\chi}}$ generated by single-valued solutions to the pair of eigenvalue equations with eigenvalues $(\chi,\bar{\chi})$ are contained in $\mathcal{H}_G$, at most one-dimensional, and non-vanishing only if $\bar{\chi}$ is complex conjugate to χ.
- The Hilbert spaces $\mathcal{H}_G$ admit an orthogonal decomposition into the spaces $\mathcal{H}_{\chi,\bar{\chi}}$ (completeness)

EFK furthermore introduce a family of integral operators called Hecke operators, roughly of the form

$$(\mathbf{H}_\lambda f)(\mathcal{E}) := \int_{Z_\lambda(\mathcal{E},P')} q_1^*(f),$$

- $Z_\lambda(\mathcal{E},P')$ – space of all possible λ-Hecke modifications of bundle $\mathcal{E}$ at point P', isomorphic to closure $\overline{\mathrm{Gr}}_\lambda$ of orbit $\mathrm{Gr}_\lambda = G[\![t]\!]\cdot\lambda(t)\,/\,G[\![t]\!]$.
- $q_1^*(f)$ pull-back of f under correspondence between bundles defined by Hecke modifications.

The work of Etingof, Frenkel and Kazhdan III

EFK conjecture and prove in some cases that

- the Hecke operators extend to a family of commuting compact normal operators on $\mathcal{H}_G$,
- $\mathcal{H}_G$ decomposes into eigenspaces of the Hecke operators,
- the eigenspaces coincide with the eigenspaces $\mathcal{H}_{\chi,\bar{\chi}}$ of the Hitchin Hamiltonians,
- and the eigenspaces $\mathcal{H}_{\chi,\bar{\chi}}$ are non-trivial only if χ is a real oper.

It seems quite remarkable that the conditions of single-valuedness and $\bar{\chi}$ being the complex conjugate of χ turn out to be equivalent to single-valuedness and square-integrability in this case.

The work of EFK uses many results from previous work on geometric Langlands as groundwork or input for the functional-analytic extension studied in their work.

Expect further fruitful interplay between algebro-geometric and analytic aspects.

Analytic Langlands correspondence: A brief summary

Analytic Langlands correspondence:

$$\boxed{\text{Real opers}} \quad \leftrightarrow \quad \boxed{\left\{ \begin{array}{c} \text{single-valued} \\ L^2\text{-normalisable} \end{array} \right\} \text{Hitchin eigen-}\mathcal{D}\text{-modules}}$$

Intepretation from the perspective of integrable models:

Analytic Langlands	**Quantum Integrable models**
$\mathcal{D}$-modules	Local IM: Commuting Hamiltonians
Hecke operators	Non-local IM: Baxter Q-operators
Uniformising oper	Ground state
Grafting	**Creation operators**

Part 3:

Quantum Analytic Langlands correspondence

and the relation to

$\mathcal{N} = 2$, $d = 4$ supersymmetric field theories

Quantum geometric Langlands correspondence

One-parameter deformations of the geometric Langlands correspondence have been proposed both in mathematics[7] and physics[8].

Such deformations are sometimes called quantum geometric Langlands correspondence.

Although less well developed at this point, they offer the attractive perspective to understand the geometric Langlands correspondence as a limiting case of a theory in which the Langlands duality is realised in a more transparent way.

As a basic feature of the quantum geometric Langlands correspondence one may expect to find on top of the usual "magnetic" Hecke functors labelled by weights λ of ${}^{\mathrm{L}}\mathfrak{g}$ a dual family of "electric" Hecke functors labelled by weights $\check{\lambda}$ of $\mathfrak{g}$.

Natural question: Is there an **analytic** version of the **quantum** geometric Langlands correspondence?

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[7] Frenkel '05; Gaitsgory '07

[8] Kapustin-Witten '06; J.T. '10

The H_3^+-WZNW model I – Physics perspective

The path integral defining the correlation functions of the H_3^+ WZNW model[9] can be represented schematically as

$$\Big\langle \prod_{k=1}^{n} \Phi^{j_r}(x_r|z_r) \Big\rangle_{\mathrm{WZW}_\kappa} := \int_{h:C\to\mathbb{H}_3^+} \mathcal{D}[h]\; e^{-S_{\mathrm{WZ}}[h]} \prod_{r=1}^{n} \phi^{j_r}(h(z_r);x_r),$$

where

$$S_{\mathrm{WZ}}[h] = \frac{\kappa}{\pi}\int dx\,\left(\partial\phi\bar{\partial}\phi + |\bar{\partial}\gamma|^2 e^{2\phi}\right), \qquad h = \begin{pmatrix} e^{-\phi}+|\gamma|^2 e^{\phi} & \bar{\gamma}e^{\phi} \\ \gamma e^{\phi} & e^{\phi} \end{pmatrix},$$

and the fields $\Phi^j(x|z)$ get represented by the functions $\phi^j(h;x)$ defined as

$$\phi^j(h;x) = \frac{2j+1}{\pi}\left((1,-x)\cdot h \cdot \begin{pmatrix} 1 \\ -\bar{x} \end{pmatrix}\right)^{2j}.$$

[9]K. Gawedzki '89

The H_3^+-WZNW model II – Translation to a math problem

The space of states is believed to be $\mathcal{H}_{\mathbb{H}_3^+} = \int_{\mathbb{R}_+} dj\; \mathcal{P}_j$, where $\mathcal{P}_j$: principal series representation of $\widehat{sl}(2,\mathbb{C})_\kappa \sim \widehat{sl}^{\,\mathrm{l}}_{2,-\kappa} \oplus \widehat{sl}^{\,\mathrm{r}}_{2,-\kappa}$, with

$$\widehat{sl}^{\,\mathrm{l}}_{2,-\kappa} \;\text{ generated by current } J^a(z) = \sum_n J^a_n z^{-n-1},$$

$$\widehat{sl}^{\,\mathrm{r}}_{2,-\kappa} \;\text{ generated by current } \bar{J}^a(\bar{z}) = \sum_n \bar{J}^a_n \bar{z}^{-n-1}.$$

This can be used to argue[10] that the correlation functions $\big\langle \prod_{r=1}^n \Phi^{j_r}(x_r|z_r) \big\rangle_{\mathrm{WZW}_\kappa}$

(i) satisfy the **KZB equations** for $\widehat{\mathfrak{sl}}_{2,-\kappa}$, and their complex conjugates,

(ii) are **single-valued** and real analytic away from wobbly loci,

(iii) have only rather **mild singular behaviour** near boundary of $\mathcal{M}(C)$.

$$\Big\langle \prod_{k=1}^n \Phi^{j_r}(x_r|z_r) \Big\rangle_{\mathrm{WZW}_\kappa} = \textbf{Single-valued tempered conformal block of } \widehat{sl}(2,\mathbb{C})_\kappa$$

[10] J.T. '97, J.T. '99, and work in progress with Duong Dinh

Quantum analytic Langlands correspondence in the H_3^+-WZNW model

Recap:

- Analytic GL $\rightsquigarrow$ find single valued eigenfunctions of Hitchin Hamiltonians
- KZB – natural deformation of Hitchin eigenvalue equations, in the sense that

$$\begin{bmatrix} \mathsf{K}_r \Psi(\mathbf{x},\mathbf{q}) = (k+h^\vee)\dfrac{\partial}{\partial q_r}\Psi(\mathbf{x},\mathbf{q}) \\ \Psi(\mathbf{x},\mathbf{q}) = e^{-b^2 S(\mathbf{q})}\psi(\mathbf{x},\mathbf{q})(1+\mathcal{O}(b^{-2})) \end{bmatrix} \underset{b^{-2}=-k-h^\vee}{\Rightarrow} \begin{bmatrix} \dfrac{\partial}{\partial q_r} S(\mathbf{q}) = E_r(\mathbf{q}), \\ \mathsf{H}_r \psi = E_r(\mathbf{q})\psi. \end{bmatrix}$$

So correlation functions of H_3^+ – natural deformations of Hitchin eigenfunctions.

Immediate puzzles:

- Do these guys exist mathematically?
- How to construct correlation functions classified by integer laminations?

Relation to Liouville theory I

The solutions to this problem can be represented in the following form[11]

$$\left\langle \prod_{r=1}^{n} \Phi^{j_r}(x_r|z_r) \right\rangle_{\mathrm{WZW}_\kappa} = \int d\mathbf{u}\; K_\kappa(\mathbf{x},\mathbf{u};\mathbf{z})\mathcal{Z}(\mathbf{u},\mathbf{z},b)\;, \tag{5}$$

with $K_\kappa(\mathbf{x},\mathbf{u};\mathbf{z})$ being an explicitly known kernel, if $\mathcal{Z}(\mathbf{u},\mathbf{z},b)$ is a solution of the BPZ-equations[12]

$$\left(b^2\frac{\partial^2}{\partial u_r^2} - \mathsf{T}_r(u_r)\right)\mathcal{Z}(\mathbf{u},\mathbf{z},b) = 0, \qquad b^{-2} = \kappa - 2 = -k-2,$$

with $\mathsf{T}_r(u_r)$ being certain first order differential operators in ∂_{u_l} and ∂_{z_r}. $\mathcal{Z}(\mathbf{u},\mathbf{z},b)$ satisfies the conditions of **single-valuedness** and real-analyticity away from the singularities of $\mathsf{T}_r(u_r)$ with mild singular behavior there and near $\partial\mathcal{M}(C)$.

The transformation in (5) intertwines KZB-equation and BPZ-equations satisfied by Liouville correlation function. It is a **deformation** of the **SOV-transformation**, and the differential operators $b^2\frac{\partial^2}{\partial u^2} - \mathsf{T}_k(u)$ are **quantum opers**, canonical deformations of the opers $\frac{\partial^2}{\partial u^2} - t(u)$!

[11] Feigin-Frenkel-Stoyanovsky; Ribault-J.T. '05; Frenkel-Gukov-J.T.

[12] BPZ: Belavin-Polyakov-Zamolodchikov

Relation to Liouville theory II

The deformed SOV-transformation could map single-valued solutions to BPZ to single valued solutions $\mathcal{Z}(\mathbf{u},\mathbf{z},b)$ of KZB with certain singularities.

Single-valued solutions to BPZ with the necessary properties exist:

$$\mathcal{Z}(\mathbf{u},\mathbf{z},b) = \left\langle \prod_{r=1}^{n} V_{\alpha_r}(z_r) \prod_{l=1}^{h} V_{-1/2b}(u_l) \right\rangle_{\text{Liou}},$$

$$\text{where} \quad \left\{ \begin{aligned} &\left\langle \prod_{r=1}^{n} V_{\alpha_r}(z_r) \right\rangle_{\text{Liou}} := \int_{\phi:C\to\mathbb{R}} \mathcal{D}[\phi]\; e^{-S_{\mathrm{L}}[\phi]} \prod_{r=1}^{n} e^{2\alpha_r\phi(z_r)}, \\ &S_{\mathrm{L}} = \frac{1}{4\pi}\int_C d^2x \left(|\nabla\phi(x)|^2 + 4\pi\mu e^{2b\phi(x)} \right). \end{aligned} \right\} \tag{6}$$

The path integral in (6) has been constructed **rigorously**[13]. The result of this construction satisfies the BPZ equations together with the conditions of single-valuedness and mild singular behaviour[14] which are necessary for producing single-valued solutions to the KZB equations via quantum SOV.

[13] David, Kupiainen, Rhodes, Vargas 2014

[14] Kupiainen, Rhodes, Vargas '15-'17; Guillarmou, Kupiainen, Rhodes, Vargas '20

Hecke operators

Beilinson and Drinfeld had identified certain representations of $\hat{\mathfrak{g}}_k$ as the objects representing Hecke functors in conformal blocks,

$$W_\lambda = \Gamma\big(\mathrm{Gr}, \mathrm{IC}_\lambda \otimes \mathcal{L}^{\otimes k}\big), \quad \lambda: \text{ weight of } {}^{\mathrm{L}}\mathfrak{g}.$$

These representations exist away from the critical level $k = -h^\vee$. They coincide with the irreducible highest-weight modules of $\hat{\mathfrak{g}}_k$ with highest weight $-(k+h^\vee)\lambda$.

The representations W_λ have null-vectors. If $\mathfrak{g} = \mathfrak{sl}_2$ and $\lambda = \frac{1}{2}$, one has $J^-_{-1} w_\lambda = 0$, for example.

In the H_3^+-WZW model there are fields $\Phi^{-(k+h^\vee)\lambda}$ associated to the representation W_λ. These fields satisfy additional differential equations (null vector decoupling) on top of the KZB-equations. Such fields are usually called degenerate.

The quantum SOV transformation maps these fields to degenerate fields of the Virasoro algebra of type $(1, n)$ satisfying the BPZ-equations.[15] Such fields will be called "magnetic" in the following.

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[15] J.T. 2011

Magnetic line operators and Hecke eigenvalue property

Following E. Verlinde one can define certain operators on spaces of conformal blocks by pair creation of degenerate fields (bubbling), analytic continuation of the position of one of them along a closed curve, and pair annihilation (fusion). It is known that

> Verlinde line operators generate an algebra representing a quantisation of the algebra of functions on the character variety of flat $PSL(2)$-connections.

Applying this sequence of operations to degenerate fields of type $(1, n)$ of Liouville theory (the (pre-)images of the Hecke fields W_λ under the SOV transformation) yields **magnetic Verlinde line operators**. In the critical level limit $k \to -h^\vee$, one recovers the commutative algebra of functions on the character variety.

Electric line operators and grafting

There is another family of degenerate $\widehat{\mathfrak{sl}}_{2,k}$ representations, associated to the fields Φ^j, $2j \in \mathbb{Z}_+$ in the H_3^+-WZNW model. The SOV transformation maps them to Virasoro degenerate fields of type $(m,1)$, called "electric".[16]

One can again define Verlinde line operators using electric VOs. Such line operators will leave a strong effect on the correlation functions in the limit $k \to -h^\vee$. The definition is easily generalised from simple closed curves to integer-measured laminations Λ.

We have evidence[17] for the following **claim:**

> Insertion of an electric Verlinde line operator associated to a lamination Λ yields for $k \to -h^\vee$ the eigenfunction of the Hitchin Hamiltonians associated to Λ.

[16] Frenkel-Gukov-J.T.

[17] Work in progress, based on previous computations of Verlinde line operators by Alday-Gaiotto-Gukov-Tachikawa-Verlinde and Drukker-Gomis-Okuda-J.T., and analysis of critical level limit using CFT bootstrap methods.

Gauge theory interpretation I

The AGT-correspondence relates the conformal blocks of Liouville theory to instanton partition functions of supersymmetric gauge theories of class $\mathcal{S}$. Such gauge theories $\mathcal{G}_{\mathfrak{g}}(C)$ are labelled a Lie algebra $\mathfrak{g}$ and a possibly punctured Riemann surface C. Restricting attention to $\mathfrak{g} = \mathfrak{sl}_2$ allows us to abbreviate to $\mathcal{G}(C)$.

There are various extensions that have been proposed and sometimes proven by now, obtained by introducing defects of various types.

Of particular interest are partition functions on squashed four-spheres $E^4_{\epsilon_1\epsilon_2}$

$$x_0^2 + \epsilon_1^2|w_1|^2 + \epsilon_2^2|w_2|^2 = 1, \qquad w_1 = x_1 + \mathrm{i}x_2, \quad w_2 = x_3 + \mathrm{i}x_4.$$

Based on important work of V. Pestun, AGT predicted that[18]

$$\mathcal{Z}_{\mathcal{G}(C)}(E^4_{\epsilon_1\epsilon_2}) \simeq \left\langle \, \prod_{r=1}^{n} V_{\alpha_r}(z_r) \, \right\rangle_{\mathrm{L}},$$

assuming a suitable dictionary between the parameters involved.

[18] $\simeq$ means equality up to a boring factor.

Gauge theory interpretation II

Interesting generalisations are concerned with certain defects. Surface defects with support on the ellipsoids $E^2_{\epsilon_1} = \{x \in E^4_{\epsilon_1\epsilon_2}; w_2 = 0\}$ or $\check{E}^2_{\epsilon_2} = \{x \in E^4_{\epsilon_1\epsilon_2}; w_1 = 0\}$ will be denoted by D_{ϵ_1} and $\check{D}_{\epsilon_2}$, respectively.

Surface defect type I - points on C: There is evidence for the following conjecture:

$$\mathcal{Z}_{\mathcal{G}(C)}\big(E^4_{\epsilon_1\epsilon_2}, D^I_{\epsilon_1}(\mathbf{u}), \check{D}^I_{\epsilon_2}(\mathbf{v})\big) \simeq \Big\langle \prod_{k=1}^{n} V_{\alpha_k}(z_k) \prod_{i=1}^{l} V_{-1/2b}(u_i) \prod_{j=1}^{m} V_{-b/2}(v_j) \Big\rangle_{\mathrm{L}}$$

Surface defect type II - wrapped on C: There is evidence[19] for

$$\mathcal{Z}_{\mathcal{G}(C)}\big(E^4_{\epsilon_1\epsilon_2}, D^{II}_{\epsilon_1}(\mathbf{x})\big) \simeq \Big\langle \prod_{k=1}^{n} \Phi^{j_r}(x_r|z_r) \Big\rangle^{\mathcal{E}_{\mathbf{x}}}_{\mathrm{W}}$$

with parameters $\mathbf{x}$ being related to singular behaviour of gauge fields near $E^2_{\epsilon_1}$.

Line defects: Supports can be $S^1_{\epsilon_1} = \{x \in E^2_{\epsilon_1}; x_0 = 0\}$ or $\check{S}^1_{\epsilon_2} = \{x \in \check{E}^2_{\epsilon_2}; x_0 = 0\}$. Such line defects are represented by Verlinde line operators in Liouville theory.[20]

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[19] Alday-Tachikawa; Kozcaz-Pasquetti-Wyllard; Nawata; Frenkel-Gukov-J.T.; Nekrasov; Nekrasov-Tsymbaliuk;....

[20] Alday-Gaiotto-Gukov-Tachikawa-Verlinde; Drukker-Gomis-Okuda-J.T.; Gomis-Okuda-Pestun; Ito-Okuda-Taki,....

Gauge theory interpretation III

Proposal: It makes sense to consider simultaneously surface defects $D^{II}_{\epsilon_1}(\mathbf{x})$ on $E^2_{\epsilon_1}$ and line defects $\check{L}_{\epsilon_2}(\Lambda)$ on $\check{S}^1_{\epsilon_2}$ – no intersection of supports. The known results immediately suggest the following conjecture:

Conjecture: The partition functions $\mathcal{Z}_{\mathcal{G}(C)}\big(E^4_{\epsilon_1\epsilon_2}, D^{II}_{\epsilon_1}(\mathbf{x}), \check{L}_{\epsilon_2}(\Lambda)\big)$ are

$$\mathcal{Z}_{\mathcal{G}(C)}\big(E^4_{\epsilon_1\epsilon_2}, D^{II}_{\epsilon_1}(\mathbf{x}), \check{L}_{\epsilon_2}(\Lambda)\big) \simeq \left\langle \prod_{k=1}^{n} \Phi^{j_r}(x_r|z_r) \right\rangle^{\mathcal{E}_{\mathbf{x}}}_{\mathrm{W},\Lambda}$$

where $\langle \dots \rangle^{\mathcal{E}_{\mathbf{x}}}_{\mathrm{W},\Lambda}$ is obtained from $\langle \dots \rangle^{\mathcal{E}_{\mathbf{x}}}_{\mathrm{W}}$ by the action of electric Verlinde line operators.

Gauge theory interpretation IV

The emerging picture looks particularly appealing from the point of view of the Nekrasov-Shatashvili program relating SUSY-QFTs to integrable models:

Analytic Langlands	**Integrable models**	**SUSY QFT**
Uniformising oper	Ground state	Surface defect $D^{II}_{\epsilon_1}$ only
$\mathcal{D}$-modules	Hamiltonians	Local observables
Hecke operators	Q-operators	Surface defect $D^{I}_{\epsilon_1}$
Grafting	**Creation operators**	**Line defects on** $\check{S}^1_{\epsilon_2}$
Potential for opers	**Yang-Yang-function**	**Free energy** $\mathcal{F}_{\epsilon_1}(\mathbf{z};\Lambda)$,

where $\mathcal{F}_{\epsilon_1}(\mathbf{z};\Lambda) = \lim_{\epsilon_2\to 0} \epsilon_2 \log \mathcal{Z}_{\mathcal{G}(C)}(E^4_{\epsilon_1\epsilon_2}, \check{L}_{\epsilon_2}(\Lambda))$

The integrable structures of $\mathcal{N}=2$, $d=4$ SUSY QFTs becomes fully visible through the **interplay** of various defect observables.