Lecture 2: Quantum state reduction and feedback stabilization

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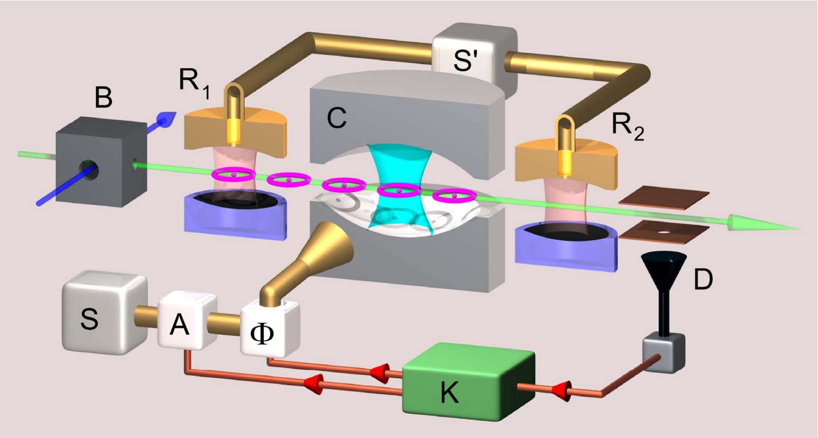
The closed-loop QED experiment[2]

[1]



Sampling time ($\sim$ 100 μs) long enough for feedback computations.

[1] Courtesy of Igor Dotsenko

[2] C. Sayrin et al., Nature, 1-September 2011

LKB photon box: Controlled Markov chain

- $\rho_{k+1} = \frac{D(\alpha_k) M_{\mu_k} \rho_k M_{\mu_k}^\dagger D(-\alpha_k)}{\mathrm{tr}(M_{\mu_k} \rho_k M_{\mu_k}^\dagger)}$ with $\mu_k = g$ or e.

- Maximum of photon number in the cavity is $n^{\max}$

- $M_g = \cos(\frac{\phi_R + \phi}{2} + N\phi)$ and $M_e = \sin(\frac{\phi_R + \phi}{2} + N\phi)$, where $N = a^\dagger a$ is the photon number operator, $N = (\mathrm{diag}(n))_{0 \leq n \leq n^{\max}}$, a is an upper-triagular matrix with $(\sqrt{n})_{1 \leq n \leq n^{\max}}$ as upper diagonal

- $D(\alpha_k)$ is the displacement operator describing the coherent pulse injection $D(\alpha_k) = \exp(\alpha_k(a^\dagger - a))$ with α_k a real parameter corresponding to the control input

Remark. Measurement operators are diagonal in the basis $|n\rangle \langle n|$, i.e., Quantum Non-Demolition (QND) measurements

Feedback stabilization

Aim. Stabilize a particular photon number state $\bar{\rho} = |\bar{n}\rangle \langle\bar{n}|$.

Define $V(\rho_k) = 1 - \mathrm{tr}(\bar{\rho}\rho_k)$.

Theorem (Mirrahimi, Dotsenko, Rouchon, 2009). Take the feedback controller

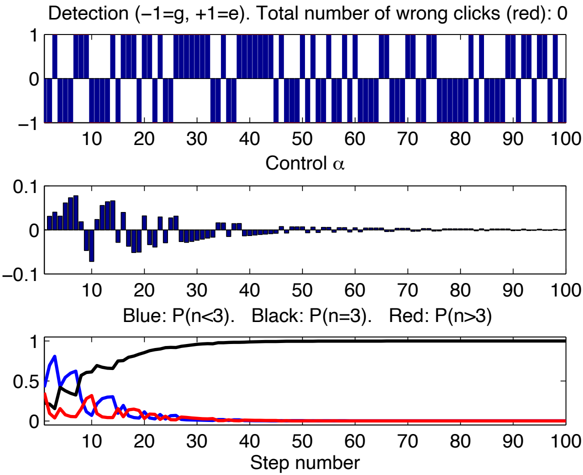
$$u_n = \begin{cases} \underset{\alpha\in[-\bar{\alpha},\bar{\alpha}]}{\mathrm{argmax}}\ \mathrm{tr}\left(\bar{\rho}D(\alpha)\rho_{k+1/2}D(-\alpha)\right) & \text{if } V(\rho) > 1-\epsilon \\ c_1\ \mathrm{tr}\left([\bar{\rho}, a^\dagger - a]\rho_{k+1/2}\right) & \text{if } V(\rho) \leq 1-\epsilon \end{cases}$$

Assume that for all $n \in \{0, \cdots, n^{\max}\}$, we have that $\frac{\phi_R+\phi}{2} + n\phi \neq 0 \mod \pi/2$. For small enough $c_1 > 0$ and $\epsilon > 0$, the trajectories converge almost surely towards the target Fock state $\bar{\rho}$.

Sketch of the proof

- Step1: the trajectories starting within the set $\{\rho|\, V(\rho) > 1-\epsilon\}$ reach in one step the set $\{\rho|\, V(\rho) \leq 1-2\epsilon\}$ with probability one
- Step 2: the trajectories starting within the set $\{\rho|\, V(\rho) \leq 1-2\epsilon\}$ will never hit the set $\{\rho|\, V(\rho) > 1-\epsilon\}$ with a uniformly non-zero probability $p > 0$
- Step 3: quantum trajectories converge toward the state $\bar{\rho}$

Monte-Carlo simulations



A better convergence

The Markov model

The random evolution of the state ρ_k at time step k,

$$\rho_{k+1} = \mathbb{U}_{u_k}(\mathbb{M}_{\mu_k}(\rho_k))$$

- Consider a finite dimensional quantum system (the Hilbert space $\mathcal{H} \in \mathbb{C}^d$ is of dimension $d > 0$) and $\mathcal{D}(\mathcal{H}) := \{\rho \in \mathbb{C}^{d\times d} \mid \quad \rho = \rho^\dagger, \quad \text{tr}\,(\rho) = 1, \quad \rho \geq 0\}$,
- $u_k \in \mathbb{R}$ is the control at step k,
- μ_k is a random variable taking values μ in $\{1, \cdots, m\}$ with probability $p_{\mu,\rho_k} = \text{tr}\left(M_\mu \rho_k M_\mu^\dagger\right)$,

- The set of Kraus operator satisfies the constraint $\sum_{\mu=1}^{m} M_\mu^\dagger M_\mu = I$,
- $\mathbb{U}_u$ is the super-operator $\mathbb{U}_u : \quad \mathcal{D}(\mathcal{H}) \ni \rho \mapsto U_u \rho U_u^\dagger \in \mathcal{D}(\mathcal{H})$ where $U_u = \exp(-iuH)$, where H is an Hermitian operator $H \in \mathbb{C}^{d\times d}$ with $H^\dagger = H$,
- For each μ, $\mathbb{M}_\mu$ is the super-operator $\mathbb{M}_\mu : \quad \rho \mapsto \frac{M_\mu \rho M_\mu^\dagger}{\mathrm{tr}\left(M_\mu \rho M_\mu^\dagger\right)} \in \mathcal{D}(\mathcal{H})$ defined for $\rho \in \mathcal{D}(\mathcal{H})$ such that $p_{\mu,\rho} = \mathrm{tr}\left(M_\mu \rho M_\mu^\dagger\right) \neq 0$.

Assumption 1 (QND measurement)

The measurement operators M_μ are diagonal in the same ortho-normal basis $\{ |n\rangle \mid \quad n \in \{1, \cdots, d\}\}$, therefore $M_\mu = \sum_{n=1}^{d} c_{\mu,n} |n\rangle \langle n|$ with $c_{\mu,n} \in \mathbb{C}$.

Since $\sum_{\mu=1}^{m} M_\mu^\dagger M_\mu = I$, we have $\sum_{\mu=1}^{m} |c_{\mu,n}|^2 = 1$ for all $n \in \{1, \cdots, d\}$.

Assumption 2

For all $n_1 \neq n_2$ in $\{1, \cdots, d\}$, there exists a $\mu \in \{1, \cdots, m\}$ such that $|c_{\mu,n_1}|^2 \neq |c_{\mu,n_2}|^2$.

A fundamental theorm

Theorem (Kushner, 1971). Let X_k be a Markov chain on the compact state space S. Suppose, there exists a non-negative function $V(X)$ satisfying

$$\mathbb{E}\left(V(X_{k+1})|X_k\right) - V(X_k) = Q(X_k),$$

where $Q(X)$ is a positif continuous function of X, then the ω-limit set Ω (in the sense of almost sure convergence) of X_k is included in the following set

$$I := \{X| \quad Q(X) = 0\}.$$

Convergence of the open loop dynamics

When $u_k = 0, \quad \forall k$, the dynamics is simply given by

$$\rho_{k+1} = \mathbb{M}_{\mu_k}(\rho_k) = \frac{M_\mu \rho_k M_\mu^\dagger}{\mathrm{tr}\left(M_\mu \rho_k M_\mu^\dagger\right)},$$

with the operator $M_\mu = \mathrm{diag}(c_{\mu,n})_{1 \leq n \leq d}$.

Theorem (A., Rouchon, Mirrahimi, 2011) Consider a Markov process ρ_k obeying the dynamics given in above with an initial condition ρ_0 in $\mathcal{D}(\mathcal{H})$. Then

- with probability one, ρ_k converges to one of the d states $|n\rangle \langle n|$ with $n \in \{1, \cdots, d\}$.
- the probability of convergence towards the state $|n\rangle \langle n|$ depends only on the initial condition ρ_0 and is given by

$$\mathrm{tr}\left(\rho_0 |n\rangle \langle n|\right) = \langle n| \rho_0 |n\rangle .$$

Elements of a proof:

- Step 1. Taking the following Lyapunov function

$$V(\rho) := -\sum_{n=1}^{d} f(\operatorname{tr}(|n\rangle\langle n|\rho)), \quad \text{where} \quad f(x) = \frac{x^2}{2}.$$

- Step 2. Show that $\operatorname{tr}(|n\rangle\langle n|\rho_{k+1})$ is a martingale.

$$\mathbb{E}\left(\operatorname{tr}(|n\rangle\langle n|\rho_{k+1})\,|\rho_k\right) = \operatorname{tr}(|n\rangle\langle n|\rho_k).$$

- Step 3. The function $-f$ being concave, thus $V(\rho)$ is a super-martingale

$$\mathbb{E}\left(V(\rho_{k+1})|\rho_k\right) \leq V(\rho_k).$$

More precisely, we obtain

$$\mathbb{E}\left(V(\rho_{k+1})|\rho_k\right) - V(\rho_k) =$$
$$-\frac{1}{4}\sum_{n=1}^{d}\sum_{\mu,\nu}\mathrm{tr}\left(M_\mu\rho_k M_\mu^\dagger\right)\mathrm{tr}\left(M_\nu\rho_k M_\nu^\dagger\right)\times$$
$$\left(\frac{|C_{\mu,n}|^2\langle n|\rho_k|n\rangle}{\mathrm{tr}\left(M_\mu\rho_k M_\mu^\dagger\right)} - \frac{|C_{\nu,n}|^2\langle n|\rho_k|n\rangle}{\mathrm{tr}\left(M_\nu\rho_k M_\nu^\dagger\right)}\right)^2 := Q_1(\rho_k).$$

The ω-limit set Ω is a subset of the following set

$$\{\rho_\infty| \quad \frac{|C_{\mu,n}|^2\langle n|\rho_\infty|n\rangle}{\mathrm{tr}\left(M_\mu\rho_\infty M_\mu^\dagger\right)} - \frac{|C_{\nu,n}|^2\langle n|\rho_\infty|n\rangle}{\mathrm{tr}\left(M_\nu\rho_\infty M_\nu^\dagger\right)} = 0\},$$

$\Longrightarrow$ Assume $\exists\bar{n}_1 \neq \bar{n}_2 \in \{1,\ldots,d\}$: $\langle\bar{n}_1|\,\rho_\infty\,|\bar{n}_1\rangle > 0$ and $\langle\bar{n}_2|\,\rho_\infty\,|\bar{n}_2\rangle > 0$. Then

$$\forall\mu, \quad |c_{\mu,\bar{n}_1}|^2 = |c_{\mu,\bar{n}_2}|^2.$$

By Assumption 2, $\exists \bar{\mu} \in \{1, \cdots, m\} : |c_{\bar{\mu},\bar{n}_1}|^2 \neq |c_{\bar{\mu},\bar{n}_2}|^2$. Then there exists a unique $\bar{n}$ such that $\langle \bar{n} | \rho_\infty | \bar{n} \rangle \neq 0$.

- Step 4. The probability measure of the random variable ρ_k converges to $\sum_{n=1}^{d} p_n \delta(|n\rangle \langle n|)$, where,
  - $\delta(|n\rangle \langle n|)$ denotes the dirac distribution at $|n\rangle \langle n|$.
  - p_n is the probability of convergence towards $|n\rangle \langle n|$.

  In particular,

$$\mathbb{E}\left(\operatorname{tr}\left(\rho_k |n\rangle \langle n|\right)\right) \longrightarrow p_n.$$

  But $\operatorname{tr}\left(|n\rangle \langle n| \rho_k\right)$ is a martingale and $\mathbb{E}\left(\operatorname{tr}\left(|n\rangle \langle n| \rho_k\right)\right) = \mathbb{E}\left(\operatorname{tr}\left(|n\rangle \langle n| \rho_0\right)\right)$ Then $p_n = \langle n | \rho_0 | n \rangle$.

Feedback Stabilization

Recall:

$$\rho_{k+1} = \mathbb{U}_{u_k}(\mathbb{M}_{\mu_k}(\rho_k)).$$

Goal. Design a feedback law that globally stabilizes the Markov chain ρ towards a target state $|\bar{n}\rangle \langle \bar{n}|$ with $\bar{n} \in \{1, \cdots, d\}$.

Astuce. Construction of a strict Lyapunov function which is based on the connectivity of the graph attached to H and inverting a Laplacian matrix derived from H.

Lemma (A., Rouchon, Mirrahimi, 2011). Consider the $d \times d$ real matrix R^H, defined by

$$R^H_{n_1,n_2} = 2|\langle n_1|\, H\, |n_2\rangle|^2 - 2\delta_{n_1,n_2} \langle n_1|\, H^2\, |n_2\rangle \, .$$

Then R^H is a Laplacian matrix. Assume the graph G^H of the Laplacian matrix R^H is connected, Then,
for any positive reals λ_n, $n \in \{1, \ldots, d\}$, $n \neq \bar{n}$, there exists a vector $\sigma = (\sigma_n)_{n \in \{1,\ldots,d\}}$ of $\mathbb{R}^d$ such that $R^H \sigma = \lambda$ where λ is the vector of $\mathbb{R}^d$ of components λ_n for $n \neq \bar{n}$ and $\lambda_{\bar{n}} = -\sum_{n \neq \bar{n}} \lambda_n$.

Now choose σ by $R^H\sigma = \lambda$ and construct the function $W_0(\rho) = \sum_{n=1}^{d} \sigma_n \langle n| \rho |n\rangle$, we have

$$\left.\frac{\partial^2 W_0\left(\mathbb{U}_u(|n\rangle\langle n|)\right)}{\partial u^2}\right|_{u=0} = \sum_{l=1}^{d} \sigma_l \,\mathrm{tr}\left([H, |n\rangle\langle n|][H, |l\rangle\langle l|]\right).$$

Since,

$$\mathrm{tr}\left([H, |n\rangle\langle n|][H, |l\rangle\langle l|]\right) = R^H_{n,l}$$

Thus

$$\left.\frac{\partial^2 W_0\left(\mathbb{U}_u(|n\rangle\langle n|)\right)}{\partial u^2}\right|_{u=0} = \sum_{l=1}^{d} R^H_{n,l}\sigma_l = \lambda_n.$$

Then W_0 is convex for $n \neq \bar{n}$ and concave for $n = \bar{n}$.

The global stabilizing feedback

Consider the controlled Markov chain $\rho_{k+1} = \mathbb{U}_{u_k}(\mathbb{M}_{\mu_k}(\rho_k))$. Construct $W_\epsilon(\rho) = \sum_{n=1}^{d} \left(\sigma_n \langle n| \rho |n\rangle + \epsilon \left(\langle n| \rho |n\rangle\right)^2\right)$, where the parameter $\epsilon > 0$ should be not too large to ensure that $\forall n \in \{1, \ldots, d\}/\{\bar{n}\},\ \lambda_n + \epsilon \left(\left(\langle n| H |n\rangle\right)^2 - \langle n| H^2 |n\rangle\right) > 0.$

Theorem (A., Rouchon, Mirrahimi, 2011) Denote by $\rho_{k+\frac{1}{2}} = \mathbb{M}_{\mu_k}(\rho_k)$. Take $\bar{u} > 0$ and consider the following feedback law

$$u_k = K(\rho_{k+\frac{1}{2}}) = \underset{u \in [-\bar{u}, \bar{u}]}{\operatorname{argmax}} \Big(W_\epsilon\big(\mathbb{U}_u\big(\rho_{k+\frac{1}{2}}\big)\big)\Big),$$

Then, for any $\rho_0 \in \mathcal{D}(\mathcal{H})$, the closed-loop trajectory ρ_k converges almost surely to the pure state $|\bar{n}\rangle \langle \bar{n}|$.

Elements of a proof

$$\mathbb{E}\left(W_\epsilon(\rho_{k+1})|\rho_k\right) - W_\epsilon(\rho_k) := Q_1(\rho_k) + Q_2(\rho_k).$$

With

$$Q_1(\rho_k) := \sum_\mu p_{\mu,\rho_k}\Big(W_\epsilon\left(\mathbb{M}_\mu(\rho_k)\right) - W_\epsilon(\rho_k)\Big),$$

and

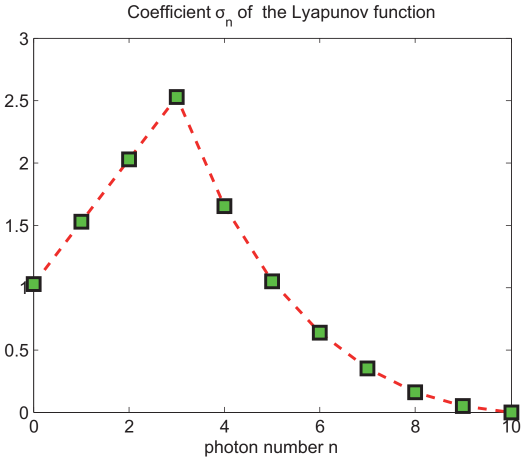
$$Q_2(\rho_k) := \sum_\mu p_{\mu,\rho_k}\Big(\max_{u\in[-\bar{u},\bar{u}]}\Big(W_\epsilon\left(\mathbb{U}_u(\mathbb{M}_\mu(\rho_k))\right)\Big) - W_\epsilon\left(\mathbb{M}_\mu(\rho_k)\right)\Big).$$

These functions are both positive continuous functions of ρ_k.
Then, the ω-limit set Ω is included in the following set

$$\{\rho \in \mathcal{D}(\mathcal{H})|\ Q_1(\rho) = 0\} \ \cap\ \{\rho \in \mathcal{D}(\mathcal{H})|\ Q_2(\rho) = 0\}.$$

- $Q_1(\rho) = 0 \Longrightarrow \rho = |n\rangle\langle n|$
- $(Q_1(\rho) = 0) + (Q_2(\rho) = 0) \Longrightarrow \rho = |\bar{n}\rangle\langle\bar{n}|.$

The control Lyapunov function used for the LKB photon box



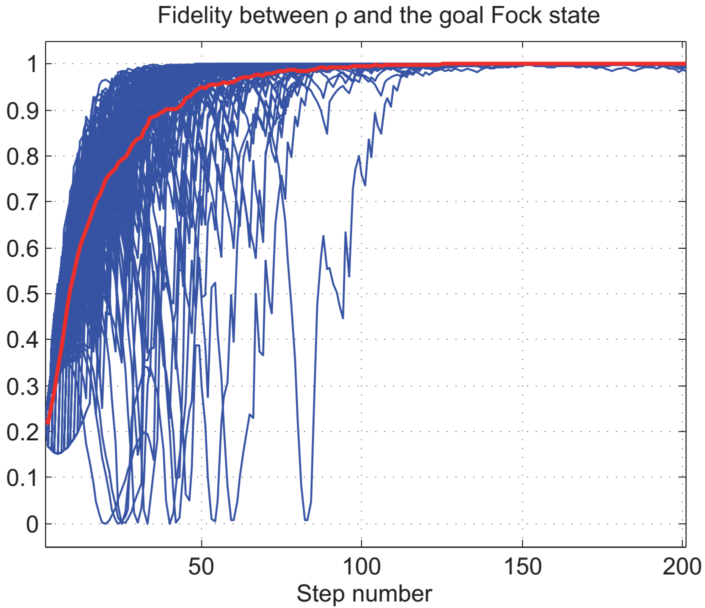
$$W_\epsilon(\rho) = \sum_{n=0}^{9} \left(\epsilon \langle n | \rho | n \rangle^2 + \sigma_n \langle n | \rho | n \rangle \right) \text{ with } \epsilon = 0 \text{ in fact}$$

LKB photon box

- Here $\mathcal{H} = \mathbb{C}^{n^{\max}+1}$
- Take θ/π irrational (then assumption there exists μ such that $|c_{\mu,n_1}|^2 \neq |c_{\mu,n_2}|^2$ is verified)
- The Hamiltonian is $H = i(a^\dagger - a)$
- The graphs G^H is connected (and consequently R^H)
- $R^H_{nn} = 4n + 2$, $R^H_{n-1,n} = -2n$, and $R^H_{n+1,n} = -2n - 2$
- Take $\bar{n} = 3$
- Take $\lambda_n = 1$ for any $n \neq \bar{n}$
- Take $\epsilon < \frac{1}{2n^{\max}+1}$
- For u small enough, by Baker-Campbell-Hausdorff:

$$\mathbb{U}_u(\rho) = \rho - iu[H,\rho] - u^2[H,[H,\rho]] + \mathcal{O}(|u|^3).$$

Convergence toward the goal state $|3\rangle\langle 3|$



LKB photon box: imperfections [3]

$$\rho_{n+1} = \frac{\mathbf{\Phi}_{i_n}(\rho_n)}{\mathrm{tr}\left(\mathbf{\Phi}_{i_n}(\rho_n)\right)}$$

with $\mathbf{\Phi}_i(\rho) = \sum_{i\in\mathcal{Y}} \eta_{i,j} V_j \rho V_j^\dagger$. The elements η_{ij} of the correlation matrix for $i, j \in \{no, g, e, gg, ge, ee\}$ is

$i\backslash j$	no	g	e	gg	ee	ge or eg
no	1	$1-\varepsilon_d$	$1-\varepsilon_d$	$(1-\varepsilon_d)^2$	$(1-\varepsilon_d)^2$	$(1-\varepsilon_d)^2$
g	0	$\varepsilon_d(1-\eta_g)$	$\varepsilon_d\eta_e$	$2\varepsilon_d(1-\varepsilon_d)(1-\eta_g)$	$2\varepsilon_d(1-\varepsilon_d)\eta_e$	$\varepsilon_d(1-\varepsilon_d)(1-\eta_g+\eta_e)$
e	0	$\varepsilon_d\eta_g$	$\varepsilon_d(1-\eta_e)$	$2\varepsilon_d(1-\varepsilon_d)\eta_g$	$2\varepsilon_d(1-\varepsilon_d)(1-\eta_e)$	$\varepsilon_d(1-\varepsilon_d)(1-\eta_e+\eta_g)$
gg	0	0	0	$\varepsilon_d^2(1-\eta_g)^2$	$\varepsilon_d^2\eta_e^2$	$\varepsilon_d^2\eta_e(1-\eta_g)$
ge	0	0	0	$2\varepsilon_d^2\eta_g(1-\eta_g)$	$2\varepsilon_d^2\eta_e(1-\eta_e)$	$\varepsilon_d^2((1-\eta_g)(1-\eta_e)+\eta_g\eta_e)$
ee	0	0	0	$\varepsilon_d^2\eta_g^2$	$\varepsilon_d^2(1-\eta_e)^2$	$\varepsilon_d^2\eta_g(1-\eta_e)$

$$V_{no} = \sqrt{p_0}\, I, \quad V_g = \sqrt{p_1}\cos\phi_N, \quad V_e = \sqrt{p_1}\sin\phi_N$$

$$V_{gg} = \sqrt{p_2}\cos^2\phi_N, \quad V_{ge} = V_{eg} = \sqrt{p_2}\cos\phi_N\sin\phi_N,$$

$$V_{ee} = \sqrt{p_2}\sin^2\phi_N,$$

[3] See the proof of robustness in Amini et al., Automatica, 2013.

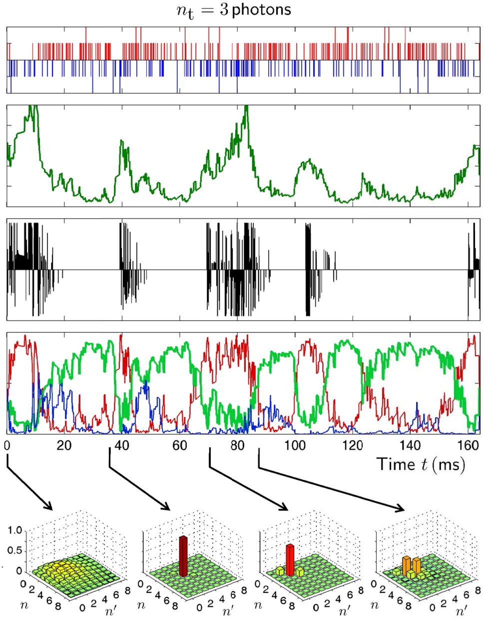
Experimental data

An open-loop trajectory starting from coherent state with an average of 3 photons relaxes towards vacuum (decoherence due to finite photon life time around 70 ms) and a closed-loop trajectory

Detection efficiency 40%
Detection error rate 10%
Delay 4 sampling periods

Truncation to 9 photons

Stabilization around 3-photon state



Thank you !