3-point K-theoretic Gromov-Witten invariants and the quantum Bruhat graph

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Basic Notation

G : a connected, simply-connected, simple algebraic grp/$\mathbb{C}$

T : a maximal torus of G

B : a Borel subgroup of G containing T

$W = \langle s_i \mid i \in I \rangle$: Weyl group, s_i : simple reflection

$Q^{\vee,+} := \sum_{j \in I} \mathbb{Z}_{\geq 0} \alpha_j^{\vee}$, $\{\alpha_i^{\vee}\}_{i \in I}$: the simple coroots

$X := G/B$: the (full) flag manifold

$K_T(X)$: T-equiv. K-theory ring of X

$\mathcal{O}_w$, $\mathcal{O}^w \in K_T(X)$ for $w \in W$:

Schubert and opposite Schubert class

with $\dim \mathcal{O}_w = \operatorname{codim} \mathcal{O}^w = \ell(w)$ (the length of w)

Main Theorem (Conjectured by Buch-Mihalcea)

Let $i \in I$, $u, v \in W$, and $d = \sum_{j \in I} d_j \alpha_j^\vee \in Q^{\vee,+}$.

1. We have

$$\underbrace{\langle \mathcal{O}^{s_i}, \mathcal{O}^u, \mathcal{O}_v \rangle_d}_{\substack{\text{3-pt (T-equiv)} \\ \text{K-theoretic} \\ \text{Gromov-Witten (KGW)} \\ \text{invariant}}} = \underbrace{\langle \mathcal{O}^u, \mathcal{O}_v \rangle_d}_{\substack{\text{2-pt (T-equiv)} \\ \text{KGW invariant}}} - \underbrace{\mathsf{C}(i; u, v, d)}_{\substack{\text{Correction Term,} \\ \text{we describe} \\ \text{in terms of} \\ \text{the QBG}}}$$

$$\mathsf{C}(i; u, v, d) := \sum_{\mathbf{p} \in \mathbf{R}^{\lhd}_{u,v,d}} (-1)^{\ell(\mathbf{p})} \mathbf{e}^{-\varpi_i + \mathrm{wt}(\eta_{\mathbf{p}})}$$

2. If $d_i = 0$, then $\langle \mathcal{O}^{s_i}, \mathcal{O}^u, \mathcal{O}_v \rangle_d = \langle \mathcal{O}^{s_i} \bullet \mathcal{O}^u, \mathcal{O}_v \rangle_d$, where $\bullet$ denotes the ordinary product in $K_T(X)$.

Remark

In many cases, $\mathbf{R}^{\lhd}_{u,v,d} = \emptyset$, and hence $\mathsf{C}(i; u, v, d) = 0$.

Let $d = \sum_{j \in I} d_j \alpha_j^{\vee} \in Q^{\vee,+}$ be such that $d_i > 0$.

If $\langle \varpi_i, \theta^{\vee} \rangle = 1$, where ϖ_i is the i-th fundamental weight and $\theta \in \Delta^+$ is the highest root, then $\mathbf{R}^{\lhd}_{u,v,d} = \emptyset$ for all $u, v \in W$.

In particular, if $\mathfrak{g}$ is of type A or C, then $\mathbf{R}^{\lhd}_{u,v,d} = \emptyset$.

K-theoretic Gromov-Witten (KGW) invariant

ϖ_i : the i-th fundamental weight for $i \in I$

$\Lambda := \bigoplus_{i \in I} \mathbb{Z}\varpi_i$: the integral weight lattice

♠ We identify the representation ring $R(T)$ of T with the group algebra $\mathbb{Z}[\Lambda] = \{\mathbf{e}^\nu \mid \nu \in \Lambda\}$.

$K_T(X)$: T-equiv. K-theory ring of X

♣ $K_T(X)$ is a free $R(T)$-module of finite rank.

For $d \in Q^{\vee,+}$ and classes $\gamma_k \in K_T(X)$, $1 \le k \le m$,

$$\overbrace{\langle \gamma_1, \gamma_2, \ldots, \gamma_m \rangle_d}^{m\text{-pt } (T\text{-equiv}) \text{ KGW invariant}}$$
$$:= \chi^T_{\overline{\mathcal{M}}_{0,m}(X,d)}(\mathrm{ev}_1^*(\gamma_1) \cdot \cdots \cdot \mathrm{ev}_m^*(\gamma_m)) \in K_T(\mathrm{pt}) = R(T),$$

$\overline{\mathcal{M}}_{0,m}(X,d)$: the Kontsevich moduli space parametrizing all m-point, genus 0, degree d stable maps to X, which is equipped with evaluation maps $\mathrm{ev}_k : \overline{\mathcal{M}}_{0,m}(X,d) \to X$, $1 \le k \le m$, where ev_k sends a stable map to its image of the k-th marked point in its domain

$\chi^T_{\overline{\mathcal{M}}_{0,m}(X,d)}$: the pushforward along the structure morphism $\overline{\mathcal{M}}_{0,m}(X,d) \to \{\mathrm{pt}\}$

♠ Today I will not use this geometric definition of KGW invariants; I will use a combinatorial description of them in our special case.

Fact

The KGW invariant $\langle \gamma_1, \gamma_2, \ldots, \gamma_m \rangle_d$ is $R(T)$-linear in each argument γ_j.

♣ So we are interested in $R(T)$-bases of $K_T(X)$.

B^- : the Borel subgroup containing T that is opposite to B

For $w \in W$,

$X_w := \overline{BwB/B} \subset X$: Schubert variety
$X^w := \overline{B^- wB/B} \subset X$: opposite Schubert variety

$\mathcal{O}_w := [\mathcal{O}_{X_w}] \in K_T(X)$: Schubert class
$\mathcal{O}^w := [\mathcal{O}_{X^w}] \in K_T(X)$: opposite Schubert class

Fact

1. $\{\mathcal{O}_w\}_{w\in W}$ forms an $R(T)$-basis of $K_T(X)$.
2. $\{\mathcal{O}^w\}_{w\in W}$ forms an $R(T)$-basis of $K_T(X)$.

♣ Fix $i \in I$. We focus on

$$\langle \mathcal{O}^{s_i}, \mathcal{O}^u, \mathcal{O}_v \rangle_d \quad \text{for } u, v \in W \text{ and } d \in Q^{\vee,+},$$

$$\langle \mathcal{O}^u, \mathcal{O}_v \rangle_d \quad \text{for } u, v \in W \text{ and } d \in Q^{\vee,+}.$$

Quantum Bruhat Graph

$\ell : W \to \mathbb{Z}_{\geq 0}$: the length function on W

Δ^+ : the set of positive roots of $\mathfrak{g}$

$\alpha^\vee$: the coroot of $\alpha \in \Delta^+$

$s_\alpha \in W$: the reflection w.r.t. $\alpha \in \Delta^+$

$\rho := (1/2) \sum_{\alpha \in \Delta^+} \alpha$

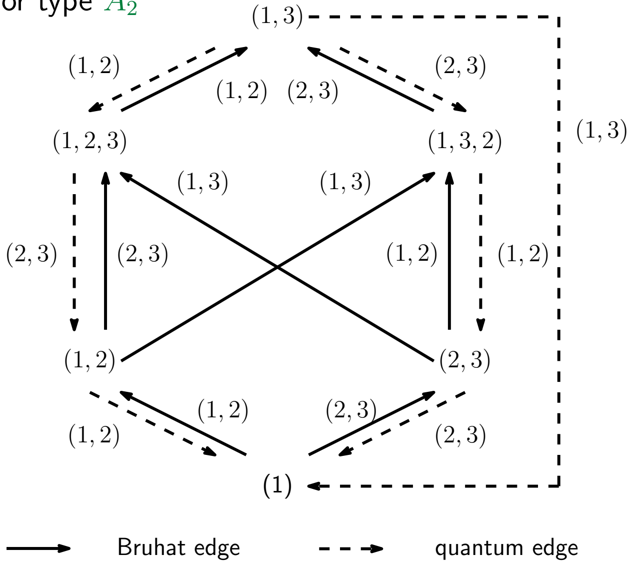
Definition (Quantum Bruhat graph)

The quantum Bruhat graph of W, denoted by $\mathrm{QBG}(W)$, is the Δ^+-labeled directed graph s.t.

- the set of vertices $= W$;
- for $x, y \in W$ and $\alpha \in \Delta^+$, $x \xrightarrow{\alpha} y$ if $y = x s_\alpha$ and
  - (B) $\ell(y) = \ell(x) + 1$, or
  - (Q) $\ell(y) = \ell(x) + 1 - 2\langle \rho, \alpha^\vee \rangle$.

An edge satisfying (B) (resp., (Q)) is called a Bruhat edge (resp., quantum edge).

♣ QBG for type A_2



$w_\circ = s_1 s_2 s_1 = s_2 s_1 s_2$: the longest element of W

For a directed path

$$\mathbf{q} : y_0 \xrightarrow{\beta_1} y_1 \xrightarrow{\beta_2} \cdots \xrightarrow{\beta_r} y_r$$

in $\mathrm{QBG}(W)$, we set

$$\mathrm{qwt}(\mathbf{q}) := \sum_{\substack{1 \le k \le r \\ y_{k-1} \xrightarrow{\beta_k} y_k \text{ is} \\ \text{a quantum edge}}} \beta_k^\vee \in Q^{\vee,+}.$$

Let $v, w \in W$, and let $\mathbf{q}$ be a shortest directed path from v to w in $\mathrm{QBG}(W)$. We set $\mathrm{qwt}(v \Rightarrow w) := \mathrm{qwt}(\mathbf{q})$, which does not depend on the choice of a shortest directed path $\mathbf{q}$.

2-point KGW invariant

Theorem [Buch-Chung-Li-Mihalcea (2020)]

For $u, v \in W$ and $d \in Q^{\vee,+}$,

$$\langle \mathcal{O}^u, \mathcal{O}_v \rangle_d = \begin{cases} 1 & \text{if } d \geq \mathrm{qwt}(u \Rightarrow v), \\ 0 & \text{otherwise,} \end{cases}$$

where for $\xi, \zeta \in Q^{\vee,+}$, we write $\xi \geq \zeta$ if $\xi - \zeta \in Q^{\vee,+}$.

Example

If $\mathfrak{g}$ is of type A_2, then $\mathrm{qwt}(w_\circ \Rightarrow s_1) = \alpha_1^\vee + \alpha_2^\vee$. Hence,

$$\langle \mathcal{O}^{w_\circ}, \mathcal{O}_{s_1} \rangle_0 = \langle \mathcal{O}^{w_\circ}, \mathcal{O}_{s_1} \rangle_{\alpha_1^\vee} = \langle \mathcal{O}^{w_\circ}, \mathcal{O}_{s_1} \rangle_{\alpha_2^\vee} = 0,$$

$$\langle \mathcal{O}^{w_\circ}, \mathcal{O}_{s_1} \rangle_{\alpha_1^\vee + \alpha_2^\vee} = 1.$$

3-point KGW invariant

$R(T)[\![\mathsf{Q}]\!]$: the ring of formal power series with coefficients in $R(T)$ in the (Novikov) variables $\mathsf{Q}_j := \mathsf{Q}^{\alpha_j^\vee}$ for $j \in I$

$QK_T(X) = K_T(X) \otimes_{R(T)} R(T)[\![\mathsf{Q}]\!]$: the (small) T-equiv quantum K-theory ring defined by Givental and Lee

We define the ($R(T)[\![\mathsf{Q}]\!]$-bilinear) quantum K-metric $(\!(\cdot,\,\cdot)\!) : QK_T(X) \times QK_T(X) \to R(T)[\![\mathsf{Q}]\!]$ by

$$(\!(\gamma_1,\,\gamma_2)\!) = \sum_{\xi \in Q^{\vee,+}} \mathsf{Q}^{\xi} \langle \gamma_1, \gamma_2 \rangle_{\xi} \quad \text{for } \gamma_1, \gamma_2 \in K_T(X), \qquad \text{(A)}$$

where $\mathsf{Q}^{\xi} := \prod_{j \in I} \mathsf{Q}_j^{\xi_j}$ for $\xi = \sum_{j \in I} \xi_j \alpha_j^\vee \in Q^{\vee,+}$.

The quantum product $\star$ in $QK_T(X)$ is determined by:

$$(\!(\gamma_1 \star \gamma_2,\ \gamma_3)\!) = \sum_{\zeta \in Q^{\vee,+}} \mathsf{Q}^{\zeta} \langle \gamma_1, \gamma_2, \gamma_3 \rangle_{\zeta} \qquad \text{(B)}$$

for $\gamma_1, \gamma_2, \gamma_3 \in K_T(X)$.

Fact

The quantum product $\star$ is $R(T)[\![\mathsf{Q}]\!]$-bilinear, commutative, and associative, with $1 \in K_T(X)$ the multiplicative identity.

Let $i \in I$, and $u \in W$.

For $z \in W$, we define $a^{z}_{s_i,u}(\mathsf{Q}) \in R(T)[\![\mathsf{Q}]\!]$ by:

$$\mathcal{O}^{s_i} \star \mathcal{O}^{u} = \sum_{z \in W} a^{z}_{s_i,u}(\mathsf{Q}) \mathcal{O}^{z}. \qquad \text{(C)}$$

We have

$$\begin{aligned}
\sum_{\zeta \in Q^{\vee,+}} \mathsf{Q}^{\zeta} \langle \mathcal{O}^{s_i}, \mathcal{O}^{u}, \mathcal{O}_v \rangle_{\zeta} &\overset{\text{(B)}}{=} (\!(\mathcal{O}^{s_i} \star \mathcal{O}^{u},\ \mathcal{O}_v)\!) \\
&\overset{\text{(C)}}{=} \sum_{z \in W} a^{z}_{s_i,u}(\mathsf{Q}) (\!(\mathcal{O}^{z},\ \mathcal{O}_v)\!) \\
&\overset{\text{(A)}}{=} \sum_{z \in W} a^{z}_{s_i,u}(\mathsf{Q}) \sum_{\xi \in Q^{\vee,+}} \mathsf{Q}^{\xi} \langle \mathcal{O}^{z}, \mathcal{O}_v \rangle_{\xi}.
\end{aligned}$$

♣ We describe $a^{z}_{s_i,u}(\mathsf{Q})$ in terms of $\mathrm{QBG}(W)$.

Pieri type formula for $\mathcal{O}^{s_i} \star \mathcal{O}^u$

Fix $i \in I$, and set $J := I \setminus \{i\}$. We set

$$\Delta_J^+ := \Delta^+ \cap \bigoplus_{j \in J} \mathbb{Z}\alpha_j = \{\alpha \in \Delta^+ \mid \langle \varpi_i, \alpha^\vee \rangle = 0\}.$$

Definition

A total order $\lhd$ on Δ^+ is called a reflection order if it satisfies the condition that for every $\alpha, \beta \in \Delta^+$ such that $\alpha + \beta \in \Delta^+$, either of $\alpha \lhd \alpha + \beta \lhd \beta$ or $\beta \lhd \alpha + \beta \lhd \alpha$ holds.

$\exists$ reflection order $\lhd$ on Δ^+ satisfying the condition

$$\gamma \lhd \beta \quad \text{for all } \gamma \in \Delta_J^+ \text{ and } \beta \in \Delta^+ \setminus \Delta_J^+;$$

fix such a reflection order $\lhd$ arbitrarily.

For $w \in W$, let $\mathbf{QBG}_w^{\lhd}$ denote the set of all directed paths $\mathbf{q}$ in $\mathrm{QBG}(W)$ of the following form:

$$\mathbf{q} : \underbrace{w = z_0 \xrightarrow{\beta_1} z_1 \xrightarrow{\beta_2} \cdots \xrightarrow{\beta_s} z_s =: \mathrm{end}(\mathbf{q}),}_{\text{directed path in } \mathrm{QBG}(W)} \quad \text{where}$$

$$\begin{cases} \ell(\mathbf{q}) := s \geq 0, \\ \beta_k \in \Delta^+ \setminus \Delta_J^+ \text{ for } 1 \leq k \leq s, \\ \beta_1 \lhd \beta_2 \lhd \cdots \lhd \beta_s. \end{cases}$$

Definition

Let $a \in \mathbb{Q}$. We define $\mathrm{QBG}_{a\varpi_i}(W)$ to be the subgraph of $\mathrm{QBG}(W)$ such that

- the set of vertices $= W$;
- an edge $x \xrightarrow{\alpha} y$ in $\mathrm{QBG}(W)$ is an edge in $\mathrm{QBG}_{a\varpi_i}(W)$ iff $a\langle \varpi_i, \alpha^\vee \rangle \in \mathbb{Z}$.

♣ Fix $N \in \mathbb{Z}_{>0}$ s.t.

$$\frac{N}{\langle \varpi_i, \alpha^\vee \rangle} \in \mathbb{Z} \quad \text{for all } \alpha \in \Delta^+ \setminus \Delta_J^+. \qquad \text{(N)}$$

Example

1. ϖ_i is minuscule if and only if $N = 1$ satisfies (N).
2. If $\mathfrak{g}$ is of classical type, and ϖ_i is not minuscule, then $N = 2$ satisfies (N).

For $u \in W$,

we define $\mathbf{QLS}_u^{\triangleleft}$ to be the set of all $\mathbf{p} = (\mathbf{q}_N, \dots, \mathbf{q}_2, \mathbf{q}_1)$ satisfying the condition that for each $1 \le k \le N$,

- $\mathbf{q}_k \in \mathbf{QBG}^{\triangleleft}_{\mathrm{end}(\mathbf{q}_{k+1})}$, where $\mathrm{end}(\mathbf{q}_{N+1}) := u$, and
- $\mathbf{q}_k$ is a directed path in $\mathrm{QBG}_{((k-1)/N)\varpi_i}(W)$.

For $\mathbf{p} = (\mathbf{q}_N, \dots, \mathbf{q}_2, \mathbf{q}_1) \in \mathbf{QLS}_u^{\triangleleft}$, we set

$$\mathrm{wt}(\eta_{\mathbf{p}}) := \frac{1}{N} \sum_{k=2}^{N+1} \mathrm{end}(\mathbf{q}_k)\varpi_i \in \Lambda, \quad \mathrm{qwt}(\mathbf{p}) := \sum_{k=1}^{N} \mathrm{qwt}(\mathbf{q}_k)$$

$$\ell(\mathbf{p}) := \sum_{k=1}^{N} \ell(\mathbf{q}_k), \qquad \mathrm{end}(\mathbf{p}) := \mathrm{end}(\mathbf{q}_1).$$

Theorem (Naito-Orr-S, Lenart-Naito-S)

For $z \in W$, we have

$$a^{z}_{s_i,u}(\mathsf{Q}) = \delta_{z,u} - (-1)^{\ell(z)-\ell(u)} \sum_{\substack{\mathbf{p}\in\mathbf{QLS}^{\lhd}_{u} \\ \mathrm{end}(\mathbf{p})=z}} \mathbf{e}^{-\varpi_i+\mathrm{wt}(\eta_{\mathbf{p}})}\mathsf{Q}^{\mathrm{qwt}(\mathbf{p})}.$$

In particular, $a^{z}_{s_i,u}(\mathsf{Q}) \in R(T)[\mathsf{Q}]$.

Recall that

$$\sum_{\zeta\in Q^{\vee,+}} \mathsf{Q}^{\zeta}\langle \mathcal{O}^{s_i}, \mathcal{O}^{u}, \mathcal{O}_{v}\rangle_{\zeta} = \sum_{z\in W} a^{z}_{s_i,u}(\mathsf{Q}) \sum_{\xi\in Q^{\vee,+}} \mathsf{Q}^{\xi}\langle \mathcal{O}^{z}, \mathcal{O}_{v}\rangle_{\xi},$$

By combining these formulas and comparing the coefficients of Q^{d} on both sides, we obtain ...

$$\langle \mathcal{O}^{s_i}, \mathcal{O}^u, \mathcal{O}_v \rangle_d$$

$$= \langle \mathcal{O}^u, \mathcal{O}_v \rangle_d - \sum_{\substack{\mathbf{p} \in \mathbf{QLS}_u^{\lhd} \\ \mathrm{qwt}(\mathbf{p}) \le d}} (-1)^{\ell(\mathbf{p})} \mathbf{e}^{-\varpi_i + \mathrm{wt}(\eta_{\mathbf{p}})} \langle \mathcal{O}^{\mathrm{end}(\mathbf{p})}, \mathcal{O}_v \rangle_{d - \mathrm{qwt}(\mathbf{p})}$$

$$= \langle \mathcal{O}^u, \mathcal{O}_v \rangle_d - \sum_{\substack{\mathbf{p} \in \mathbf{QLS}_u^{\lhd} \\ \mathrm{qwt}(\mathrm{end}(\mathbf{p}) \Rightarrow v) \le d - \mathrm{qwt}(\mathbf{p})}} (-1)^{\ell(\mathbf{p})} \mathbf{e}^{-\varpi_i + \mathrm{wt}(\eta_{\mathbf{p}})}$$

$$= \langle \mathcal{O}^u, \mathcal{O}_v \rangle_d - \sum_{\mathbf{p} \in \mathbf{QLS}_{u,v,d}^{\lhd}} (-1)^{\ell(\mathbf{p})} \mathbf{e}^{-\varpi_i + \mathrm{wt}(\eta_{\mathbf{p}})}, \qquad \text{(Z)}$$

where

$$\mathbf{QLS}_{u,v,d}^{\lhd} := \left\{ \mathbf{p} \in \mathbf{QLS}_u^{\lhd} \mid \mathrm{qwt}(\mathrm{end}(\mathbf{p}) \Rightarrow v) \le d - \mathrm{qwt}(\mathbf{p}) \right\}.$$

♠ The rightmost-hand side of (Z) contains cancellations. By removing cancellations, we obtain part (1) of the main theorem.

Main Theorem

Theorem (Lenart-Naito-S-Xu)

Let $i \in I$, $u, v \in W$, and $d \in Q^{\vee,+}$. We have

$$\begin{aligned} &\langle \mathcal{O}^{s_i}, \mathcal{O}^u, \mathcal{O}_v \rangle_d \\ &= \langle \mathcal{O}^u, \mathcal{O}_v \rangle_d - \sum_{\mathbf{p} \in \mathbf{R}^{\lhd}_{u,v,d}} (-1)^{\ell(\mathbf{p})} \mathbf{e}^{-\varpi_i + \mathrm{wt}(\eta_{\mathbf{p}})}, \end{aligned}$$

where $\mathbf{R}^{\lhd}_{u,v,d}$ is the subset of $\mathbf{QLS}^{\lhd}_{u,v,d}$ consisting of all elements $\mathbf{p} = (\mathbf{q}_N, \ldots, \mathbf{q}_2, \mathbf{q}_1) \in \mathbf{QLS}^{\lhd}_{u,v,d}$ which satisfy the conditions that

$$\ell(\mathbf{q}_1) = 0, \quad \mathrm{end}(\mathbf{p}) \in vW_J, \quad \langle \varpi_i,\, d - \mathrm{qwt}(\mathbf{p}) \rangle = 0.$$

♠ For $z \in W$, we define $c^{z}_{s_i,u} \in R(T)$ by

$$\mathcal{O}^{s_i} \bullet \mathcal{O}^{u} = \sum_{z \in W} c^{z}_{s_i,u} \mathcal{O}^{z};$$

recall that $\bullet$ denotes the ordinary product in $K_T(X)$. We know that

$$c^{z}_{s_i,u} = a^{z}_{s_i,u}(\mathsf{Q})|_{\mathsf{Q}=0}. \qquad \text{(X)}$$

For $\mathbf{p} \in \mathbf{QLS}^{\lhd}_{u}$, we see that

$$\mathrm{qwt}(\mathbf{p}) > 0 \iff \mathsf{Q}^{\mathrm{qwt}(\mathbf{p})} \in \mathsf{Q}_i R(T)[\mathsf{Q}] \qquad \text{(Y)}$$

By using the formulas on page 20, together with (X) and (Y), we deduce .....

Theorem (Lenart-Naito-S-Xu)

Let $d = \sum_{j\in I} d_j \alpha_j^\vee \in Q^{\vee,+}$ with $d_i = 0$. We have

$$\langle \mathcal{O}^{s_i}, \mathcal{O}^u, \mathcal{O}_v \rangle_d = \sum_{z\in W} c^z_{s_i,u} \langle \mathcal{O}^z, \mathcal{O}_v \rangle_d = \langle \mathcal{O}^{s_i} \bullet \mathcal{O}^u,\ \mathcal{O}_v \rangle_d.$$

Formulas for Line Bundles

$\mathcal{O}(\lambda) \in K_T(X)$ for $\lambda \in \Lambda$: line bundle over X ass. to λ

Since $\mathcal{O}^{s_i} = 1 - \mathbf{e}^{-\varpi_i}\mathcal{O}(-\varpi_i)$ for $i \in I$, we obtain

Corollary of [LNSX]

Let $i \in I$, $u, v \in W$, and $d \in Q^{\vee,+}$. We have

$$\langle \mathcal{O}(-\varpi_i), \mathcal{O}^u, \mathcal{O}_v \rangle_d = \sum_{\mathbf{p} \in \mathbf{R}^{\lhd}_{u,v,d}} (-1)^{\ell(\mathbf{p})} \mathbf{e}^{\mathrm{wt}(\eta_{\mathbf{p}})}.$$

Fix minuscule $\lambda \in \Lambda$. We set

$$\Delta^+_{-1} := \{\alpha \in \Delta^+ \mid \langle \lambda, \alpha^\vee \rangle = -1\},$$
$$\Delta^+_0 := \{\alpha \in \Delta^+ \mid \langle \lambda, \alpha^\vee \rangle = 0\},$$
$$\Delta^+_1 := \{\alpha \in \Delta^+ \mid \langle \lambda, \alpha^\vee \rangle = 1\}.$$

♠ We fix a reflection order $\prec$ satisfying the condition that $\beta \prec \alpha \prec \gamma$ for all $\beta \in \Delta^+_{-1}$, $\alpha \in \Delta^+_0$, $\gamma \in \Delta^+_1$.

For $w \in W$, we denote by $\mathbf{M}_w$ the set of all directed paths $\mathbf{m}$ in $\mathrm{QBG}(W)$ of the form :

$$\mathbf{m} : \overbrace{w = x_0 \xrightarrow{\beta_1} x_1 \xrightarrow{\beta_2} \cdots \xrightarrow{\beta_s} x_s}^{=:\mathbf{m}_\beta}$$
$$= \underbrace{y_t \xrightarrow{\gamma_t} y_{t-1} \xrightarrow{\gamma_{t-1}} \cdots \xrightarrow{\gamma_1} y_1}_{=:\mathbf{m}_\gamma},$$

with $s, t \geq 0$ and $\beta_1 \prec \cdots \prec \beta_s \prec \gamma_t \prec \cdots \prec \gamma_1$.

Theorem (recent joint work with Naito)

Assume that λ is minuscule. Let $u, w \in W$, and $d \in Q^{\vee,+}$. We have

$$\langle \mathcal{O}(-\lambda), \mathcal{O}^u, \mathcal{O}_v \rangle_d = \sum_{\mathbf{m} \in \mathbf{S}_{u,v,d}} \mathbf{e}^{\mathrm{end}(\mathbf{m}_\beta)\lambda},$$

where $\mathbf{S}_{u,v,d}$ is the subset of $\mathbf{M}_u$ consisting of all elements $\mathbf{m} \in \mathbf{M}_u$ which satisfy the conditions that

$$\mathrm{qwt}(\mathrm{end}(\mathbf{m}) \Rightarrow v) \leq d - \mathrm{qwt}(\mathbf{m}),$$

$$\ell(\mathbf{m}_\gamma) = 0, \quad \iota(\mathrm{end}(\mathbf{m}) \Rightarrow v) \notin \Delta_1^+,$$

$$\langle \varpi_k, \mathrm{qwt}(\mathrm{end}(\mathbf{m}) \Rightarrow v) \rangle \leq \langle \varpi_k, d - \mathrm{qwt}(\mathbf{m}) \rangle$$

for all $k \in I$ s.t. $\langle \lambda, \alpha_k^\vee \rangle = 1$.