Conformal Rep^{ns} & Character

Conformal group in d-dimension is
$SO(d,2) \simeq SO(d+2, \mathbb{C})$
$\mapsto Spin(d+2, \mathbb{C})$
(covering group

* $SO(d)$ is the usual rotation group in Euclidean space $\mathbb{R}^d$, having rank $\lfloor d/2 \rfloor = r$

Rep^{ns} are labelled by r. spin qtm. #s
$\Rightarrow (l_1, \ldots, l_r) = \underline{l}$.

* The extra 2. dim in $SO(d,2)$ increases rank by <u>one unit</u> $\Rightarrow$ another label (qtm #)
Δ (scaling dim, associated w/ dilatation op)

* <u>$SO(d)$ is compact</u> $\Rightarrow$ $\underline{l}$ is quantized

Thus, $l_i \in \frac{1}{2}\mathbb{Z}$.

* Scaling dim. is associated w/ non-compact direction $\Rightarrow \Delta \in \mathbb{R}$ (continuous)

* Irrep. of CA are labelled by $(r+1)$ qtm numbers $\Rightarrow (-\Delta, l_1, \dots, l_r)$

  as we are working w/ $SO(d+2, \mathbb{C})$

Unitary Rep^{ns} of $SO(d+2, \mathbb{C})$

* choose a HW state and filled it by applying lowering ops.

* In addition to raising and lowering ops, $SO(d+2, \mathbb{C})$ has translation generators P_μ (lowering ops) and special conformal gens. K_μ (raising ops $= P_\mu^\dagger$)

* Due to non-compact nature, lowering op (P_μ) can be applied infinitely (w/out annihilating

any lowest weight state)

$\Rightarrow$ explains why unitary rep^{ns} are infinite dimensional.

* Unitary irreps consist of some operator O_ℓ

$\Rightarrow$ primary.

$O_\ell \Rightarrow$ spin-ℓ and scaling dim. Δ.

infinite tower of derivatives acting on O_ℓ

$\Rightarrow$ descendants.

$$R_{[\Delta;\ell]} \sim \begin{bmatrix} O_\ell \\ \partial_{\mu_1} O_\ell \\ \partial_{\mu_1}\partial_{\mu_2} O_\ell \\ \vdots \end{bmatrix}$$

* Rep^n is unitary put constraints on Δ, ℓ

* The conditions on $\underline{\ell}$ for finite dimensional irreps of $SO(d)$:

(i) a unitary irrep is labelled by

$\underline{l} = (l_1, l_2, \ldots, l_r)$, $l_i \in \frac{1}{2}\mathbb{Z}$

and $(l_i - l_{i+1}) \in \mathbb{Z}$ ∧

$l_1 \geq l_2 \geq \cdots \geq l_{r-1} \geq |l_r|$ for $SO(2r)$

Then, for each $\underline{l}$ satisfying these conditions, Δ needs to satisfy a lower bound $\Delta \geq \Delta_l$ for the irrep. to be unitary.

Such, unitary bounds read as

$$\Delta \geq \Delta_l = \begin{cases} (d-2)/2 & \text{for } l = (0, \ldots, 0) \\ (d-1)/2 & \text{for } l = (\frac{1}{2}, \ldots, \frac{1}{2}) \\ l_1 + d - p_l - 1 & \text{for all others } \underline{l}. \end{cases}$$

where, $1 \leq p_l \leq r$ denotes the position of the last component in $l = (l_1, l_2, \ldots, l_r)$ satisfying $|l_1| = |l_2| = \cdots = |l_p| > |l_{p_l+1}|$

* Physically, the unitary bounds impose rep^{ns} to having scaling dim $\geq$ that for free fields or currents.

When a bound is saturated, i.e. $\Delta = \Delta_l$ some descendants obtained by applying ∂_μ is annihilated $\Rightarrow$ leads to null and subsequently negative norm state.

These states need to be removed and the irrep is called short-rep^n.

long rep^{ns} $\Rightarrow$ that are not short

* Consider, $l = (n, 0, \ldots 0)$, $n \in \mathbb{N}$

(traceless symm. tensors w/ n-indices)

unitary bounds

$$\Delta_l = \begin{cases} (d-2)/2 & \text{for } n = 0 \\ n + d - 2 & \text{for } n > 0 \end{cases}$$

* saturation corresponding free scalar
  for $n=0$

* For $n>0$, we have conserved currents

  e.g. $n=1 \Rightarrow j_\mu$

  $n=2 \Rightarrow T_{\mu\nu}$

  $n\geq 3 \Rightarrow$ higher spin conserved current.

* In each cases, some descendants are annihilated by derivative action.

  e.g. Free scalar field $\partial^2\phi=0$ (EOM)

  $T_{\mu\nu}$ conservation $\Rightarrow \partial_\mu T^{\mu\nu}=0$

* For $d=2r$, the building blocks of composite ops are scalars, chiral-spinors, (anti) self-dual $\frac{d}{2}$-form field strength, & higher spins.

<u>Rep^{ns}</u>: $l = (s, \ldots, s, \pm s)$

$\rightsquigarrow$ $s \in \mathbb{N}/2$

<u>Free field EOM</u>:

$$\partial^2 \phi = 0, \quad \not{\partial} \psi = 0$$

$\rightsquigarrow$ free - fields are short rep^{ns}.

Long Representations

Consider a unitary irrep $R_{[\Delta;l]}$ of scaling dim Δ and spin $l = (l_1, \ldots, l_r)$.
When the irrep doesn't saturate a unitary bound ($\Delta > \Delta_l$), then we have

		Scaling-dim	Spin
$R_{[\Delta;l]} =$	O_l	Δ	l
	$\partial_\mu O_l$	$\Delta+1$	$\mathrm{Sym}^1(\square)\otimes l$
	$\partial_{\mu_1}\partial_{\mu_2} O_l$	$\Delta+2$	$\mathrm{Sym}^2(\square)\otimes l$
	$\vdots$	$\vdots$	$\vdots$

$\mathrm{Sym}^n(\square) \Rightarrow$ repⁿ formed by n^{th} symm. product of vector repⁿ ($\square$)

$\square \Leftrightarrow l = (1, 0, \ldots 0)$ of $SO(d)$

* Derivative is an $SO(d)$ vector and symmetrization works due to commuting nature of partial derivs.

The character is given as

$$\chi^{(d)}_{[\Delta;l]}(q;x) = \sum_{n=0}^{\infty} q^{\Delta+n}\, \chi^{(d)}_{\text{sym}^n(\square)}(x)\, \chi^{(d)}_{l}(x)$$

$$= q^{\Delta}\, \chi^{(d)}_{l}\, P^{(d)}(q;x)$$

$\chi^{(d)}_{l}(x) \Rightarrow$ character of spin-l repⁿ of $SO(d)$

$$\chi_{R_1 \otimes R_2} = \chi_{R_1} \times \chi_{R_2}$$

* Momentum generating fⁿ

$$P^{(d)}(q;x) = \sum_{n=0}^{\infty} q^{n}\, \chi^{(d)}_{\text{sym}^n(\square)}(x)$$

$q^{\Delta}\chi_l^{(d)} \Rightarrow$ Contribution from primary block.

$P^{(d)}(q;x) \Rightarrow$ generates contribution from all descendants

Short Representation

character formula is modified when a unitary bound is saturated, i.e., $\boxed{\Delta = \Delta_l}$

Consider the short-repⁿ formed by the free scalar field φ w/ $\Delta = \Delta_0 = (d-2)/2$ and $\underline{l} = \underline{0} = (0, \ldots, 0)$.

* EOM provides the shortening condition

$$\boxed{\partial^2 \varphi = 0}$$

$\Rightarrow \exists$ only traceless symmetric components $\partial_{\{\mu_1} \cdots \partial_{\mu_n\}} \varphi$ in the descendants.

then the repⁿ looks like,

		Scaling-dim	Spin
$R_{[\Delta_0; \underline{0}]} =$	φ	Δ_0	0
	$\partial_{\mu_1}\varphi$	$\Delta_0 + 1$	$\square$
	$\partial_{\{\mu_1}\partial_{\mu_2\}}\varphi$	$\Delta_0 + 2$	$\square\square$
	$\vdots$	$\vdots$	

where, $\square\square\cdots\square$ represents sym-traceless rep^n of $SO(d)$, w/ n-indices, corresponding to $\lambda = (n, 0, 0, \cdots, 0)$

e.g. For $SU(2)$: $2 \equiv \square \equiv \alpha + 1/\alpha$

$$2 \otimes 2 = 3 \oplus 1$$

$$\mathrm{Sym}^2(2) = 3 \equiv \alpha^2 + 1/\alpha^2 + 1$$

The set of components $\partial_{\{\mu_1} \cdots \partial_{\mu_n\}} \varphi$ is obtained from $\partial_{\mu_1} \cdots \partial_{\mu_n} \varphi$ by contracting two indices and leaving others fully symmetric, i.e.

$$\chi^{(d)}_{(n,0,\cdots 0)}(x) = \begin{cases} \chi^{(d)}_{\mathrm{Sym}^n(\square)}(x) & , n < 2 \\ \chi^{(d)}_{\mathrm{Sym}^n(\square)}(x) - \chi^{(d)}_{\mathrm{Sym}^{n-2}(\square)} & , n \geq 2 \end{cases}$$

Now, character of the short-rep^n is given as

$$\tilde{\chi}^{(d)}_{[\Delta_0,\underline{0}]}(q;x) = \sum_{n=0}^{\infty} q^{\Delta_0+n}\,\chi^{(d)}_{(n,0,\dots 0)}(x)$$

$$= q^{\Delta_0}(1-q^2)\sum_{n=0} q^n\,\chi^{(d)}_{\mathrm{sym}^n(\square)}(x)$$

$$= q^{\Delta_0}(1-q^2)\,P^{(d)}(q;x)$$

Thus, for free scalar φ

$$\tilde{\chi}^{(d)}_{[\Delta_0,\underline{0}]}(q;x) = \chi^{(d)}_{[\Delta_0;\underline{0}]} - \chi^{(d)}_{[\Delta_0+2;\,\underline{0}]}$$

$\Rightarrow$ subtracting off the states

$(\partial^2\varphi,\ \partial_\mu\partial^2\varphi,\ \partial_{\mu_1}\partial_{\mu_2}\partial^2\varphi,\ \dots)$ from long-rep^n

$(\varphi,\ \partial_\mu\varphi,\ \partial_{\mu_1}\partial_{\mu_2}\varphi,\ \dots)$

- Consider, conserved current $\hat{j}_\mu$.

$$\Delta = d-1, \quad l = (1, 0, \dots 0)$$

current conservation $\Rightarrow \partial_\mu j^\mu = 0$

$(\partial^\mu j_\mu)$ is an op. w/ $\Delta = d, \; l = \underline{0}$

The character of j_μ is given as,

$$\tilde{\chi}^{(d)}_{[d-1,(1,0\dots,0)]} = \chi^{(d)}_{[d-1,(1,\dots,0)]} - \chi^{(d)}_{[d,\underline{0}]}$$

- Consider, Left-handed field strength (in $d=4$)

$$F_{L\mu\nu} = F_{\mu\nu} + \tilde{F}_{\mu\nu}, \quad \text{w/ } \Delta = 2, \; l = (1,1)$$

EOM + Bianchi-identity $\Rightarrow \partial^\mu F_{L\mu\nu} = 0$

Now, $(\partial^\mu F_{L\mu\nu})$ is an op. w/ $\Delta = 3, \; l = (1,0)$

Again, j_μ has same $\Delta = 3, \; l = (1,0)$.

Thus, $(\partial^\mu F_{L\mu\nu})$ saturates unitarity bound itself.

Here, $\partial^{\mu} (\partial \cdot F_L)_{\mu}$ vanishes automatically due to anti-symmetry.

Thus, character of $F_{L\mu\nu}$ in $d=4$, is

$$\tilde{\chi}^{(4)}_{[2;(1,1)]} = \chi^{(4)}_{[2,(1,1)]} - \tilde{\chi}^{(4)}_{[3,(1,0)]}$$

$$= \chi^{(4)}_{[2,(1,1)]} - \left\{ \chi^{(4)}_{[3,(1,0)]} - \chi^{(4)}_{[4,\underline{0}]} \right\}$$

$$= \chi^{(4)}_{[2,(1,1)]} - \chi^{(4)}_{[3,(1,0)]} + \chi^{(4)}_{[4,(0,0)]}$$

Explicit Examples w/ Quantum Fields

① **Scalar**: $\Delta = 1$, $j_1 = j_2 = 0$ w/ $q = D$ (identify)

$$\chi_{[1,(0,0)]}(q;\alpha,\beta) = D^1 P(D;\alpha,\beta)(1 - D^2)$$

② **Fermion**: $\Delta = 3/2$, $j_1 = 1/2$, $j_2 = 0$
$\qquad\qquad\qquad\qquad = 0$, $\qquad = 1/2$

$$\chi_{[3/2;(1/2,0)]} = D^{3/2} P(D;\alpha,\beta) \left[\alpha + 1/\alpha - D(\beta + 1/\beta)\right]$$

$$\chi_{[3/2;(0,1/2)]} = D^{3/2} P(D;\alpha,\beta) \left[\beta + 1/\beta - D(\alpha + 1/\alpha)\right]$$

③ **Field Tensor**: $\Delta = 2$, $j_1 = 1$, $j_2 = 0$
$\qquad\qquad\qquad\quad = 0$, $\qquad = 1$

$$\chi_{[2;(1,0)]} = D^2 P(D;\alpha,\beta)\left[\alpha^2 + \frac{1}{\alpha^2} + 1 - D(\alpha + 1/\alpha)(\beta + 1/\beta) + D^2\right]$$

$$\chi_{[2;(0,1)]} = \chi_{[2;(1,0)]}\Big|_{\alpha \leftrightarrow \beta}$$

Modified Plethystics

Boson

$$PE[\varphi, D, R] = \exp\left[\sum_{r=1}^{\infty}\left(\frac{\varphi}{D^{\Delta\varphi}}\right)^r \frac{\chi_R(z_j^r, \alpha^r, \beta^r)}{r}\right]$$

Fermion

$$PE[\psi, D, R] = \exp\left[\sum_{r=1}^{\infty}(-1)^{r+1}\left(\frac{\psi}{D^{\Delta\psi}}\right)^r \frac{\chi_R(z_j^r, \alpha^r, \beta^r)}{r}\right]$$

Haar Measure

$$\int d\mu_{Lorentz} = \int \frac{1}{P^{(4)}(D;\alpha,\beta)}\, d\mu_{SU(2)}(\alpha)\, d\mu_{SU(2)}(\beta)$$

$$\left[P^{(4)}(D;\alpha,\beta)\right]^{-1}$$

$$= (1-D\alpha\beta)(1-D/\alpha\beta)(1-D\alpha/\beta)(1-D\beta/\alpha).$$

* $D^{-\Delta\phi}$ in PE gets cancelled by $D^{\Delta\varphi}$ from $\chi_{[\Delta\varphi; (j_1, j_2)]}$

Example within SM-gauge group

$$\mathcal{G}_{SM} = U(1)_Y \times SU(2)_L \times SU(3)_C$$

$$Q \equiv (1/6, 2, 3)$$

$$\chi^{U(1)}_{1/6}(z_1) = z_1^{1/6}, \quad \chi^{SU(2)}_{2}(z_2) = z_2 + 1/z_2$$

$$\chi^{SU(3)}_{3}(z_3, z_4) = z_3 + \frac{z_4}{z_3} + \frac{1}{z_4}$$

$$\chi^{gauge}_{Q} = \chi_{1/6}(z_1)\,\chi_2(z_2)\,\chi_3(z_3, z_4)$$

$$\chi_Q = \chi_{[\frac{3}{2};(\frac{1}{2},0)]}(D;\alpha,\beta)\,\chi^{gauge}_{Q}$$

$$PE\left[\frac{Q}{D^{3/2}}\chi_Q\right] = \exp\left[\sum_{r=1}^{\infty}\frac{(-1)^{r+1}}{r}\frac{Q^r}{(D^{3/2})^r}\right.$$

$$\left.\chi_Q[D^r;\alpha^r,\beta^r;z_1^r,z_2^r,z_3^r,z_4^r]\right]$$

$$\int d\mu_{U(1)_Y} = \frac{1}{2\pi i} \oint_{|z_1|=1} \frac{dz_1}{z_1}$$

$$\int d\mu_{SU(2)_L} = \frac{1}{2\cdot(2\pi i)} \oint_{|z_2|=1} \frac{dz_2}{z_2} \left(1-z_2^2\right)\left(1-\frac{1}{z_2^2}\right)$$

$$\int d\mu_{SU(3)_c} = \frac{1}{6\cdot(2\pi i)^2} \oint_{|z_3|=1} \oint_{|z_4|=1} \frac{dz_3}{z_3} \frac{dz_4}{z_4}$$

$$(1-z_3z_4)\left(1-\frac{z_3^2}{z_4}\right)\left(1-\frac{z_4^2}{z_3}\right)\left(1-\frac{1}{z_3z_4}\right)$$

$$\left(1-\frac{z_4}{z_3^2}\right)\left(1-\frac{z_3}{z_4^2}\right)$$

$$\int d\mu_{SU(2)_l\times SU(2)_r} = \frac{1}{[2\cdot(2\pi i)]^2} \oint_{|\alpha|=1} \oint_{|\beta|=1} \frac{d\alpha}{\alpha} \frac{d\beta}{\beta}$$

$$(1-\alpha^2)\left(1-\frac{1}{\alpha^2}\right)(1-\beta^2)\left(1-\frac{1}{\beta^2}\right)$$